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A Differential Game between Government and Firms: Comments

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It is argued here that the Nash equilibrium calculated by Gradus (1989) is not the Nash equilibrium found when applying the definition generally employed in differential game theory. The model presented by Gradus would call for a modified Nash equilibrium concept as outlined in this note.

In a paper recently published in this *Journal,* Gradus (1989) analyses the interaction between a government and firms using differential game techniques. In this note I do not intend to go into the economic merits of this approach but I will concentrate on the derivation of the results, in particular the Nash equilibrium. It is argued here that the Nash equilibrium calculated by Gradus is not the Nash equilibrium found when applying the definition generally employed in differential game theory. The description of a non-cooperative game involves a. o. a specification of the players and of the strategy space of each player. In a standard formulation (see e. g. Başar and Olsder, 1982) these spaces are independent in the sense that the action of one player does not affect the range of strategies open to the other(s). The model presented by Gradus lacks this property and would therefore call for a modified Nash equilibrium concept, preserving the basic idea that each player maximizes its pay-offs given the strategies followed by the other(s). In the case at hand this is relatively easy as will be outlined below in detail.

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In the model the government decides on the tax rate (τ) it imposes on the (aggregate) firm. The firm takes this rate as given and selects labour (L) and investments (I) so as to maximize shareholders' discounted welfare. The firm's problem is defined as follows:

$$\max_{L,I,D} \int_0^T e^{-rs} D(s)\, ds + bK(T)\, e^{-rT}$$

subject to

$$pF(K(t), L(t)) - wL(t) - TX(t) = D(t) + I(t), \tag{1}$$

$$TX(t) = \tau(t)\,(pF(K(t), L(t)) - wL(t) - aK(t)), \tag{2}$$

$$\dot{K}(t) = I(t) - aK(t),\ K(0) = K_0 \text{ given}, \tag{3}$$

$$D(t) \geq 0, \tag{4}$$

$$I(t) \geq 0, \tag{5}$$

where the symbols have the following meaning:

T	the fixed finite horizon
r	the constant rate of discount
b	the value shareholders attach to a unit of capital left at T
p	the constant price of output
w	the constant wage rate
K	capital
L	labour
F	production function
TX	taxes
D	dividends
I	investments
a	the constant rate of depreciation.

It is assumed by Gradus that $0 \leq b \leq 1$. One could argue that it would be interesting to consider the case $b > 1$ as well, but this will not be gone into here. The same applies to the condition that investments are non-negative.

The government's problem can be formulated as follows:

$$\max_{\tau} \int_0^T e^{-rs} \tau(s)\,(pF(K(s), L(s)) - wL(s) - aK(s))\, ds$$

subject to

$$0 < \tau_1 \leq \tau(t) \leq \tau_2 < 1. \tag{6}$$

So Gradus assumes that the tax rate is strictly positive and strictly smaller than unity. This is not motivated. We shall deal with the case $\tau_1 = 0$, $\tau_2 = 1$ below.

Clearly the strategy space of the firm consists of the mappings L, I and D from $[0, T]$ into $\mathbb{R}$ satisfying (1)—(5) and this space is affected by the tax rate τ appearing in (2). The strategy space of the government is the set of mappings $\tau: [0, T] \to \mathbb{R}$ satisfying (6). This space is independent of the strategy space of the firm. It seems perfectly in line with the traditional Nash equilibrium concept to define a Nash equilibrium in the model at hand as a set of mappings $(L^*, I^*, D^*, \tau^*): [0, T]^4 \to \mathbb{R}^4$ such that L^*, I^*, D^* maximizes the firm's pay-off subject to (1)—(5) with τ replaced by τ^* and such that τ^* maximizes the government's pay-off, with L^*, I^*, D^* and hence K^* inserted, subject to (6).

It is not difficult to solve for this equilibrium. Consider the government first. (We omit * in the sequel.) For any moment in time we have

$$\begin{aligned} \tau &= \tau_1 && \text{if} \quad pF - wL - aK < 0, \\ \tau_1 \le \tau &\le \tau_2 && \text{if} \quad pF - wL - aK = 0, \\ \tau &= \tau_2 && \text{if} \quad pF - wL - aK > 0. \end{aligned}$$

The firm takes the tax rate as given. After some simple manipulations the Lagrangean of the firm's problem reads

$$V = e^{-rt} \{(1 - \tau)(pF - wL) + \tau a K - I\} + \lambda (I - aK) + \mu_1 I + \\ + \mu_2 \{(1 - \tau)(pF - wL) + \tau a K - I\}$$

and the necessary conditions are

$$pF_L = w,$$

$$-e^{-rt} + \lambda + \mu_1 - \mu_2 = 0,$$

$$\mu_1 \ge 0, \ \mu_1 I = 0,$$

$$\mu_2 \ge 0, \ \mu_2 D = \mu_2 ((1 - \tau)(pF - wL) + \tau a K - I) = 0,$$

$$-\dot{\lambda} = (e^{-rt} + \mu_2)((1 - \tau) pF_K + \tau a) - \lambda a, \tag{7}$$

$$\lambda (T) = b e^{-rT}. \tag{8}$$

It is assumed here that p and w are such that, for any positive K, they allow for positive gross profits. In view of the homogeneity of F we have along a solution

$$pF - wL - aK = pF_K K + pF_L L - wL - aK = \\ = (pF_K - a)K := (q - a)K,$$

where q is a constant.

Let us make a distinction between several possible regimes.

$\underline{I > 0, D > 0.}$

Then $\mu_1 = \mu_2 = 0$ and $\lambda = e^{-rt}$. Furthermore

$$-\dot{\lambda} = re^{-rt} = e^{-rt}((1-\tau)q + \tau a).$$

So $r = (1-\tau)q + \tau a$. If $q - a = 0$, this implies $r = a$. If $a \neq q$ then $\tau = \tau_1$ or $\tau = \tau_2$ and $r = (1-\tau_i)q + \tau_i a$ ($i = 1$ or $i = 2$). We shall assume, as Gradus implicitly does, that none of these rather special conditions is satisfied. So the case $I < 0, D > 0$ will not occur.

$\underline{I = D = 0.}$

$D = ((1-\tau)q + \tau a)K \neq 0$ under the conditions given above.

$\underline{I > 0, D = 0.}$

Then $\mu_1 = 0$ and $\lambda - e^{-rt} = \mu_2 \geq 0$. Furthermore

$$-\dot{\lambda} = \lambda(1-\tau)(q - a). \tag{9}$$

$\underline{I = 0, D > 0.}$

Then $\mu_2 = 0$ and $\lambda - e^{-rt} = -\mu_1 \leq 0$. Furthermore

$$-\dot{\lambda} = e^{-rt}(1-\tau)(q-a) + e^{-rt}a - \lambda a.$$

The analysis from here is quite simple. Since $pF - wL - aK = (q-a)K$, we have $\tau = \tau_1$ if $(q-a) < 0$ and $\tau = \tau_2$ if $(q-a) > 0$.

If $(q-a) < 0$, then there is no interval of time with $I > 0$ and $D = 0$ because otherwise it follows from the continuity of λ and the fact that λ is increasing in such intervals (see (9)) that $\lambda(T) > be^{-rT}$ (recall that $b \leq 1$). But this contradicts (8). So, if $(q-a) < 0$, $\tau(t) = \tau_1$ and $I(t) = 0$ for all $0 \leq t \leq T$.

The interesting case is of course $(q-a) > 0$. Then $\tau(t) = \tau_2$ for all $0 \leq t \leq T$. A necessary condition for positive investment is

$$(1-\tau_2)(q-a) > r, \tag{10}$$

because otherwise (5) is not satisfied. However, condition (10), which is imposed by Gradus, is by no means sufficient for the existence of a phase with positive investment. To see this, consider the differential equation (7) with $\tau = \tau_2$ and $\lambda(T) = be^{-rt}$. The solution is

$$\lambda(t) = \frac{(1-\tau_2)(q-a)+a}{r+a} e^{-rt} + \\ + \left(b - \frac{(1-\tau_2)(q-a)+a}{r+a}\right) e^{-(r+a)T+at}.$$

It could well be that $\lambda(0) < 1$, in which case $I(t) = 0$ for all $0 \leq t \leq T$. The results are summarized in the following

Proposition: If $(q-a) < 0$, then along the Nash equilibrium $\tau(t) = \tau_1$, $I(t) = 0$ for all t. If $(q-a) > 0$, then there exists t_1 $(0 \leq t_1 \leq T)$ with t_1 possibly equal to zero such that along the Nash equilibrium

$$\tau(t) = \tau_2,\ I(t) > 0 \quad 0 \leq t \leq t_1,$$

$$\tau(t) = \tau_2,\ I(t) = 0 \quad t_1 \leq t \leq T. \qquad \square$$

These results are in sharp contrast with those obtained by Gradus, where (for $q > a$) there is an initial phase with the tax rate at the minimum level.

Finally, consider the case with $\tau_1 = 0$ and $\tau_2 = 1$. Clearly $\tau = 0$ if $(q-a) < 0$. So $(q-a) > 0$ implies $\tau > 0$ and, in particular, $\tau = 1$. But $(1-\tau)(q-a)r > 0$ is a necessary condition for positive investment. Therefore, along the Nash equilibrium $\tau(t) = 1$ and $I(t) = 0$ for all t.

The conclusion is that Gradus employs an equilibrium concept in which the government does *not* take the firm's actions as given. This is not to say that Gradus confuses Nash and Stackelberg equilibria, because his Stackelberg equilibrium seems to be correct. It is not clear however what equilibrium concept in the Nash sense has been used.

Another conclusion going beyond this particular model, is that, since there exist many economic models where strategy spaces are interdependent, these must be handled with great care if one is looking for a Nash-like equilibrium.

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