Tôhoku Math. J.
47 (1995), 15–29

A DIFFERENTIABLE SPHERE THEOREM BY CURVATURE PINCHING II

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

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(Received May 31, 1993, revised August 19, 1994)

Abstract. We give a new diffeotopy theorem on the standard sphere, and an estimate for some geometric invariants concerning positively curved Riemannian manifold. By using these results we prove that a complete, simply connected and 0.654-pinched Riemannian manifold is diffeomorphic to the standard sphere.

Introduction. Let (M^n, g) be a complete, simply connected and δ-pinched Riemannian n-manifold. In this paper we prove that if $\delta=0.654$, then M is diffeomorphic to the standard sphere S^n.

For a $\delta(>1/4)$-pinched Riemannian n-manifold, an orientation preserving diffeomorphism f of S^{n-1} is naturally defined, and is used in the proof of the differentiable sphere theorem [3, 4]. In fact, if there exists a diffeotopy from f to an isometry f_1 of S^{n-1}, then M is diffeomorphic to the standard sphere. In order to find the minimum of such δ's it is important to construct a diffeotopy in as many different ways as possible. In this paper, we propose a new construction of a diffeotopy. The statement of our diffeotopy theorem and the construction of diffeotopy in it are fairly simple in comparison with these in [4]. Furthermore, by giving new estimates concerning f and its differential df we prove the differentiable sphere theorem above. In this paper we use the same notation as in [4, §2–§6].

The author would like to thank the referees for careful reading of the previous versions of this paper and for valuable suggestions for improvements.

1. $\delta(>1/4)$-pinched Riemannian manifolds. Let (M^n, g) be a complete, simply connected and $\delta(>1/4)$-pinched Riemannian n-manifold, i.e., the sectional curvature K of M satisfies $\delta \leq K \leq 1$ everywhere. We denote by D the Levi-Civita connection induced by the Riemannian metric g. First, we review the definitions of the diffeomorphism f, mentioned in the Introduction, and the differentiable map $\alpha: S^{n-1} \ni x \mapsto \alpha_x \in SO(n, \boldsymbol{R})$, which is regarded as an approximation of df, and related results in (A) and (B) below (cf. [4]). Let S^{n-1} be the standard sphere with sectional curvature 1, i.e., $S^{n-1}=S^{n-1}(1)$. We denote by $d_s(x, y)$ the distance between x and y

1991 *Mathematics Subject Classification*. Primary 53C20.

on S^{n-1}. Secondly, we estimate $d_s((df)_x X/\|(df)_x X\|, \alpha_x X)$ and $d_s(\alpha_x V, \alpha_{-x} V)$ for any $x \in S$ and any unit vectors $X \in T_x(S^{n-1})$ and $V \in \boldsymbol{R}^n$ in (C) and (D), which are necessary for the diffeotopy theorem.

(A) Diffeomorphism $f: S^{n-1} \to S^{n-1}$. The manifold M is homeomorphic to the standard sphere by the sphere theorem. In particular, we use the following properties of M. Let q_0 and q_1 be a pair of points with maximal distance $d_M(q_0, q_1)$ on M, where d_M denotes the distance function induced by g. We put

$$M_0 = \{ p \in M \mid d_M(p, q_0) \leq d_M(p, q_1) \},$$

$$M_1 = \{ p \in M \mid d_M(p, q_0) \geq d_M(p, q_1) \},$$

$$C = \{ q \in M \mid d_M(q, q_0) = d_M(q, q_1) \}.$$

Let S_0 and S_1 denote the unit spheres in the tangent spaces $T_{q_0}(M)$ and $T_{q_1}(M)$ of points q_0 and q_1, respectively. The exponential maps Exp_0 and Exp_1 with centers at q_0 and q_1, respectively, are bijective maps if restricted to an open ball of radius π. Then we can define a diffeomorphism $f: S_0 \to S_1$ by requiring the geodesics $\mathrm{Exp}_0(tx)$ and $\mathrm{Exp}_1[tf(x)]$ to coincide at some $t = t(x)$ satisfying $\pi/2 \leq t(x) \leq \pi/2\sqrt{\delta}$. We put $q(x) = \mathrm{Exp}_0[t(x)x]$. Note that $q(x) \in C$ for $x \in S_0$.

We indentify $T_{q_0}(M)$ with $T_{q_1}(M)$ by fixing their orthonormal bases. Then we can regard f as a diffeomorphism of S^{n-1}. We fix a minimal geodesic $\gamma = \gamma(t)$ joining $q_0 = \gamma(0)$ to $q_1 = \gamma[d(q_0, q_1)]$. Let $\{X_1, X_2, \dots, X_n\}$ be an orthonormal basis of $T_{q_0}(M)$ with $X_n = \dot{\gamma}(0)$. The orthonormal basis $\{X_1, \dots, X_n\}$ of $T_{q_1}(M)$ is now defined by the parallel translation with respect to D of $\{X_1, \dots, X_{n-1}, -X_n\}$ ($\subset T_{q_0}(M)$) along γ. Then we have the following Proposition. We denote S^{n-1} simply by S from now on.

PROPOSITION 1 (cf. [3] and [4]). *Let f be a diffeomorphism of S as above. We assume that there exists a differentiable map F: $[0, 1] \times S \to S$ satisfying the following conditions*:

(1) $F(0, \cdot) = f$.

(2) $F_1 = F(1, \cdot)$ *is an isometry of* S.

(3) $F_t = F(t, \cdot): S \to S$ *is a diffeomorphism for each* $t \in [0, 1]$.

Then M is diffeomorphic to the standard sphere S^n.

We call F in Proposition 1 a diffeotopy constructed from f.

REMARK 1. We can replace the assumption of Proposition 1 by the following: There exists a differentiable map F: $[0, 1] \times S \to \boldsymbol{R}^n - \{0\}$ satisfying the following conditions (1), (2) and (3).

(1) $F_0 = f$.

(2) F_1 is the restriction to S of a linear automorphism of $\boldsymbol{R}^n$.

(3) $\Pi \circ F_t$: $S \to S$ is a diffeomorphism for each $t \in [0, 1]$, where Π: $\boldsymbol{R}^n - \{0\} \to S$ is the natural projection.

This fact was pointed out in the discussion with Takashi Sakai, Tetsuya Ozawa and Atsushi Katsuda, and turned out to be useful for our construction of the diffeotopy.

(B) The properties of f. Let τ_x^i $(i=0, 1)$ be a geodesic defined by $\tau_x^i(t)=\mathrm{Exp}_i(tx)$ for $x \in S$. Let $V \in T_{q(x)}(C)$. We define tangent vectors V^0 and V^1 at $q(x)$ for $V \in T_{q(x)}(C)$ by

$$\begin{cases} V^0 = V - g[V, \dot{\tau}_x^0(t(x))]\dot{\tau}_x^0(t(x)) , \\ V^1 = V - g[V, \dot{\tau}_{f(x)}^1(t(x))]\dot{\tau}_{f(x)}^1(t(x)) , \end{cases}$$

respectively. We extend V^0 and V^1 to the Jacobi fields along the geodesics τ_x^0 and $\tau_{f(x)}^1$, respectively, satisfying $V_{q(x)}^0 = V^0$, $V_{q(x)}^1 = V^1$ and $V_{q_0}^0 = V_{q_1}^1 = 0$. By the definition of f, we have

$$(df)_x(D_x V^0) = D_{f(x)} V^1 \qquad \text{for} \quad V \in T_{q(x)}(C) .$$

The Toponogov comparison theorem yields the following estimates:

$$(1.1) \quad \begin{cases} d_s(f(x), f(y)) \geq \sqrt{\delta}\, \sin(\pi/2\sqrt{\delta}\,) d_s(x, y) \qquad \text{for} \quad (x, y) \in S \times S . \\ \sqrt{\delta}\, \sin(\pi/2\sqrt{\delta}\,) \leq \dfrac{\|(df)_x X\|}{\|X\|} \leq [\sqrt{\delta}\, \sin(\pi/2\sqrt{\delta}\,)]^{-1} \qquad \text{for} \quad X \neq 0 \in T_x(S) . \end{cases}$$

We put

$$(1.2) \qquad L = L(\delta) = \sqrt{\delta}\, \sin(\pi/2\sqrt{\delta}\,) .$$

We now define a differentiable map $\alpha\colon S \ni x \mapsto \alpha_x \in SO(n, \boldsymbol{R})$ (cf. [4, Prop. 2]):

$$(1.3) \quad \begin{cases} (1) \quad \alpha_x x = f(x) \qquad \text{for} \quad x \in S , \\ (2) \quad \alpha_x([\tau_x^0]_0^{t(x)} V^0) = [\tau_{f(x)}^1]_0^{t(x)} V^1 \qquad \text{for} \quad V \in T_{q(x)}(C) , \end{cases}$$

where $[\tau_x^i]_0^{t(x)}$ denotes the parallel translation with respect to D along τ_x^i, and each vector in (1) and (2) is the component vector with respect to the basis $\{X_1, \dots, X_n\}$.

(C) The estimate for $d_s((df)_x X/\|(df)_x X\|, \alpha_x X)$.

LEMMA 1 (cf. [4, Prop. 2]). *Let $c = \sqrt{(1+\delta)/2}$. Then we have*

$$\|(df)_x X - \alpha_x X\| \leq B_0(\|X\| + \|(df)_x X\|) \qquad \textit{for} \quad X \in T_x(S) ,$$

where

$$B_0 = B_0(\delta) = \frac{1-\delta}{2(1+c^2)} \left\{ c \sinh\left(\frac{\pi}{2\sqrt{\delta}}\right) \Big/ \sin\left(\frac{c\pi}{2\sqrt{\delta}}\right) - 1 \right\} .$$

We have

$$(1.4) \qquad (df)_x X = \alpha_x X + (d_X \alpha.)x \qquad \text{for} \quad X \in T_x(S)$$

by (1.3).

PROPOSITION 2. *Assume* $B_0<1$. *Then we have*

$$d_s((df)_xX/\|(df)_xX\|,\ \alpha_xX)\le\theta_2$$

for $(x, X)\in S\times S$ *with* $\langle x, X\rangle=0$, *where* $\theta_2=\theta_2(\delta)=\cos^{-1}(1-2B_0^2)$.

PROOF. We put $u=\|(df)_xX\|$,

$$\theta=d_s\left(\frac{(df)_xX}{\|(df)_xX\|},\ \alpha_xX\right)\quad\text{and}\quad\bar\theta=d_s\left(\alpha_xX,\ \frac{(d_X\alpha.)x}{\|(d_X\alpha.)x\|}\right),$$

where $L\le u\le L^{-1}$. We have

$$\begin{aligned}u^2&=(1+\|(d_X\alpha.)x\|\cos\bar\theta)^2+\|(d_X\alpha.)x\|^2\sin^2\bar\theta\\&=1+2\|(d_X\alpha.)x\|\cos\bar\theta+\|(d_X\alpha.)x\|^2\\&\le1+2\|(d_X\alpha.)x\|\cos\bar\theta+B_0^2(1+u)^2\end{aligned}$$

by Lemma 1 and (1.4). If $B_0<1$, then we have

$$\cos\theta=\frac{1+\|(d_X\alpha.)x\|\cos\bar\theta}{u}\ge\frac{1+u^2-B_0^2(1+u)^2}{2u}\ge1-2B_0^2\,. \tag{1.5}$$

The minimum in (1.5) is attained at $u=1$. q.e.d.

REMARK 2. We have $B_0(0.373)=0.997251$. Therefore, if $\delta\ge0.373$, then $B_0(\delta)<1$ holds.

(D) The estimate for $d_s(\alpha_xV, \alpha_{-x}V)$. Let us take $(x, V)\in S\times S$. Then V can be written as $V=\sin\xi x+\cos\xi Y$, $-\pi/2\le\xi\le\pi/2$, by a unit vector $Y\in T_x(S)$ ($\subset\boldsymbol{R}^n$). For a while we assume $\cos\xi\ne0$. Let $x(t)$ $(0\le t\le\pi)$ be a geodesic joining $x=x(0)$ to $-x=x(\pi)$ with $\dot x(0)=Y$. We have $x=\cos t\,x(t)-\sin t\,\dot x(t)$, $Y=\sin t\,x(t)+\cos t\,\dot x(t)$ for $t\in[0, \pi]$. Thus we have $V=\sin(t+\xi)x(t)+\cos(t+\xi)\dot x(t)$ for $t\in[0, \pi]$. Therefore we have

$$\begin{aligned}d_s(\alpha_xV, \alpha_{-x}V)&\le\int_0^\pi\left\|\frac{d}{dt}\alpha_{x(\cdot)}V\right\|dt\\&=\int_0^\pi\|\sin(t+\xi)(d_{\dot x(t)}\alpha.)x(t)+\cos(t+\xi)(d_{\dot x(t)}\alpha.)\dot x(t)\|\,dt\,.\end{aligned}\tag{1.6}$$

We study the integrand of (1.6). We choose $N_i(\delta)$ $(i=2, 3)$ satisfying

$$\|(d_X\alpha.)x\|\le N_2(\delta)\,,\qquad\|(d_X\alpha.)X\|\le N_3(\delta)\tag{1.7}$$

for any $x\in S$ and any unit vector $X\in T_x(S)$. We can take $N_2=N_2(\delta)=B_0(1+L^{-1})$ by $(df)_xX=\alpha_xX+(d_X\alpha.)x$ and Lemma 1. Furthermore, we put $\|(d_X\alpha.)V\|\le N_1(\delta)$ for any unit vector $V\in\boldsymbol{R}^n$. As for the estimate of $N_1(\delta)$ we refer to [4, Lemma 8]. We can take $N_3(\delta)=N_1(\delta)$, but we estimate $N_3=N_3(\delta)$ more sharply in §3 and §4 below.

Now, we put $d_s((df)_x X/\|(df)_x X\|, \alpha_x X)=\theta$. Then we have

$$|\langle (d_X\alpha.)X, (d_X\alpha.)x\rangle| \le \|(df)_x X\| \, \|(d_X\alpha.)X\| \sin\theta$$

by $\langle (d_X\alpha.)X, \alpha_x X\rangle = 0$.

LEMMA 2. *Assume $B_0<1$. Then we have*

$$\|(df)_x X\| \sin\theta \le N_4 \qquad \textit{for any unit vector} \quad X\in T_x(S)\,,$$

where, taking $v=\min\{L^{-1}, (2B_0^2+1+\sqrt{8B_0^2+1})/(2(1-B_0^2))\}$,

$$N_4=N_4(\delta)=\frac{1+v}{2}\sqrt{(1-B_0^2)[B_0^2(1+v)^2-(1-v)^2]}\,.$$

PROOF. We put $u=\|(df)_x X\|$. We have

$$u\cos\theta \ge \frac{1}{2}\{1+u^2-B_0^2(1+u)^2\}$$

by (1.5). Thus we have

$$u^2\sin^2\theta \le \frac{1}{4}(1+u)^2(1-B_0^2)[B_0^2(1+u)^2-(1-u)^2]\,.$$

q.e.d.

Thus, by (1.6) and the continuity for V of $d_s(\alpha_x V, \alpha_{-x} V)$ we have the following:

PROPOSITION 3. *Assume $B_0<1$. Then we have*

$$d_s(\alpha_x V, \alpha_{-x} V) \le \theta_1 \qquad \textit{for} \quad (x, V)\in S\times S\,,$$

where $\theta_1=\theta_1(\delta)=2\int_0^{\pi/2}\sqrt{N_3^2-[N_3^2-N_2^2]\sin^2 t+2(N_3N_4)\sin t\cos t}\,dt$.

REMARK 3. (1) We have

$$d_s(\alpha_x x, \alpha_{-x} x) \le \pi(1-L(\delta)) \qquad \text{for} \quad x\in S$$

by (1.1). By the calculation in §5, we have $\pi(1-L(\delta))<\theta_1(\delta)$.

(2) We can always take $\theta_1=N_1\pi$. The estimate of θ_1 in Proposition 3 is more precise than $N_1\pi$.

2. A diffeotopy theorem. Let f be a diffeomorphism of S and α a differentiable map of S into $SO(n, \boldsymbol{R})$ with $f(x)=\alpha_x x$. We choose numbers N_1, θ_1 and θ_2 satisfying

$$\|(d_X\alpha.)V\| \le N_1\,, \quad d_s(\alpha_x V, \alpha_{-x} V) \le \theta_1\,, \quad d_s\!\left(\alpha_x X, \frac{(df)_x X}{\|(df)_x X\|}\right) \le \theta_2$$

for any $x\in S$ and any unit vectors $X\in T_x(S)$ and $V\in \boldsymbol{R}^n$.

THEOREM 1. *If $N_1\pi+\theta_1+2\theta_2<2\pi$, then there exists a diffeotopy F constructed from f.*

PROOF. We fix $x_0\in S$, and define a differentiable map $G\colon[0,1]\times S\to\boldsymbol{R}^n$ by $G(t,x)=t\alpha_{x_0}x+(1-t)\alpha_x x$. If $G(t,x)\in\boldsymbol{R}^n-\{0\}$ for $(t,x)\in[0,1]\times S$, then we define $F(t,x)=\Pi\circ G(t,x)$, where $\Pi\colon\boldsymbol{R}^n-\{0\}\to S$ is the natural projection. We have $F(0,x)=f(x)$ and $F(1,x)=\alpha_{x_0}x$. Therefore, if $F_t\colon S\to S$ is a diffeomorphism for each $t\in[0,1]$, then F is a diffeotopy constructed from f.

We put $\bar{G}(t,x)=\|x\|G(t,x/\|x\|)$ for $x\in\boldsymbol{R}^n-\{0\}$. We have

$$(2.1)\qquad \begin{cases}(d\bar{G}_t)_x x=t\alpha_{x_0}x+(1-t)\alpha_x x=\bar{G}(t,x)\,,\\ (d\bar{G}_t)_x X=t\alpha_{x_0}X+(1-t)\{(d_X\alpha.)x+\alpha_x X\}\\ \qquad\qquad =t\alpha_{x_0}X+(1-t)(df)_x X\,,\end{cases}$$

for $(x,X)\in S\times S$ with $\langle x,X\rangle=0$. If $(d\bar{G}_t)_x$ is regular for $x\in S$, then $(dF_t)_x$ is also regular by $(d\bar{G}_t)_x x=\bar{G}(t,x)$. Therefore we must show that $G(t,x)\in\boldsymbol{R}^n-\{0\}$ for $(t,x)\in[0,1]\times S$, and $(d\bar{G}_t)_x$ is regular for $(t,x)\in[0,1]\times S$. But, if $(d\bar{G}_t)_x$ is regular for $(t,x)\in[0,1]\times S$, then $G(t,x)\in\boldsymbol{R}^n-\{0\}$ for $(t,x)\in[0,1]\times S$ holds from the first equation of (2.1). Let us take a unit vector $Z\in T_x(\boldsymbol{R}^n)$ for $x\in S$, and write it as $Z=ax+bX$, $a^2+b^2=1$, by using a unit vector $X\in T_x(S)$. Then we have

$$(2.2)\qquad (d\bar{G}_t)_x Z=t\alpha_{x_0}Z+(1-t)\alpha_x Z+(1-t)b(d_X\alpha.)x$$

by (2.1). Since $|b|\le 1$ at (2.2), we have only to show $(d\bar{G}_t)_x X\neq 0$ for $(t,x,X)\in[0,1]\times S\times S$ with $\langle x,X\rangle=0$.

Let us take $(x,X)\in S\times S$ with $\langle x,X\rangle=0$ and $x\in S-\{x_0,-x_0\}$. Let $\eta=\eta(t)$ be a geodesic in S which joins $x_0=\eta(0)$ to $-x_0=\eta(\pi)$ and passes through x. Then the length of the curve $\alpha_{\eta(t)}X$ $(0\le t\le\pi)$ in S is given by

$$(2.3)\qquad d_s(\alpha_{x_0}X,\alpha_x X)+d_s(\alpha_x X,\alpha_{-x_0}X)\le\int_0^\pi\|(d_{\dot\eta(t)}\alpha.)X\|dt\le N_1\pi\,.$$

Now, we assume $\theta_1\le N_1\pi$ and $(N_1\pi+\theta_1)/2\le\pi$. We take a point $\bar{p}\in S$ which satisfies the following (1) and (2):

(1) $\bar{p}$ is on the geodesic which issues from $\alpha_{x_0}X$ and passes through $\alpha_{-x_0}X$.

(2) $d_s(\alpha_{x_0}X,\bar{p})=(N_1\pi+\theta_1)/2$.

Then we have

$$d_s(\alpha_{x_0}X,\bar{p})\ge\max\{d_s(\alpha_{x_0}X,\alpha_{\eta(t)}X)\mid 0\le t\le\pi\}$$

by (2.3) and $d_s(\alpha_{x_0}X,\alpha_{-x_0}X)\le\theta_1$. Therefore we have $d_s(\alpha_{x_0}X,\alpha_x X)\le(N_1\pi+\theta_1)/2$. Furthermore, if $(N_1\pi+\theta_1)/2+\theta_2<\pi$ holds, there exists no constant c (≥ 0) such that $\alpha_{x_0}X=-c(df)_x X$ by $d_s((df)_x X/\|(df)_x X\|,\alpha_x X)\le\theta_2$. Therefore, if $N_1\pi+\theta_1+2\theta_2<2\pi$ holds, then we have $(d\bar{G}_t)_x X\neq 0$ by (2.1). q.e.d.

3. The stabilized tangent bundle E of M. In this and the following sections we estimate $N_3=N_3(\delta)$ such that $\|(d_X\alpha.)X\|\le N_3(\delta)$. As we remarked it in § 1, (D), we estimate $N_3(\delta)$ more sharply than $N_1(\delta)$ such that $\|(d_X\alpha.)V\|\le N_1(\delta)$. To estimate $N_1(\delta)$ we used the second inequality of (3.1) below [4]. On the other hand, we use the first inequality of (3.1) to estimate a main term of $\|(d_X\alpha.)X\|$. To start with, we review several results for the stabilized tangent bundle of M in (A).

(A) A connection with small curvature on E. The stabilized tangent bundle E of M is given by $E=T(M)\oplus 1(M)$, where $T(M)$ and $1(M)$ are the tangent bundle and trivial line bundle $M\times\boldsymbol{R}$, respectively. Let $\boldsymbol{e}: M\ni p\mapsto \boldsymbol{e}_p\in E$ be a cross-section defined by $\boldsymbol{e}_p=(\boldsymbol{o}_p, 1)\in T_p(M)\times\boldsymbol{R}$. Let h be a fibre metric on E given by

$$h(X, Y)=g(X, Y)\,,\quad h(X, \boldsymbol{e}_p)=0 \quad\text{and}\quad h(\boldsymbol{e}_p, \boldsymbol{e}_p)=1$$

for $X, Y\in T_p(M)$. An h-metric connection ∇ on E is given by

$$\nabla_X Y=D_X Y-cg(X, Y)\boldsymbol{e}\,,\quad \nabla_X\boldsymbol{e}=cX$$

for $X, Y\in T(M)$, where $c=\sqrt{(1+\delta)/2}$. The connection ∇ has the curvature tensor $R^\nabla=R-c^2\bar{R}$, where R is the Riemannian curvature tensor on M and $\bar{R}$ is the algebraic expression for the curvature tensor on the standard sphere $S^n(1)$ in terms of the Riemannian metric g on M. We have $R^\nabla(X, Y)\boldsymbol{e}=0$ for $X, Y\in T(M)$ and

$$\|R^\nabla(X, Y)Y\|\le(1-\delta)/2\,,\quad \|R^\nabla(X, Y)Z\|\le 2(1-\delta)/3\,, \tag{3.1}$$

for $X, Y, Z\in T_p(M)$ with $\|X\|=\|Y\|=\|Z\|=1$. In fact, the first inequality of (3.1) implies that (M, g) is δ-pinched. As for the second inequality we refer to [2].

Let P be a principal bundle over M of $(n+1)$-frames with structure group $O(n+1, \boldsymbol{R})$ associated to E. Then the connection form ω and the curvature form Ω^∇ induced by ∇ satisfy the structure equation $d\omega=-\omega\wedge\omega+\Omega^\nabla$. We take a cross-section $u^i=(\boldsymbol{u}^i_1, \dots, \boldsymbol{u}^i_{n+1})$: $M_i\to P|_{M_i}$ $(i=0, 1)$ as follows: First we choose $u^0(q_0)=(X_1, \dots, X_n, \boldsymbol{e}_{q_0})$. Second, we define a section u^0 on M_0 by moving the $(n+1)$-frame $u^0(q_0)$ by parallel translation with respect to ∇ along the geodesic from q_0 to points in M_0. Next, we choose $u^1(q_1)=(X_1, \dots, X_n, -\boldsymbol{e}_{q_1})$. Then we can also take a cross-section u^1: $M_1\to P|_{M_1}$ analogous to u^0.

There exists a differentiable map $\mathscr{A}$: $C=M_0\cap M_1\to O(n+1, \boldsymbol{R})$ such that $u^0(q)\mathscr{A}(q)=u^1(q)$ for $q\in C$. We note

$$\mathscr{A}(q)({}^t[z_1^1, \dots, z_1^{n+1}])={}^t[z_0^1, \dots, z_0^{n+1}] \tag{3.2}$$

for $Z=\sum_{i=1}^{n+1} z_0^i(\boldsymbol{u}_i^0)_q=\sum_{i=1}^{n+1} z_1^i(\boldsymbol{u}_i^1)_q\in E_{\pi^{-1}(q)}$ $(q\in C)$. We put $\beta_x=\mathscr{A}(q(x))$ for $x\in S_0$.

(B) Relation between α_x and β_x. Let $w(x)$ be a vector of $T_{q(x)}(M)$ defined by

$$w(x)=\frac{[\tau_x^0]_{t(x)}^0 x-[\tau_{f(x)}^1]_{t(x)}^0 f(x)}{\|[\tau_x^0]_{t(x)}^0 x-[\tau_{f(x)}^1]_{t(x)}^0 f(x)\|}\,,$$

where $[\tau_x^0]_{t(x)}^0$ and $[\tau_{f(x)}^1]_{t(x)}^0$ denote the parallel translation with respect to D along the geodesics τ_x^0 and $\tau_{f(x)}^1$, respectively. Note that $[\tau_x^0]_{t(x)}^0 x - [\tau_{f(x)}^1]_{t(x)}^0 f(x) \neq 0$ for $x \in S_0$. We also denote $[u^1(q)]^{-1} w(x) \in \boldsymbol{R}^{n+1}$ by $w(x)$, where $u^1(q)\colon \boldsymbol{R}^{n+1} \to E_q$ is the natural linear isomorphism. Let us put

$$\bar{\alpha}_x = \begin{bmatrix} \alpha_x, & 0 \\ 0, & -1 \end{bmatrix} \quad \text{for} \quad x \in S_0 .$$

LEMMA 3 (cf. [4, Prop. 3 and its Cor.]). *We express* ${}^t\beta_x = [\boldsymbol{a}_1(x), \ldots, \boldsymbol{a}_{n+1}(x)]$ *and* $\bar{\alpha}_x = [\boldsymbol{b}_1(x), \ldots, \boldsymbol{b}_{n+1}(x)]$ *in terms of the column vectors* $\boldsymbol{a}_i(x)$ *and* $\boldsymbol{b}_i(x)$ *in* $\boldsymbol{R}^{n+1}$. *Then we have* $\boldsymbol{b}_i(x) = \boldsymbol{a}_i(x) - 2\langle \boldsymbol{a}_i(x), w(x) \rangle w(x)$. *In particular, we put*

$$\boldsymbol{a}_{n+1}(x) = \begin{bmatrix} \sin u(x) \cdot \boldsymbol{a}(x) \\ \cos u(x) \end{bmatrix} \quad \textit{for} \quad \boldsymbol{a}(x) \in \boldsymbol{R}^n .$$

Then we have

$$w(x) = \begin{bmatrix} \sin(u(x)/2)\boldsymbol{a}(x) \\ \cos(u(x)/2) \end{bmatrix} .$$

We have

$$\begin{aligned} \cos u(x) &= h(\boldsymbol{u}_{n+1}^0, \boldsymbol{u}_{n+1}^1)(q(x)) \\ &= -\cos^2(ct(x)) - \sin^2(ct(x)) g(\dot{\tau}_x^0, \dot{\tau}_{f(x)}^1)(q(x)) \\ &= -1 + \sin^2(ct(x))[1 - g(\dot{\tau}_x^0, \dot{\tau}_{f(x)}^1)(q(x))] \\ &\geq -1 + \sin^2(c\pi/2\sqrt{\delta})[1 - \cos(\sqrt{\delta}\,\pi)] , \end{aligned} \tag{3.3}$$

and

$$\cos^2(u(x)/2) \geq \frac{1}{2} \sin^2(c\pi/2\sqrt{\delta})[1 - \cos(\sqrt{\delta}\,\pi)] , \tag{3.4}$$

by [4, (2.1) and (2.2)]. We should correct [4, the second equation of (2.2)] as follows:

$$(\boldsymbol{u}_{n+1}^1)_{\tau_x^1(t)} = -\cos(ct)\boldsymbol{e} + \sin(ct)x . \tag{2.2}$$

We put

$$\cos(u/2) = \sin(c\pi/2\sqrt{\delta})\sqrt{(1 - \cos(\sqrt{\delta}\,\pi))/2} . \tag{3.5}$$

(C) The norms of $(d_X\alpha.)$ and $(d_X\beta.)$. Let $X \in T_x(S_0) = T_x(S)$. We also denote the $(n+1)$ vector $\begin{bmatrix} X \\ 0 \end{bmatrix}$ by X.

LEMMA 4. *For any unit vector* $X \in T_x(S_0) = T_x(S)$, *we have*

$$\|(d_X\alpha.)X\| \cos u/2 \leq \|(d_X\beta.)[\bar{\alpha}_x X - (\cos u(x)/2)^{-1} \langle \bar{\alpha}_x X, w(x) \rangle \boldsymbol{e}_{n+1}]\| .$$

PROOF. By $\|(d_X\alpha.)X\| = \|(d_X{}^t\alpha.)(\alpha_x X)\| = \|(d_X{}^t\bar{\alpha}.)(\bar{\alpha}_x X)\|$, we estimate $\|(d_X{}^t\bar{\alpha}.)(\bar{\alpha}_x X)\|$.

For $Z \in \boldsymbol{R}^{n+1}$ with $\langle Z, w(x)\rangle = 0$, we have $\|(d_X{}^t\bar{\alpha}.)Z\| \cos(u/2) \leq \|(d_X\beta.)Z\|$ by [4, (5.3) and (5.7)]. Since we have

$$(3.6)\qquad \begin{cases} (d_X{}^t\bar{\alpha}.)(\bar{\alpha}_x X) = (d_X{}^t\bar{\alpha}.)[\bar{\alpha}_x X - (\cos u(x)/2)^{-1}\langle \bar{\alpha}_x X, w(x)\rangle \boldsymbol{e}_{n+1}] \\ \langle \bar{\alpha}_x X - (\cos u(x)/2)^{-1}\langle \bar{\alpha}_x X, w(x)\rangle \boldsymbol{e}_{n+1}, w(x)\rangle = 0 \end{cases}$$

by $(d_X{}^t\bar{\alpha}.)\boldsymbol{e}_{n+1} = 0$, we are done. q.e.d.

LEMMA 5. *For any unit vector* $X \in T_x(S_0) = T_x(S)$, *we have*

$$\langle \bar{\alpha}_x X, w(x)\rangle = -\langle {}^t\beta_x X, w(x)\rangle\,, \qquad |\langle \bar{\alpha}_x X, w(x)\rangle| \leq c_1\,,$$

where $c_1 = \sqrt{(1+\cos(\sqrt{\delta}\,\pi))/2}$.

PROOF. We have $\bar{\alpha}_x X = {}^t\beta_x X - 2\langle {}^t\beta_x X, w(x)\rangle w(x)$ by Lemma 3. Therefore we have $\langle \bar{\alpha}_x X, w(x)\rangle = -\langle {}^t\beta_x X, w(x)\rangle$. Since $\beta_x \boldsymbol{a}_{n+1} = \boldsymbol{e}_{n+1}$ and

$$\begin{aligned} w(x) &= \sin(u(x)/2)\boldsymbol{a}(x) + \cos(u(x)/2)\boldsymbol{e}_{n+1} \\ &= (\sin(u(x)/2)/\sin u(x))(\boldsymbol{a}_{n+1}(x) - \cos u(x)\boldsymbol{e}_{n+1}) + \cos(u(x)/2)\boldsymbol{e}_{n+1} \\ &= (\sin(u(x)/2)/\sin u(x))\boldsymbol{a}_{n+1}(x) + (2\cos(u(x)/2))^{-1}\boldsymbol{e}_{n+1} \end{aligned}$$

by Lemma 3, we have

$$(3.7)\qquad \langle {}^t\beta_x X, w(x)\rangle = \langle X, \beta_x w(x)\rangle = \langle X, \beta_x \boldsymbol{e}_{n+1}\rangle/(2\cos(u(x)/2))$$

by $\langle X, \boldsymbol{e}_{n+1}\rangle = 0$. We put $\beta_x \boldsymbol{e}_{n+1} = {}^t[z_0^1, \ldots, z_0^{n+1}]$, then we have $(\boldsymbol{u}_{n+1}^1)_{q(x)} = \sum_{i=1}^{n+1} z_0^i(\boldsymbol{u}_i^0)_{q(x)}$ by (3.2). Thus we have

$$(3.8)\qquad -\cos(ct(x))\boldsymbol{e} + \sin(ct(x))\dot{\tau}_{f(x)}^1 = \sum_{i=1}^{n+1} z_0^i(\boldsymbol{u}_i^0)_{q(x)}$$

by [4, (2.2)]. Furthermore, we put $X = {}^t[a^1, \ldots, a^n]$, then we have

$$(3.9)\qquad \sum_{i=1}^{n} a^i(\boldsymbol{u}_i^0)_{q(x)} = [\tau_x^0]_{t(x)}^0 X$$

by [4, (2.1)] and $\langle x, X\rangle = 0$. Since (3.7), (3.8) and (3.9), we have

$$\langle {}^t\beta_x X, w(x)\rangle = (\sin(ct(x))/2\cos(u(x)/2))g(\dot{\tau}_{f(x)}^1, [\tau_x^0]_{t(x)}^0 X)\,.$$

Finally, we have

$$\begin{aligned} |\langle \bar{\alpha}_x X, w(x)\rangle| &\leq \frac{1}{\sqrt{2}}\sqrt{(1-\cos(\sqrt{\delta}\,\pi))^{-1}}\cos\left(\sqrt{\delta}\,\pi - \frac{\pi}{2}\right) \\ &= \frac{1}{\sqrt{2}}\sqrt{(1-\cos(\sqrt{\delta}\,\pi))^{-1}}\sin(\sqrt{\delta}\,\pi)\,, \end{aligned}$$

by (3.3). q.e.d.

4. Holonomy estimate for the stabilized tangent bundle. Let $X \in T_x(S_0) = T_x(S)$ with $\|X\| = 1$. We again denote the $(n+1)$-vector $\begin{bmatrix} X \\ 0 \end{bmatrix}$ by X. In this section we estimate $\|(d_X\beta.)[\bar{\alpha}_x X - (\cos u(x)/2)^{-1}\langle \bar{\alpha}_x X, w(x)\rangle e_{n+1}]\|$, which appeared in Lemma 4.

(A) Let $x(s)$ be a differentiable curve in S_0 with $x(0) = x$ and $\dot{x}(0) = X$. For the curve $q(s) = \tau^0_{x(s)}[t(x(s))]$ in C, $v^0(s)$ and $v^1(s)$ are the horizontal lifts to P of $q(s)$ satisfying $v^0(0) = u^0[q(0)]$ and $v^1(0) = u^1[q(0)]$, respectively. Then there exist $O(n+1, \boldsymbol{R})$-valued functions $b^0(s)$ and $b^1(s)$ satisfying

$$\begin{cases} v^0(s) = u^0[q(s)]b^0(s) \\ b^0(0) = E \end{cases} \quad \text{and} \quad \begin{cases} v^1(s) = u^1[q(s)]b^1(s) \\ b^1(0) = E \, . \end{cases}$$

Then we have

$$\beta_{x(s)} - \beta_x = [b^0(s) - E]\beta_x[b^1(s)]^{-1} + \beta_x[(b^1(s))^{-1} - E] \, , \tag{4.1}$$

because of $\beta_{x(s)} = b^0(s)\beta_x[b^1(s)]^{-1}$.

Let $D_i(s)$ be a surface in M_i swept out by geodesics joining q_i to $q(s)$ $(i = 0, 1)$. We have, for $i = 0, 1$.

$$b^i(s) - E = -\int_{D_i(s)} (u^i)^*\Omega^\nabla - \int_0^s (\omega - \bar{\omega})[u^i_*(\dot{q}(r))][b^i(r) - E]dr \, , \tag{4.2}$$

where $\bar{\omega}$ is a connection form which makes u^i to a parallel cross-section on each $P|_{M_i}$ (cf. [4, (6.2)]). Therefore we have, for $Z \in \boldsymbol{R}^{n+1}$,

$$\begin{aligned} \|(d_X\beta.)Z\| &\le \left\| \left(\frac{d}{ds} b^0\right)\beta_x Z \right\|_{s=0} + \left\| \left(\frac{d}{ds} b^1\right) Z \right\|_{s=0} \\ &\le \int_0^{t(x)} \|R^\nabla((Y^0)^\perp_t, \dot{\tau}^0_x(t))u^0[\tau^0_x(t)]\beta_x Z\| dt \\ &\quad + \int_0^{t(x)} \|R^\nabla((Y^1)^\perp_t, \dot{\tau}^1_{f(x)}(t))u^1[\tau^1_{f(x)}(t)]Z\| dt \, , \end{aligned} \tag{4.3}$$

by (4.1) and (4.2), where Y^0 and Y^1 are, respectively, the Jacobi fields along τ^0_x and $\tau^1_{f(x)}$ with $(Y^0)_{q_0} = (Y^1)_{q_1} = 0$ and $(Y^0)_{q(0)} = (Y^1)_{q(0)} = \dot{q}(0)$, and we simply denote $(Y^0)^\perp_{\tau^0_x(t)}$ and $(Y^1)^\perp_{\tau^1_{f(x)}(t)}$ by $(Y^0)^\perp_t$ and $(Y^1)^\perp_t$, respectively.

LEMMA 6. *We have*

$$\beta_x\left[\bar{\alpha}_x X - \frac{\langle \bar{\alpha}_x X, w(x)\rangle}{\cos u(x)/2} e_{n+1}\right] = X + \frac{\langle \bar{\alpha}_x X, w(x)\rangle}{\cos u(x)/2} e_{n+1} \, .$$

PROOF. We have

$$\beta_x\left[\bar{\alpha}_x X-\frac{\langle\bar{\alpha}_x X, w(x)\rangle}{\cos u(x)/2}\, e_{n+1}\right]={}^t\bar{\alpha}_x\left[\bar{\alpha}_x X-\frac{\langle\bar{\alpha}_x X, w(x)\rangle}{\cos u(x)/2}\, e_{n+1}\right]$$
$$=X+(\cos u(x)/2)^{-1}\langle\bar{\alpha}_x X, w(x)\rangle e_{n+1}\,,$$

by Lemma 3 and (3.6). q.e.d.

We have the following proposition by (4.3) and Lemmas 5 and 6.

PROPOSITION 4. *We have*

$$\|(d_X\beta.)[\bar{\alpha}_x X-(\cos u(x)/2)^{-1}\langle\bar{\alpha}_x X, w(x)\rangle e_{n+1}]\|$$
$$\leq\int_0^{t(x)}\|R^\nabla((Y^0)_t^\perp, \dot{\tau}_x^0(t))u^0[\tau_x^0(t)][X+(\cos u/2)^{-1}c_1 e_{n+1}]\|dt$$
$$+\int_0^{t(x)}\|R^\nabla((Y^1)_t^\perp, \dot{\tau}_{f(x)}^1(t))u^1[\tau_{f(x)}^1(t)][\bar{\alpha}_x X-(\cos u/2)^{-1}c_1 e_{n+1}]\|dt\,.$$

(B) We put

$$\begin{cases} B_1=\dfrac{(1-\delta)^2}{3(1+c^2)}\displaystyle\int_0^{\pi/2\sqrt{\delta}}\sqrt{1+\left(\frac{c^1}{\cos(u/2)}\right)^2\sin^2(ct)}\left\{\sinh t-\frac{1}{c}\sin(ct)\right\}dt\,,\\ B_2=\dfrac{(1-\delta)^2}{3c(1+c^2)}\left\{c\,\dfrac{\sinh(\pi/2\sqrt{\delta})}{\sin(c\pi/2\sqrt{\delta})}-1\right\}\displaystyle\int_0^{\pi/2\sqrt{\delta}}\sqrt{1+\left(\frac{c_1}{\cos(u/2)}\right)^2\sin^2(ct)\sin(ct)dt}\,,\\ B_3=\dfrac{(1-\delta)}{2c}\displaystyle\int_0^{\pi/2\sqrt{\delta}}\sqrt{1+\left(\frac{c_1}{\cos(u/2)}\right)^2\sin^2(ct)\sin(ct)dt}\,. \end{cases}$$

The following lemma is proved in (C) and (D) below.

LEMMA 7. *We have*

(1) $$\int_0^{t(x)}\left\|R^\nabla((Y^0)_t^\perp, \dot{\tau}_x^0(t))u^0[\tau_x^0(t)]\left[X+\frac{c_1}{\cos(u/2)}\,e_{n+1}\right]\right\|dt\leq B_1+B_3\,,$$

(2) $$\int_0^{t(x)}\left\|R^\nabla((Y^1)_t^\perp, \dot{\tau}_{f(x)}^1(t))u^1[\tau_{f(x)}^1(t)]\left[\bar{\alpha}_x X-\frac{c_1}{\cos(u/2)}\,e_{n+1}\right]\right\|dt$$
$$\leq L^{-1}(B_1+B_2)+B_2+B_3\,.$$

By Lemmas 4 and 7 and Proposition 4, we have the following:

PROPOSITION 5. *We have*

$$\cos(u/2)\cdot\|(d_X\alpha.)X\|\leq(1+L^{-1})(B_1+B_2)+2B_3$$

for any unit vector $X\in T_x(S_0)=T_x(S)$.

Thus we can take

$$N_3 = N_3(\delta) = \frac{(1+L^{-1})(B_1+B_2)+2B_3}{\cos(u/2)} . \tag{4.4}$$

(C) THE PROOF OF LEMMA 7, (1). We have

$$u^0[\tau_x^0(t)]\left(X+\frac{c_1 e_{n+1}}{\cos u/2}\right)=[\tau_x^0]_t^0 X+\frac{c_1(-\sin(ct)\dot{\tau}_x^0(t)+\cos(ct)e)}{\cos u/2}$$

by [4, (2.1)] and $g(X, x)=0$. We put $\bar{Y}_t^0=(1/c)\sin(ct)[\tau_x^0]_t^0 X$. Since $(Y^0)_0^\perp=\bar{Y}_0^0=0$ and $D_x(Y^0)^\perp=D_x(\bar{Y}^0)=X$, we have

$$\|(Y^0)_t^\perp-\bar{Y}_t^0\|\le\frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\sinh t-\frac{1}{c}\sin(ct)\right\} \tag{4.5}$$

by [4, Proof (2), (c) in Prop. 2]. We put

$$\bar{X}_t=[\tau_x^0]_t^0 X-c_1(\cos u/2)^{-1}\sin(ct)\dot{\tau}_x^0(t) .$$

Then we have

$$\|R^\nabla((Y^0)_t^\perp, \dot{\tau}_x^0(t))\bar{X}_t\|\le\|R^\nabla((Y^0)_t^\perp-\bar{Y}_t^0, \dot{\tau}_x^0(t))\bar{X}_t\|+\frac{1}{c}\sin(ct)\|R^\nabla(\bar{X}_t, \dot{\tau}_x^0(t))\bar{X}_t\|$$

$$\le\|R^\nabla((Y^0)^\perp-\bar{Y}_t^0, \dot{\tau}_x^0(t))\bar{X}_t\|+\frac{1}{c}\sin(ct)\left\|R^\nabla\left(\bar{X}_t, \dot{\tau}_x^0(t)-\langle\dot{\tau}_x^0(t), \bar{X}_t\rangle\frac{\bar{X}_t}{\|\bar{X}_t\|^2}\right)\bar{X}_t\right\|$$

$$\le\|\bar{X}_t\|\left\{\frac{(1-\delta)^2}{3(1+c^2)}\left(\sinh t-\frac{1}{c}\sin(ct)\right)+\frac{(1-\delta)}{2c}\sin(ct)\right\},$$

by (3.1). Thus we have Lemma 7, (1) by $R(X, Y)e=0$.

(D) THE PROOF OF LEMMA 7, (2). We have $u^1(\tau_{f(x)}^1(t))\bar{\alpha}_x X=[\tau_{f(x)}^1]_t^0\alpha_x X$ by $\bar{\alpha}_x X=\alpha_x X$ and $g(\alpha_x X, f(x))=0$. We put

$$\bar{Y}_t^1=\frac{1}{c}\sin(ct)[\tau_{f(x)}^1]_t^0\alpha_x X, \quad \bar{U}_t^1=\frac{\sin(ct)}{\sin(ct(x))}[\tau_{f(x)}^1]_t^{t(x)}(Y^1)_{t(x)}^\perp ,$$

$$\bar{V}_t^1=\frac{1}{c}\sin(ct)[\tau_{f(x)}^1]_t^0 D_{f(x)}(Y^1)^\perp .$$

Then we have

$$(Y^1)_t^\perp=\{(Y^1)_t^\perp-\bar{V}_t^1\}+\{\bar{V}_t^1-\bar{U}_t^1\}+\{\bar{U}_t^1-\bar{Y}_t^1\}+\bar{Y}_t^1 .$$

We have

$$\|(Y^1)_t^\perp - \bar{V}_t^1\| \leq \frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\sinh t - \frac{1}{c}\sin(ct)\right\}L^{-1}$$

by $D_{f(x)}(Y^1)^\perp = (df)_x X$ and [4, Proof (2), (c) of Prop. 2].

About $\|\bar{U}_t^1 - \bar{V}_t^1\|$: By $\bar{U}_{t(x)} = (Y^1)_{t(x)}^\perp$, $\bar{U}_0^1 = (Y^1)_0^\perp = 0$ and [4, Proof (2), (b) and (c) of Prop. 2], we note

$$\|D_{f(x)}[(Y^1)^\perp - \bar{U}^1]\| \leq \frac{1-\delta}{2(1+c^2)}\left\{c\frac{\sinh(t(x))}{\sin(ct(x))} - 1\right\}L^{-1}.$$

Therefore we have

$$\begin{aligned}\|\bar{U}_t^1 - \bar{V}_t^1\| &= \frac{1}{c}\sin(ct)\left\|\frac{c}{\sin(ct(x))}[\tau_{f(x)}^1]_t^{t(x)}(Y^1)_{t(x)}^\perp - [\tau_{f(x)}^1]_t^0 D_{f(x)}(Y^1)^\perp\right\| \\ &= \frac{1}{c}\sin(ct)\left\|\frac{c}{\sin(ct(x))}[\tau_{f(x)}^1]_0^{t(x)}(Y^1)_{t(x)}^\perp - D_{f(x)}(Y^1)^\perp\right\| \\ &= \frac{1}{c}\sin(ct)\|D_{f(x)}[\bar{U}^1 - (Y^1)^\perp]\| \\ &\leq \frac{1-\delta}{2c(1+c^2)}L^{-1}\left\{c\frac{\sinh(t(x))}{\sin(ct(x))} - 1\right\}\sin(ct).\end{aligned}$$

About $\|\bar{U}_t^1 - \bar{Y}_t^1\|$: We note

$$\begin{aligned}\|\bar{U}_{t(x)}^1 - \bar{Y}_{t(x)}^1\| &= \left\|(Y^1)_{t(x)}^\perp - \frac{\sin(ct(x))}{c}[\tau_{f(x)}^1]_{t(x)}^0 \alpha_x X\right\| \\ &= \left\|(Y^0)_{t(x)}^\perp - \frac{\sin(ct(x))}{c}[\tau_x^0]_{t(x)}^0 X\right\| = \|(Y^0)_{t(x)}^\perp - \bar{Y}_{t(x)}^0\| \\ &\leq \frac{1}{2}\frac{1-\delta}{1+c^2}\left\{\sinh(t(x)) - \frac{1}{c}\sin(ct(x))\right\},\end{aligned}$$

by §1, (B) and (4.5). Therefore we have

$$\|\bar{U}_t^1 - \bar{Y}_t^1\| = \frac{\sin(ct)}{\sin(ct(x))}\|\bar{U}_{t(x)}^1 - \bar{Y}_{t(x)}^1\| \leq \frac{1-\delta}{2c(1+c^2)}\left\{c\frac{\sinh(t(x))}{\sin(ct(x))} - 1\right\}\sin(ct).$$

Thus we have Lemma 7, (2) in the same way as in (C).

5. A differentiable sphere theorem.

THEOREM 2. *Let (M, g) be a complete, simply connected and* 0.654*-pinched Riemannian manifold. Then M is diffeomorphic to the standard sphere.*

We calculate the numbers $N_1=N_1(\delta)$, $\theta_1=\theta_1(\delta)$ and $\theta_2=\theta_2(\delta)$. The number $N_1=N_1(\delta)$ is given by

$$N_1=\frac{2}{3}\frac{1-\delta}{\delta}\frac{1}{\cos^2(u/2)}\{1+L^{-1}\}, \tag{5.1}$$

(cf. [4, Lemma 8]). We denote

$$\bar{v}=(2B_0^2+1+\sqrt{8B_0^2+1})/(2(1-B_0^2)) \tag{5.2}$$

in the culculation in Tables 1 and 2.

TABLE 1.

δ	def.	0.652	0.653	0.654
L	(1.2)	0.751487	0.752502	0.753516
L^{-1}	(1.2)	1.3307	1.3289	1.32711
$\cos(u/2)$	(3.5)	0.93611	0.936546	0.936981
B_0	Lemma 1	0.207329	0.20625	0.205177
B_1	§4, (B)	0.0259119	0.0256736	0.0254372
B_2	§4, (B)	0.0721266	0.0714361	0.0707516
B_3	§4, (B)	0.260914	0.259767	0.258624
$\bar{v}$	(5.2)	1.17304	1.17122	1.16942
N_1	(5.1)	0.946395	0.940627	0.934895
N_2	(1.7)	0.48322	0.480336	0.47747
N_3	(4.4)	0.801536	0.796216	0.790935
N_4	Lemma 2	0.442148	0.439556	0.43698
θ_1	Prop. 3	2.53017	2.51406	2.49805
θ_2	Prop. 2	0.417687	0.415483	0.413289

TABLE 2.

δ	$2\pi-(N_1\pi+\theta_1+2\theta_2)$
0.652	-0.0555507
0.653	-0.0169031
0.654	0.0214957

We have Theorem 2 by Tables 1 and 2, Theorem 1 and Proposition 1.

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