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A DIFFERENTIAL GEOMETRIC SETTING FOR DYNAMIC EQUIVALENCE AND DYNAMIC LINEARIZATION

JEAN-BAPTISTE POMET

I.N.R.I.A., 2004 Route des Lucioles, B.P. 93, 06902 Sophia Antipolis, France
E-mail: pomet@sophia.inria.fr

Abstract. This paper presents an (infinite-dimensional) geometric framework for control systems, based on infinite jet bundles, where a system is represented by a single vector field and dynamic equivalence (to be precise: equivalence by endogenous dynamic feedback) is conjugation by diffeomorphisms. These diffeomorphisms are very much related to Lie-Bäcklund transformations.

It is proved in this framework that dynamic equivalence of single-input systems is the same as static equivalence.

1. Introduction.

For a control system

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the input, what one usually means by a dynamic feedback is a system with a certain state z, input (x, v) and output u:

$$\begin{aligned} \dot{z} &= g(x, z, v), \\ u &= \gamma(x, z, v). \end{aligned} \tag{2}$$

When applying this dynamic feedback to system (1), one gets a system with state (x, z) and input $v : \dot{x} = f(x, \gamma(x, z, v))$, $\dot{z} = g(x, z, v)$. This system may be transformed with a change of coordinates $X = \phi(x, z)$ in the extended variables to a system $\dot{X} = h(X, v)$. The problem of dynamic feedback linearization is stated in [7] by B. Charlet, J. Lévine and R. Marino as the one of finding g, γ and ϕ such that $\dot{X} = h(X, v)$ be a linear controllable system. When z is not present, γ and ϕ define a static feedback transformation in the usual sense. This transformation is

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said to be invertible if ϕ is a diffeomorphism and γ is invertible with respect to v; these transformations form a group of transformations. On the contrary, when z is present, the simple fact that the general "dynamic feedback transformation" (2), defined by g, γ and ϕ increases the size of the state prevents dynamic feedbacks in this sense from being "invertible".

In [12, 13], M. Fliess, J. Lévine, P. Martin and P. Rouchon introduced a notion of equivalence in a differential algebraic framework where two systems are *equivalent by endogenous dynamic feedback* if the two corresponding differential fields are algebraic over one another. This is translated in a state-space representation by some (implicit algebraic) relations between the "new" and the "old" state, output and many derivatives of outputs transforming one system into the other and vice-versa. It is proved that equivalence to controllable linear system is equivalent to *differential flatness*, which is defined as existence of m elements in the field which have the property to be a "linearizing output" or "flat output". In [21, "Point de vue analytique"], P. Martin introduced the notion of *endogenous dynamic feedback* as a dynamic feedback (2) where, roughly speaking, z is a function of $x, u, \dot{u}, \ddot{u} \dots$. He proved that a system may be obtained from another one by nonsingular endogenous feedback if and only if there exists a transformation of the same kind as in [12, 13] but explicit and analytic which transforms one system into the other. This is called equivalence by endogenous dynamic feedback as in the algebraic case. These transformations may either increase or decrease the dimension of the state.

B. Jakubczyk gives in [18, 19] a notion of dynamic equivalence in terms of transformations on "trajectories" of the system; different types of transformations are defined there in terms of infinite jets of trajectories. One of them is proved there to be exactly the one studied here. See after definition 1 for further comparisons.

In [27], W. F. Shadwick makes (prior to [12, 13, 18, 19]) a link between dynamic feedback linearization and the notion of absolute equivalence defined by E. Cartan for pfaffian systems. It is not quite clear that this notion of equivalence coincides with equivalence in the sense of [12, 13] or [18, 19], the formulation is very different.

The contribution of the present paper—besides Theorem 3 which states that dynamic equivalent single input systems with the same number of states are static equivalent—is to give a geometric meaning to transformations which are exactly these introduced by P. Martin in [21] (endogenous dynamic feedback transformations). Our system is represented by a single vector field on a certain "infinite-dimensional manifold", and our transformations are diffeomorphisms on this manifold. Then the action of these transformations on systems is translated by the usual transformation diffeomorphisms induce on vector fields. There are of course many technical difficulties in defining vector fields, diffeomorphisms or smooth functions in these "infinite-dimensional manifolds". The original motivation was to "geometrize" the constructions made in [2, 24]; it grew up into the present framework which, we believe, has some interest in itself, the geometric exposition of [2, 24] is contained in [3], which is somehow "part 2" of the present paper.

Note finally that the described transformations are very closely related to *infinite order contact transformations* or *Lie-Bäcklund transformations* or $\mathcal{C}$*-transformations*, see [16, 1] and that the geometric context we present here is the one of infinite jet spaces used in [23, 20, 28, 26] for example to describe and study Lie-Bäcklund transformations. These presentations however are far from being unified, for instance smooth functions do not have to depend only on a finite number of variables in [26], and are not explicitly defined in [1]. They also had to be adapted for many reasons in order to get a technically workable framework; for instance, we prove an inverse function theorem which characterizes local diffeomorphisms without having to refer to an inverse mapping which is of the same type. The language of jet spaces and differential systems has been used already in control theory by M. Fliess [11] and by J.-F. Pommaret [25], with a somewhat different purpose.

Some recent work by M. Fliess [10] (see also a complete exposition on this topic in E. Delaleau's [9]) points out that a more natural state-space representation than (1) for a nonlinear system involves not only x and u, but also an arbitrary number of time-derivatives of u; this is referred to as "generalized-state" representation, and we keep this name for the infinite-dimensional state-manifold, see section 3. In [10, 9], the "natural" state-space representation is $F(x, \dot{x}, u, \dot{u}, \ddot{u}, \ldots, u^{(J)}) = 0$ rather than (1). Here not only do we suppose that $\dot{x}$ is an explicit function of the other variables ("explicit representation" according to [10, 9]) but also that $J = 0$ ("classical representation"). Almost everything in this paper may be adapted to the "non-classical" case, i.e. to the case where some time-derivatives of the input would appear in the right-hand side of (1); we chose the classical representation for simplicity and because, as far as dynamic equivalence is concerned, a non-classical system is equivalent to a classical one by simply "adding some integrators"; on the contrary, the implicit case is completely out of the scope of this paper, see the end of section 2.

Very recently the authors of [12, 13] have independently proposed a "differential geometric" approach for dynamic equivalence, see [14, 15], which is similar in spirit to the present approach, although the technical results do differ. This was brought to the attention of the author too late for a precise comparison between the two approaches.

The paper is organized as follows: section 2 presents briefly the point of view of jet spaces and contact structure for system (1) considered as a differential relation $\dot{x} - f(x, u) = 0$ (no theoretical material from this section is used elsewhere in the paper). Section 3 presents in details the differential structure of the "generalized state-space manifold" where coordinates are $x, u, \dot{u}, \ldots$, where we decide to represent a system by a single vector field. Section 4 defines in this context dynamic equivalence and relates it to notions already introduced in the literature.

Section 5 deals with static equivalence. Section 6 is devoted to the single-input case, and states the result that dynamic equivalence and static equivalence are then the same. Finally section 7 is devoted to dynamic linearization, it introduces in a geometric way the "linearizing outputs" defined for dynamic linearization in [12, 13, 21].

2. Control systems as differential relations. This section is only meant to relate the approach described subsequently to some better known theories. It does not contain rigorous arguments.

In the spirit of the work of J. Willems [29], or also of M. Fliess [10], one may consider that the control system (1) is simply a differential relation on the functions of time $x(t), u(t)$ and that the object of importance is the set of solutions, i.e. of functions $t \mapsto (x(t), u(t))$ such that $\frac{dx}{dt}(t)$ is identically equal to $f(x(t), u(t))$. Of course this description does not need precisely a state-space description like (1).

The geometric way of describing the solution of this first order relation in the "independent variable" t (time) and the "dependent variables" x and u is to consider, as in [1, 25, 20, 28, 23], the fibration

$$\mathbb{R} \times \mathbb{R}^{n+m} \xrightarrow{\pi} \mathbb{R}, \qquad (t, x, u) \mapsto t, \tag{3}$$

and its first jet manifold $J^1(\pi)$, which is simply $T(\mathbb{R}^n \times \mathbb{R}^m) \times \mathbb{R}$. A canonical set of coordinates on $J^1(\pi)$ is $(t, x, u, \dot{x}, \dot{u})$. The relation $R(t, x, u, \dot{x}, \dot{u}) = \dot{x} - f(x, u) = 0$ defines a submanifold $\mathcal{R}$ of the fiber bundle (3), which is obviously a subbundle. The contact module on $J^1(\pi)$ is the module of 1-forms (or the codistribution) generated by the 1-forms $\mathrm{d}x_i - \dot{x}_i \mathrm{d}t$ and $\mathrm{d}u_j - \dot{u}_j \mathrm{d}t$, $1 \le i \le n$, $1 \le j \le m$. A "solution" of the differential system is a section $t \mapsto (t, x(t), u(t), \dot{x}(t), \dot{u}(t))$ of the subbundle $\mathcal{R}$, which annihilates the contact forms (this simply means that $\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$ and $\frac{\mathrm{d}u}{\mathrm{d}t} = \dot{u}$, i.e. that this section is the jet of a section of (3)).

Since we wish to consider some transformations involving an arbitrary number of derivatives, we need the infinite jet space $J^\infty(\pi)$ of the fibration (3). For short, it is the projective limit of the finite jet spaces $J^k(\pi)$, and some natural coordinates on this "infinite-dimensional manifold" are $(t, x, u, \dot{x}, \dot{u}, \ddot{x}, \ddot{u}, x^{(3)}, u^{(3)}, \ldots\ldots)$. The contact forms are

$$\mathrm{d}x_i^{(j)} - x_i^{(j+1)}\mathrm{d}t, \quad \mathrm{d}u_k^{(j)} - u_k^{(j+1)}\mathrm{d}t \quad \begin{cases} i = 1, \ldots, n, \\ k = 1, \ldots, m, \\ j \ge 0. \end{cases} \tag{4}$$

This infinite-dimensional "manifold" is described in [20] for example, and we will recall in next section what we really need. The "Cartan distribution" is the one annihilated by all these forms, it is spanned by the single vector field

$$\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ldots \tag{5}$$

where $\dot{x}\frac{\partial}{\partial x}$ stands for $\sum_i \dot{x}_i \frac{\partial}{\partial x_i}$, $\dot{u}\frac{\partial}{\partial u}$ for $\sum_i \dot{u}_i \frac{\partial}{\partial u_i}, \ldots$ The relation R has to be replaced by its infinite prolongation, i.e. R itself plus all its "Lie derivatives" along (5):

$$
\begin{aligned}
R(t,x,u,\dot x,\dot u) &= \dot x - f(x,u) = 0,\\
R_1(t,x,u,\dot x,\dot u,\ddot x,\ddot u) &= \ddot x - \frac{\partial f}{\partial x}\dot x - \frac{\partial f}{\partial u}\dot u = 0,\\
R_2(t,x,u,\dot x,\dot u,\ddot x,\ddot u,x^{(3)},u^{(3)}) &= x^{(3)} - \ldots = 0,\\
&\vdots
\end{aligned}
\tag{6}
$$

This defines a subbundle $\mathcal{R}_\infty$ of $J^\infty(\pi)$. A "solution" of the differential system is a section $t \mapsto (t, x(t), u(t), \dot x(t), \dot u(t), \ddot x(t), \ddot u(t), \ldots)$ of the subbundle $\mathcal{R}_\infty$, which annihilates the contact forms; it is obviously defined uniquely by $x(t)$ and $u(t)$ such that $\frac{dx}{dt}(t) = f(x(t), u(t))$ with the functions $u^{(j)}$ and $x^{(j)}$ obtained by differentiating $x(t)$ and $u(t)$.

$\mathcal{R}_\infty$ is a subbundle of $J^\infty(\pi)$ which has a particular form: since the relations allow one to explicitly express all the time-derivatives $\dot x, \ddot x, x^{(3)}, \ldots$ of x as functions of $x, u, \dot u, \ddot u, u^{(3)}, \ldots$, a natural set of coordinates on this submanifold is $(t, x, u, \dot u, \ddot u, \ldots)$; note that if, instead of the explicit form (1), we had an implicit system $f(x, u, \dot u) = 0$, this would not be true. The vector field (5), which spans the Cartan distribution is tangent to $\mathcal{R}_\infty$, and its expression in the coordinates $(t, x, u, \dot u, \ddot u, \ldots)$ *considered as coordinates on* $\mathcal{R}_\infty$ is

$$
\frac{\partial}{\partial t} + f(x,u)\frac{\partial}{\partial x} + \dot u\frac{\partial}{\partial u} + \ddot u\frac{\partial}{\partial \dot u} + \ldots + u^{(k+1)}\frac{\partial}{\partial u^{(k)}} + \ldots \tag{7}
$$

and the restriction of the contact forms are $\mathrm{d}x - f\mathrm{d}t$, $\mathrm{d}u^{(j)} - u^{j+1}\mathrm{d}t$, $j \geq 0$. The subbundles $\mathcal{R}_\infty$ obtained for different systems are therefore all diffeomorphic to a certain "canonical object" independent of the system, and where coordinates are $(t, x, u, \dot u, \ddot u, \ldots)$, let this object be $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ where $\mathcal{M}_\infty^{m,n}$ is described in more details in next section and the first factor $\mathbb{R}$ is time, with an embedding ψ of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ into $J_\infty(\pi)$ which defines a diffeomorphism between $\mathcal{R}_\infty$ and $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$; this embedding depends on the system and completely determines it; it pulls back the contact module on $J^\infty(\pi)$ to a certain module of forms on $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ and the Cartan vector field (5) into (7). The points in $J^\infty(\pi)$ which are outside $\mathcal{R}_\infty$ are not really of interest to the system, so that we only need to retain $\mathcal{R}_\infty$, and it turns out that all the information is contained in $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ and the vector field (7) which translates the way the contact module is pulled back by the embedding of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ into $J_\infty(\pi)$ whose image is $\mathcal{R}_\infty$. This is the point of view defended in [28] for example where such a manifold endowed with what it inherits from the contact structure on $J^\infty(\pi)$ is called a "diffiety". It is only in the special case of explicit systems like (1) that all diffieties can be parameterized by $x, u, \dot u, \ldots$ and therefore can all be represented by the single object $\mathcal{M}_\infty^{m,n}$, endowed with a contact structure, or a Cartan vector field, which of course depends on the system.

Finally, since everything is time-invariant, one may "drop" the variable t (or quotient by time-translations, or project on the submanifold $\{t = 0\}$ which is possible because all objects are invariant along the fibers) and work with the coordinates $(x, u, \dot u, \ddot u, \ldots)$ only, with $f\frac{\partial}{\partial x} + \dot u \frac{\partial}{\partial u} + \ddot u \frac{\partial}{\partial \dot u} + \ldots$ instead of (7); solutions are curves which are tangent to this vector field. This is the point of view we adopt here, and this is described in detail in the next section.

3. The generalized state-space manifold. The phrase "generalized state" denotes the use of many derivatives of the input as in [10, 9]. The "infinite-dimensional manifold" $\mathcal{M}_\infty^{m,n}$ we are going to consider is parameterized by x, u, $\dot u$, $\ddot u, \ldots$; in order to keep things simple, we define it in coordinates, i.e. a point of $\mathcal{M}_\infty^{m,n}$ is simply a sequence of numbers, as in [22] for example. It may be extended to x and u living in arbitrary manifolds via local coordinates, but, since dynamic equivalence is local in nature, the present description is suitable.

3.1. *The manifold, functions and mappings.* For $k \geq -1$, let $\mathcal{M}_k^{m,n}$ be $\mathbb{R}^n \times (\mathbb{R}^m)^{k+1}$ ($\mathcal{M}_{-1}^{m,n}$ is $\mathbb{R}^n$), and let us denote the coordinates in $\mathcal{M}_k^{m,n}$ by

$$(x, u, \dot u, \ddot u, \ldots, u^{(k)})$$

where x is in $\mathbb{R}^n$ and $u, \dot u, \ldots$ are in $\mathbb{R}^m$. $\mathcal{M}_\infty^{m,n}$ is the space of infinite sequences

$$(x, u, \dot u, \ddot u, \ldots, u^{(j)}, u^{(j+1)}, \ldots).$$

For simplicity, we shall use the following notation:

$$\mathcal{U} = (u, \dot u, \ddot u, u^{(3)}, \ldots), \qquad \mathcal{X} = (x, \mathcal{U}) = (x, u, \dot u, \ddot u, u^{(3)}, \ldots). \tag{8}$$

Let, for $k \geq -1$, the projection π_k, from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_k^{m,n}$ be defined by

$$\pi_k(\mathcal{X}) = (x, u, \dot u, \ldots, u^{(k)}), \quad k \geq 0, \quad \pi_{-1}(\mathcal{X}) = x. \tag{9}$$

$\mathcal{M}_\infty^{m,n}$ may be constructed as the projective limit of $\mathcal{M}_k^{m,n}$, and this naturally endows it with the weakest topology such that all these projections are continuous (product topology); a basis of the topology are the sets

$$\pi_k^{-1}(O), \qquad O \text{ open subset of } \mathcal{M}_k^{m,n}.$$

This topology makes $\mathcal{M}_\infty^{m,n}$ a topological vector space, which is actually a Fréchet space (see for instance [4]). It is easy to see that continuous linear forms are these which depend only on a finite number of coordinates. This leads one to the (false) idea that there is a natural way of defining differentiability so that differentiable functions depend only on a finite number of variables, which is exactly the class of smooth functions we wish to consider (as in most of the literature on differential systems and jet spaces [1, 20, 22, 23, 28]), since they translate into realistic dynamic feedbacks from the system theoretic point of view. It is actually possible to define a very natural notion of differentiability in Fréchet spaces (see for instance the very complete [17]) but there is nothing wrong in this framework with smooth functions depending on infinitely many variables. For instance the function mapping $(u, \dot u, \ddot u, u^{(3)}, \ldots)$ to $\sum_{j=0}^{\infty} \frac{1}{2^j} \rho(\frac{u^{(j)}}{j})$, with ρ a smooth function with compact support containing 0 vanishing at 0 as well as its derivatives of all orders depends on *all* the variables at zero, but it is smooth in this framework. It is hard to imagine a local definition of differentiability which would classify this function non-smooth.

Here, we do not wish to consider smooth functions or smooth maps depending on infinitely many variables; we therefore define another differentiable structure, which agrees with the one usually used for differential systems [23, 1, 22, 20, 28]:

• A function h from an open subset V of $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ (or to any finite-dimensional manifold) is a *smooth function* at $\mathcal{X} \in V$ if and only if, locally at each point, it depends only on a finite number of derivatives of u and, as a function of a finite number of variables, it is smooth (of class $\mathcal{C}^\infty$); more technically: if and only if there exists an open neighborhood U of $\mathcal{X}$ in V, an integer ρ, and a smooth function h_ρ from an open subset of $\mathcal{M}_\rho^{m,n}$ to $\mathbb{R}$ (or to the finite-dimensional manifold under consideration) such that $h(\mathcal{Y}) = h_\rho \circ \pi_\rho(\mathcal{Y})$ for all $\mathcal{Y}$ in U. It is a smooth function on V if it is a smooth function at all $\mathcal{X}$ in V. The highest ρ such that h actually depends on the ρth derivative of u on any neighborhood of $\mathcal{X}$ (-1 if it depends on x only on a certain neighborhood of $\mathcal{X}$) we will call the *order* of h at $\mathcal{X}$, and we denote it by $\delta(h)(\mathcal{X})$. It is also the largest integer such that $\frac{\partial h}{\partial u^{(\rho)}}$ (this may be defined in coordinates and is obviously a smooth function) is not identically zero on any neighborhood of $\mathcal{X}$. Note that $\delta(h)$ may be unbounded on $\mathcal{M}_\infty^{m,n}$. We denote by $\mathcal{C}^\infty(V)$ the algebra of smooth functions from V to $\mathbb{R}$, $\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n})$ if $V = \mathcal{M}_\infty^{m,n}$.

• A *smooth mapping* from an open subset V of $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a map φ from V to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ such that, for any ψ in $\mathcal{C}^\infty(\mathcal{M}_\infty^{\tilde{m},\tilde{n}})$, $\psi \circ \varphi$ is in $\mathcal{C}^\infty(V)$. It is a smooth mapping at $\mathcal{X}$ if it is a smooth mapping from a certain neighborhood of $\mathcal{X}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$. Of course, in coordinates, it is enough that this be true for ψ any coordinate function. For such a map and for all k, there exists locally an integer ρ_k and a (unique) smooth map φ_k from $\pi_{\rho_k}(V) \subset \mathcal{M}_{\rho_k}^{m,n}$ to $\mathcal{M}_k^{\tilde{m},\tilde{n}}$ such that

$$\pi_k \circ \varphi = \varphi_k \circ \pi_{\rho_k}. \tag{10}$$

The smallest possible ρ_k at a point $\mathcal{X}$ is $\delta(\pi_k \circ \varphi)(\mathcal{X})$.

• A *diffeomorphism* from an open subset V of $\mathcal{M}_\infty^{m,n}$ to an open subset $\widetilde{V}$ of $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a smooth mapping φ from V to $\widetilde{V}$ which is invertible and is such that φ^{-1} is a smooth mapping from $\widetilde{V}$ to V.

• A *static diffeomorphism* φ from an open subset V of $\mathcal{M}_\infty^{m,n}$ to an open subset $\widetilde{V}$ of $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a diffeomorphism from V to $\widetilde{V}$ such that for all k, $\delta(\pi_k \circ \varphi)(\mathcal{X})$ is constant equal to k.

• A (*local*) *system of coordinates* on $\mathcal{M}_\infty^{m,n}$ (at a certain point) is a sequence $(h_\alpha)_{\alpha \geq 0}$ of smooth functions (defined on a neighborhood of the point under consideration) such that the smooth mapping $\mathcal{X} \mapsto (h_\alpha(\mathcal{X}))_{\alpha \geq 0}$ is a local diffeomorphism onto an open subset of $\mathbb{R}^\mathbb{N}$, considered as $\mathcal{M}_\infty^{1,0}$.

Note that the functions $x_1,\ldots,x_n,u_1,\ldots,u_m,\dot u_1,\ldots,\dot u_m,\ldots$ are coordinates in this sense. Actually, this makes all the "manifolds" $\mathcal{M}^{m,n}_\infty$ globally diffeomorphic to $\mathcal{M}^{1,0}_\infty$, so that they are all diffeomorphic to one another (this can be viewed as renumbering the natural coordinates). The following proposition shows that static diffeomorphisms are much more restrictive: they preserve n and m.

PROPOSITION 1. *Let φ be a static diffeomorphism from an open set U of $\mathcal{M}^{m,n}_\infty$ to an open set V of $\mathcal{M}^{\tilde m,\tilde n}_\infty$. Its inverse φ^{-1} is also a static diffeomorphism and φ induces, for all $k\ge 0$, a diffeomorphism φ_k from $\mathcal{M}^{m,n}_k$ to $\mathcal{M}^{\tilde m,\tilde n}_k$ (from $\mathbb{R}^n$ to $\mathbb{R}^{\tilde n}$ for $k=-1$). Its existence therefore implies $\widetilde n=n$ and $\widetilde m=m$.*

Proof. For all $k\ge -1$, since $\delta(\varphi\circ\pi_k)=k$, there exists a mapping φ_k from $\pi_k(U)$ to $\pi_k(V)$ satisfying (10) with $\rho_k=k$. All these mappings are onto because if one of them was not onto, (10) would imply that φ is onto either. Now let us consider φ^{-1}; it is a diffeomorphism from V to U and there exists therefore, for all k, an integer σ_k and a smooth map $\left(\varphi^{-1}\right)_k$ from $\pi_{\sigma_k}(V)\subset\mathcal{M}^{\tilde m,\tilde n}_{\sigma_k}$ to $\mathcal{M}^{m,n}_k$ such that

$$\pi_k\circ\varphi^{-1}=(\varphi^{-1})_k\circ\pi_{\sigma_k}. \tag{11}$$

Applying φ on the right to both sides and using the fact that $\pi_{\sigma_k}\circ\varphi=\varphi_{\sigma_k}\circ\pi_{\sigma_k}$, we get

$$\pi_k=\left(\varphi^{-1}\right)_k\circ\varphi_{\sigma_k}\circ\pi_{\sigma_k}. \tag{12}$$

Applied to $(x,u,\dot u,\ldots)$, this means

$$\begin{gathered}(x,u,\dot u,\ldots,u^{(k)})=(\varphi^{-1})_k(y,v,\dot v,\ldots,v^{(k)},\ldots,v^{(\sigma_k)})\\ \text{with } (y,v,\dot v,\ldots,v^{(k)},\ldots,v^{(\sigma_k)})=\varphi_{\sigma_k}(x,u,\dot u,\ldots,u^{(k)},\ldots,u^{(\sigma_k)})\end{gathered} \tag{13}$$

Since φ_{σ_k} is onto and each $v^{(j)}$ depends only on $x,u,\ldots,u^{(j)}$, (13) implies that $\left(\varphi^{-1}\right)_k$ depends only on $y,v,\dot v,\ldots,v^{(k)}$. Therefore σ_k might have been taken to be k, and then one has (12) with $\sigma_k=k$ and therefore

$$(\varphi^{-1})_k\circ\varphi_k=\mathrm{Id}_{\mathcal{M}^{\tilde m,\tilde m}_k} \tag{14}$$

which proves that each φ_k is a diffeomorphism and ends the proof. ∎

Let us define, as examples of diffeomorphisms, the (non static!) diffeomorphisms $\Upsilon_{n,(p_1,\ldots,p_m)}$ from $\mathcal{M}^{m,n}_\infty$ $\mathcal{M}^{m,n+p_1+\ldots+p_m}_\infty$ which "adds p_k integrators on the kth input":

$$\begin{aligned}&\Upsilon_{n,(p_1,\ldots,p_m)}(x,\mathcal{U})=(z,\mathcal{V})\\ &\quad\text{with } z=(x,u_1,\dot u_1\ldots u_1^{(p_1-1)},\ldots,u_m,\dot u_m,\ldots u_m^{(p_m-1)}),\quad v_k^{(j)}=u_k^{(j+p_k)}.\end{aligned} \tag{15}$$

It is invertible: one may define $\Upsilon_{N,(-p_1,\ldots,-p_m)}$ from $\mathcal{M}^{m,N}_\infty$ to $\mathcal{M}^{m,N-p_1-\ldots-p_m}_\infty$ for $N\ge p_1+\ldots+p_m$ by $\Upsilon_{n,(-p_1,\ldots,-p_m)}(z,\mathcal{V})=(x,\mathcal{U})$ where x is the $N-p_1-\ldots-p_m$ first coordinates of z, and $u_k^{(j)}$ is $v_k^{(j-p_k)}$ if $j\ge p_k$ and one of the remaining components of z if $0\le j\le p_k-1$, so that $\Upsilon_{n,(p_1,\ldots,p_m)}\circ\Upsilon_{n,(-p_1,\ldots,-p_m)}=\mathrm{Id}$.

3.2. *Vector fields and differential forms.* The "tangent bundle" to the infinite-dimensional manifold $\mathcal{M}_\infty^{m,n}$ is, since $\mathcal{M}_\infty^{m,n}$ is a vector space, $\mathcal{M}_\infty^{m,n} \times \mathcal{M}_\infty^{m,n}$, which is a (trivial) vector bundle over $\mathcal{M}_\infty^{m,n}$. A *smooth vector field* is a smooth (as a mapping from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,n} \times \mathcal{M}_\infty^{m,n}$, considered as $\mathcal{M}_\infty^{2m,2n}$) section of this bundle. It is of the form

$$F = f\frac{\partial}{\partial x} + \sum_0^\infty \alpha_j \frac{\partial}{\partial u^{(j)}} \tag{16}$$

where f is a smooth function from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}^n$ and the α_j's are smooth functions from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}^m$, where $f\frac{\partial}{\partial x}$ stands for $\sum_i f_i \frac{\partial}{\partial x_i}$ and $\alpha_j \frac{\partial}{\partial u^{(j)}}$ for $\sum_i \alpha_{j,i} \frac{\partial}{\partial u_i^{(j)}}$, and the $\frac{\partial}{\partial x_i}$'s and $\frac{\partial}{\partial u_i^{(j)}}$'s are the canonical sections corresponding to the "coordinate vector fields" associated with the canonical coordinates. Vector fields obviously define smooth differential operators on smooth functions: in coordinates, $L_F h$ is an infinite sum with finitely many nonzero terms.

Smooth differential forms are smooth sections of the cotangent bundle, which is simply $\mathcal{M}_\infty^{m,n} \times (\mathcal{M}_\infty^{m,n})^*$ where $(\mathcal{M}_\infty^{m,n})^*$ is the topological dual of $\mathcal{M}_\infty^{m,n}$, i.e. the space of infinite sequences with only a finite number of nonzero entries; they can be written:

$$\omega = g\mathrm{d}x + \sum_{\text{finite}} \beta_j \mathrm{d}u^{(j)}. \tag{17}$$

This defines the $\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n})$ module $\Lambda^1(\mathcal{M}_\infty^{m,n})$ of smooth differential forms on $\mathcal{M}_\infty^{m,n}$. One may also define differential forms of all degree.

Of course, one may apply a differential form to a vector field according to $\langle \omega, F\rangle = fg + \sum \alpha_j \beta_j$ (compare (16)–(17)), where the sum is finite because finitely many β_j's are nonzero. One may also define the *Lie derivative* of a smooth function h, of a differential form $\omega, \ldots$ along a vector field F, which we denote by $L_F h$ or $L_F \omega$. The *Lie bracket* of two vector fields may also be defined. All this may be defined exactly as in the finite-dimensional case because, on a computational point of view, all the sums to be computed are finite.

Finally, note that a diffeomorphism carries differential forms, vector fields, functions from a manifold to another, exactly as in the finite dimensional case; for example, if φ is a diffeomorphism from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$, F is given by (16) and $z, v, \dot{v}, \ddot{v}, \ldots$ are the canonical coordinates on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$, the vector field $\varphi_* F$ on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is given by $\sum_i \widetilde{f}_i \frac{\partial}{\partial x_i} + \sum_{j,k} \widetilde{\alpha}_{j,k} \frac{\partial}{\partial u_k^{(j)}}$ with $\widetilde{f}_i = (L_F(z_i \circ \varphi)) \circ \varphi^{-1}$ and $\widetilde{\alpha}_{j,k} = (L_F(v_k^{(j)} \circ \varphi)) \circ \varphi^{-1}$.

3.3. *Systems.* A *system* is a vector field F on $\mathcal{M}_\infty^{m,n}$—with $n \geq 0$ and $m \geq 1$ some integers—of the form

$$F(\mathcal{X}) = f(x,u)\frac{\partial}{\partial x} + \sum_{j=0}^{+\infty} u^{(j+1)} \frac{\partial}{\partial u^{(j)}}, \tag{18}$$

i.e. the x-component of F is a function of x and u only, and its $u^{(j)}$-component is $u^{(j+1)}$. This may be rewritten, in a more condensed form,

$$F = f + C \tag{19}$$

where C is the *canonical vector field* on $\mathcal{M}_\infty^{m,n}$, given by

$$C = \sum_0^\infty u^{(j+1)} \frac{\partial}{\partial u^{(j)}}, \tag{20}$$

and the vector field f is such that

$$\begin{aligned} \langle \mathrm{d}u_i^{(j)}, f\rangle &= 0, \qquad i = 1, \ldots, m, j \geq 0, \\ [\partial/\partial u_i^{(j)}, f] &= 0, \qquad i = 1, \ldots, m, j \geq 1. \end{aligned} \tag{21}$$

m will be called the *number of inputs* of the system, and n its *state dimension*. Note that in the (explicit) non-classical case [10, 9] (i.e. the case when some derivatives of u would appear in the right-hand side of (1), there would be no restriction on f, besides being smooth, i.e. the second relation in (21) would no longer be there (note however that any smooth vector field has zero Lie bracket with $\frac{\partial}{\partial u^{(j)}}$ for j large enough, or in other words f depending on infinitely many time-derivatives of u in (1) is ruled out).

In the special case where $n = 0$, there is only one system (with "no state") on $\mathcal{M}_\infty^{m,0}$. We call this system the *canonical linear system* with m inputs; it is simply represented by the canonical vector field C given by (20).

In section 2, a system was an embedding of $\mathbb{R} \times \mathcal{M}_\infty^{m,n}$ as a subbundle of J^∞; this defines canonically the vector field F on $\mathcal{M}_\infty^{m,n}$ as, more or less, the pull back of the Cartan vector field (annihilating the contact forms) in $J^\infty(\pi)$.

F is the vector field defining the "total derivation along the system", i.e. the derivative of a smooth function (depending on $x, u, \dot{u}, \ldots, u^{(j)}$) knowing that $\dot{x} = f(x, u)$ is exactly its Lie derivative along this vector field. In [18], B. Jakubczyk attaches a differential algebra to the smooth system (1) which is exactly $\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n})$ endowed with the Lie derivative along the vector field F. Of course, this is very much related to the differential algebraic approach introduced in control theory by M. Fliess [10], based on differential Galois theory, and where a system is represented by a certain differential field. In the analytic case, as explained in [8], this differential field may be realized as the field of fractions of the integral domain $\mathcal{C}^\omega(\mathcal{M}_\infty^{m,n})$. The present framework is more or less dual to these differential algebra representations since it describes the set of "points" on which the objects manipulated in differential algebra are "functions".

The following proposition gives an intrinsic definition of the number of inputs, which will be useful to prove that it is invariant under dynamic equivalence:

PROPOSITION 2. *The number of inputs m is the largest integer q such that there exists q smooth functions $h_1, \ldots, h_q$ from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that all the functions*

$$L_F^j h_k, \qquad 1 \leq k \leq q, j \geq 0,$$

are independent (*the Jacobian of a finite collection of them has maximum rank*).

Proof. On one hand, $h_k(x,\mathcal{U}) = u_k$ provides m functions enjoying this property. On the other hand, consider $m+1$ smooth functions $h_1, \ldots, h_{m+1}$, let $\rho \geq 0$ be such that they are functions only of $x, u, \dot{u}, \ldots, u^{(\rho)}$, and consider the $(m+1)(n+m\rho+1)$ functions functions

$$L_F^j h_k, \qquad 1 \leq k \leq m+1, 0 \leq j \leq n+m\rho;$$

from the form of F (see (19) and (20)), they depend only on $x, u, \dot{u}, \ldots, u^{(\rho+n+m\rho)}$, i.e. on $n + m(\rho+n+m\rho+1)$ coordinates; since this integer is strictly smaller than $(m+1)(n+m\rho+1)$, the considered functions cannot be independent. ∎

3.4. *Differential calculus*; *an inverse function theorem.* All the identities from differential calculus involving functions, vector fields, differential forms apply on the "infinite-dimensional manifold" $\mathcal{M}_\infty^{m,n}$ exactly as if it were finite-dimensional: if it is an equality between functions or forms, it involves only a finite number of variables (i.e. both sides are constant along the vector fields $\frac{\partial}{\partial u_k^{(j)}}$ for j larger than a certain $J > 0$) so that all the vector fields appearing in the formula may be truncated (replaced by a vector fields with a zero component on $\frac{\partial}{\partial u_k^{(j)}}$ for $j > J$), and everything may then be projected by a certain π_K (K possibly larger than J), yielding an equivalent formula on the finite-dimensional manifold $\mathcal{M}_K^{m,n}$; if it is an equality between vector fields, it may be checked component by component, yielding equalities between functions, and the preceding remark applies.

Of course, theorems from differential calculus yielding existence of an object do not follow so easily, and often do not hold in infinite dimension. For instance, locally around a point where it is nonzero, a vector field on a manifold of dimension n has $n-1$ independent first integrals (functions whose Lie derivative along this vector field is zero) whereas this is false on $\mathcal{M}_\infty^{m,n}$ in general: for the vector field C on $\mathcal{M}_\infty^{m,0}$ given by (20), any function h such that $L_C h = 0$ is a constant function.

One fundamental theorem in differential calculus is the inverse function theorem stating that a smooth function from a manifold to another one whose tangent map at a certain point is an isomorphism admits locally a smooth inverse. In infinite dimensions, the situation is much more intricate, see for instance [17] for a very complete discussion of this subject and general inverse function theorems on Fréchet spaces, which are not exactly the kind of theorem we will need since more general smooth functions are considered there. Here, for a mapping φ from $\mathcal{M}_\infty^{m,n}$ (coordinates: $x, u, \dot{u}, \ldots$) to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ (coordinates: $z, v, \dot{v}, \ldots$), the function assigning to each point the tangent map to F at this point may be represented by the collection of differential forms $\mathrm{d}(z_i \circ \varphi), \mathrm{d}(v_k^{(j)} \circ \varphi)$, and a way of saying that, at all point, the linear mapping is invertible with a continuous inverse, and that it depends smoothly on the point, is to say that these forms are a basis of the module $\Lambda^1(\mathcal{M}_\infty^{m,n})$; equivalently, this tangent map might be represented by an infinite matrix whose lines are finite (each line represents one of the above differential forms), and which is invertible for matrix multiplication with an inverse having also finite lines. It is clear that for a diffeomorphism this linear invertibility holds; the additional assumption we add to get a converse is that the mapping under consideration carries a control system (as defined by (19)) on $\mathcal{M}_\infty^{m,n}$ to a control system on $\mathcal{M}_\infty^{\tilde m,\tilde n}$; note also that we require that the tangent map be invertible *in a neighborhood* of the point under consideration whereas the finite-dimensional theorem just asks for invertibility *at* the point.

Besides its intrinsic interest, the following result will be required to prove theorem 5 which characterizes "linearizing outputs" in terms of their differentials.

PROPOSITION 3. (local inverse function theorem) *Let $m, n, \widetilde m, \widetilde n$ be nonnegative integers with m and $\widetilde m$ nonzero. Let $z_1, \ldots, z_{\tilde n}, v_1, \ldots, v_{\tilde m}, \dot v_1, \ldots, \dot v_{\tilde m}, \ldots\ldots$ be the canonical coordinates on $\mathcal{M}_\infty^{\tilde m,\tilde n}$, and $\overline{\mathcal{X}} = (\bar x, \bar u, \dot{\bar u}, \ddot{\bar u}, \ldots)$ be a point in $\mathcal{M}_\infty^{m,n}$. Let φ be a smooth mapping from a neighborhood of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$ to a neighborhood of $\varphi(\overline{\mathcal{X}})$ in $\mathcal{M}_\infty^{\tilde m,\tilde n}$ such that*

1) *on a neighborhood of $\overline{\mathcal{X}}$, the following set of 1-forms on $\mathcal{M}_\infty^{m,n}$:*

$$\{\mathrm{d}(z_i \circ \varphi)\}_{1\le i\le \tilde n} \cup \{\mathrm{d}(v_k^{(j)} \circ \varphi)\}_{1\le k\le \tilde m, j\ge 0}, \tag{22}$$

forms a basis of the $\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n})$-module $\Lambda^1(\mathcal{M}_\infty^{m,n})$,

2) *there exist two control systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde F$ on $\mathcal{M}_\infty^{\tilde m,\tilde n}$ such that, for all function $\widetilde h \in \mathcal{C}^\infty(\mathcal{M}_\infty^{\tilde m,\tilde n})$, defined on a neighborhood of $\varphi(\overline{\mathcal{X}})$,*

$$(L_{\tilde F}\widetilde h)\circ\varphi = L_F(\widetilde h\circ\varphi). \tag{23}$$

Then φ is a local diffeomorphism at $\overline{\mathcal{X}}$, i.e. there exists a neighborhood U of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$, a neighborhood V of $\varphi(\overline{\mathcal{X}})$ in $\mathcal{M}_\infty^{\tilde m,\tilde n}$ and a smooth mapping (a diffeomorphism) ψ from V to U such that $\psi\circ\varphi = Id_U$ and $\varphi\circ\psi = Id_V$.

Note that (22) is a way of expressing that the tangent map to φ is invertible with a continuous inverse, and (23) is a way of expressing that φ transforms the control system F into the control system $\widetilde F$, in a dual manner since writing $\widetilde F = \varphi_* F$ would presuppose that φ is a diffeomorphism.

Proof. Let $x_1, \ldots, x_n, u_1, \ldots, u_m, \dot u_1, \ldots, \dot u_m, \ldots\ldots$ be the canonical coordinates on $\mathcal{M}_\infty^{m,n}$. The first condition implies that there exist some smooth functions a_i^k, $b_i^{k,j}$, c_i^k, $d_i^{k,j}$ such that

$$\begin{aligned}
\mathrm{d}x_i &= \sum_{k=1}^{\tilde n} a_i^k \mathrm{d}(z_k\circ\varphi) + \sum_{j=0}^{L}\sum_{i=1}^{\tilde m} b_i^{k,j}\mathrm{d}(v_k^{(j)}\circ\varphi), \qquad i = 1,\ldots,n,\\
\mathrm{d}u_i &= \sum_{k=1}^{\tilde n} c_i^k \mathrm{d}(z_k\circ\varphi) + \sum_{j=0}^{L}\sum_{i=1}^{\tilde m} d_i^{k,j}\mathrm{d}(v_k^{(j)}\circ\varphi), \qquad i = 1,\ldots,m.
\end{aligned} \tag{24}$$

Let K be the integer such that the functions $z_1\circ\varphi, \ldots, z_{\tilde n}\circ\varphi, v_1\circ\varphi, \ldots, v_{\tilde m}\circ \varphi, \ldots, v_1^{(L)}\circ\varphi, \ldots, v_{\tilde m}^{(L)}\circ\varphi$, and the functions a_i^k, $b_i^{k,j}$, c_i^k, $d_i^{k,j}$ all depend on x, u, $\dot u$, …, $u^{(K)}$ only. Then $z_1\circ\varphi, \ldots, z_{\tilde n}\circ\varphi, v_1\circ\varphi, \ldots, v_{\tilde m}\circ\varphi$ are $\widetilde n + \widetilde m$ functions of the $n+(K+1)m$ variables $x_1,\dots,x_n,u_1,\dots,u_m,\dots,u_1^{(K)},\dots,u_m^{(K)}$ which, from condition 1 in the proposition are independent because the fact the forms in (24) form a basis of the module of all forms implies in particular that a finite number of them has full rank *at all points* as vectors in the cotangent vector space. Hence, from the finite dimensional inverse function theorem, one may locally replace, in $x_1,\dots,x_n,u_1,\dots,u_m,\dots,u_1^{(K)},\dots,u_m^{(K)}$, $\widetilde{n}+\widetilde{m}$ coordinates with the functions $z_1\circ\varphi,\dots,z_{\tilde{n}}\circ\varphi,v_1\circ\varphi,\dots,v_{\tilde{m}}\circ\varphi$. In particular, there exists $n+m$ functions ξ_i and ζ_i^0 defined on a neighborhood of $(\bar{z},\bar{v},\dot{\bar{v}},\dots,\bar{v}^{(L)})$—with $\varphi(\overline{\mathcal{X}})=(\bar{z},\bar{v},\dot{\bar{v}},\ddot{\bar{v}},\dots,\bar{v}^{(L)})$— and such that

$$\begin{aligned} x_i&=\xi_i(z\circ\varphi,v\circ\varphi,\dots,v^{(L)}\circ\varphi,\mathcal{Y}), \qquad i=1,\dots,n,\\ u_i&=\zeta_i^0(z\circ\varphi,v\circ\varphi,\dots,v^{(L)}\circ\varphi,\mathcal{Y}), \qquad i=1,\dots,m, \end{aligned} \tag{25}$$

where $\mathcal{Y}$ represents some of the $n+(K+1)m$ variables $x,u,\dot{u},\dots,u^{(K)}$ (all minus $\widetilde{n}+(L+1)\widetilde{m}$ of them). $\mathrm{d}x_i$ and $\mathrm{d}u_i$ may be computed by differentiating (25); the expression involves the partial derivatives of the functions ξ_i and ζ_i and comparing with the expressions in (24), one may conclude that

$$\frac{\partial\xi_i}{\partial\mathcal{Y}}=0,\quad \frac{\partial\zeta_i^0}{\partial\mathcal{Y}}=0, \tag{26}$$

and we may write, instead of (25),

$$\begin{aligned} x_i&=\xi_i(z\circ\varphi,v\circ\varphi,\dots,v^{(L)}\circ\varphi), \qquad i=1,\dots,n,\\ u_i&=\zeta_i^0(z\circ\varphi,v\circ\varphi,\dots,v^{(L)}\circ\varphi), \qquad i=1,\dots,m. \end{aligned} \tag{27}$$

We then define the functions ζ_i^j for $j>0$ by

$$\zeta_i^j=L_{\tilde{F}}\zeta_i^0 \tag{28}$$

(note that this makes ζ_i^j a smooth function of $z,v,\dots,v^{(l+j)}$) and we define ψ by

$$\begin{aligned} \psi(z,v,\dot{v},\ddot{v},\dots)=(x,u,\dot{u},\ddot{u},\dots) \quad &\text{with}\\ x_i&=\xi_i(z,v,\dots,v^{(L)}),\\ u_i&=\zeta_i^0(z,v,\dots,v^{(L)}),\\ \dot{u}_i&=\zeta_i^1(z,v,\dots,v^{(L+1)}),\\ &\vdots \end{aligned} \tag{29}$$

It is straightforward to check that (23), (28), (29) and the fact that $L_F^j u$ is $u^{(j)}$ imply that $\varphi\circ\psi=Id$ and $\psi\circ\varphi=Id$. ■

4. Dynamic equivalence. The objective of the previous sections is the following definition. As announced in the introduction, it mimics the notion of equivalence, or *equivalence by endogenous dynamic feedback* given in [21] for analytic systems (analyticity plays no role at all in the definition of local equivalence), which coincides with the one given in [12, 13] when the transformations are algebraic. The present definition is more concise than in [21] and allows some simple geometric considerations, but the concept of equivalence is the same one. It also coincides with "dynamic equivalence" as defined in [18, 19], see below. It is proved in [21] that if two systems are equivalent in this sense then there exists a dynamic feedback in the sense of (2) which is *endogenous* and nonsingular and transforms one system into a "prolongation" of the other.

DEFINITION 1 (Equivalence). Two systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ are *equivalent* at points $\overline{\mathcal{X}} \in \mathcal{M}_\infty^{m,n}$ and $\overline{\mathcal{Y}} \in \mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ if and only if there exists a neighborhood U of $\overline{\mathcal{X}}$ in $\mathcal{M}_\infty^{m,n}$, a neighborhood V of $\overline{\mathcal{Y}}$ in $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$, and a diffeomorphism φ from U to V such that $\varphi(\mathcal{X}) = \mathcal{Y}$ and, on U,

$$\widetilde{F} = \varphi_* F. \tag{30}$$

They are *globally equivalent* if there exists a diffeomorphism φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ such that (30) holds everywhere.

Note that in the definition of *local* equivalence, the diffeomorphism is only defined locally. This might be worrying: it is not very practical to know that something may be constructed in a region which imposes infinitely many constraints on infinitely many derivatives of the input u. This actually does not occur because a neighborhood U of a point $\mathcal{X}$ contains an open set of the form $\pi_K^{-1}(U_K)$ with U_K open in $\mathcal{M}_K^{m,n}$, so that being in U imposes some constraints on $x, u, \dot{u}, \ddot{u}, \ldots, u^{(K)}$ but none on $u^{(K+1)}, u^{(K+2)}, \ldots$.

Some notions of dynamic equivalence ("dynamic equivalence" and "dynamic feedback equivalence") are also given in [18, 19]. To describe them, let us come back to the framework of section 2, where $\mathcal{M}_\infty^{m,n}$ is a subbundle of $J^\infty(\pi)$ and $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ is a subbundle of $J^\infty(\widetilde{\pi})$; the transformations considered in [18, 19] have to be defined from $J^\infty(\pi)$ to $J^\infty(\widetilde{\pi})$ whereas our diffeomorphism φ is only defined on $\mathcal{M}_\infty^{m,n}$ (and maps it onto $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$); actually, Lie-Bäcklund transformations are usually defined, like in [18, 19], all over $J^\infty(\pi)$; this is referred to as *outer* transformations, or outer symmetries if it maps a system into itself, whereas *inner* transformations are these, like our φ, defined only "on the solutions", i.e. on $\mathcal{M}_\infty^{m,n}$. Since the transformations in [19] are required to be invertible on the solutions only, it is proved there that a transformation like our φ may be extended (at least locally) to $J^\infty(\pi)$ and therefore that local equivalence in the sense of definition 1 is the same as the local version of the one called "dynamic equivalence" (and not "dynamic feedback equivalence") in [19].

It is clear that equivalence *is* an equivalence relation on systems, i.e. on vector fields of the form (19) because the composition of two diffeomorphisms is a diffeomorphism. There is not however a natural group acting on systems since a given diffeomorphism might transform a system F into a system G and transform another system F' into a vector field on $\mathcal{M}_\infty^{m,n'}$ which is not a system. For instance, for $p_1, \ldots, p_m$ nonnegative, the diffeomorphism $\Upsilon_{n,(p_1,\ldots,p_m)}$ defined in (15) transforms any system on $\mathcal{M}_\infty^{m,n}$ into a system on $\mathcal{M}_\infty^{m,n+p_1+\ldots+p_m}$ whereas the diffeomorphism $\Upsilon_{n+p_1+\ldots+p_m,(-p_1,\ldots,-p_m)}$—its inverse—transforms most systems on $\mathcal{M}_\infty^{m,n+p_1+\cdots+p_m}$ into a vector field on $\mathcal{M}_\infty^{m,n}$ which is not a "system" because it does not have the required structure on the coordinates which are called "inputs" on $\mathcal{M}_\infty^{m,n}$. Two important questions arise: what is exactly the class of diffeomorphisms which transform at least one system into another system and what is the class of vector fields equivalent to a system by such a diffeomorphism. An element of answer to the latter question is that "non-classical" systems [10, 9], i.e. these where the right-hand side of (1) depends also on some time-derivatives of u, or vector fields on which the second constraint in (21) does not hold, *are* in this class of vector fields because they are transformed by $\Upsilon_{n,(K,\ldots,K)}$, where K is the number of derivatives of the input appearing in the system, into a (classical) system, this illustrates that generalized state-space representations [10, 9] are "natural"; however, it is clear that the class of vector fields which may be conjugated to a "system" is much larger: the only system (classical or not) on $\mathcal{M}_\infty^{m,0}$ is C and very few systems on $\mathcal{M}_\infty^{m,n}$ are transformed into C by $\Upsilon_{n,(-n,0,\ldots,0)}$ for example. A partial answer to the former question is given by:

THEOREM 1. *The number of inputs m is invariant under equivalence.*

Proof. For any function h, $L_{\tilde{F}}\left(h \circ \varphi^{-1}\right) = (L_F h) \circ \varphi^{-1}$. The integer m from proposition 2 is therefore preserved by a diffeomorphism φ. ∎

Further remarks on the class of diffeomorphisms which transform at least one system into another system may be done. One may restrict attention to systems of the same dimension, i.e. to diffeomorphisms from $\mathcal{M}_\infty^{m,n}$ to itself because if φ goes from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,N}$ with $N > n$ and transforms a system into a system, $\Upsilon_{n,(N-n,0,\ldots,0)} \circ \varphi$ is a diffeomorphism of $\mathcal{M}_\infty^{m,N}$ that transforms a system into a system. In the single-input case ($m = 1$), as stated in section 6, φ must be static, which is a complete answer to the question because a static diffeomorphism transforms *any* system into a system. In the case of at least two inputs ($m > 1$), the literature ([20, Theorem 4.4.5] or [1, Theorem 3.1], but these have to be adapted since they are stated in an "outer" context) tells us that either φ is static or it does not preserve the fibers of $\pi_k : \mathcal{M}_\infty^{m,n} \to \mathcal{M}_k^{m,n}$ for any k, i.e., if φ is given by $\varphi(x, u, \dot{u}, \ddot{u}, \ldots) = (z, v, \dot{v}, \ddot{v}, \ldots)$, there is no k such that $(z, v, \dot{v}, \ldots, v^{(k)})$ is a function of $(x, u, \dot{u}, \ldots, u^{(k)})$ only. This is related to the statement [7] that, when dynamic feedback is viewed as adding some integrators plus performing a static feedback, it is inefficient to add the same number of integrators on each input.

5. Static equivalence

DEFINITION 2 (Static equivalence). Two systems F on $\mathcal{M}_\infty^{m,n}$ and $\widetilde{F}$ on $\mathcal{M}_\infty^{\tilde{m},\tilde{n}}$ are (*locally/globally*) *static equivalent* if and only if they are (locally/globally) feedback equivalent with the diffeomorphism φ in (30) being a *static* diffeomorphism.

From proposition 1, we know that a static diffeomorphism really defines an invertible static feedback transformation in the usual sense, this is summed up in the following:

THEOREM 2. *Both the number of inputs* m *and the dimension* n *of the state are invariant under static equivalence. Moreover*, $\pi_{-1} \circ \varphi$ *provides a local diffeomorphism in the classical state-space* $\mathbb{R}^n$ *and the* u *component of* $\pi_0 \circ \varphi$ *provides a nonsingular feedback transformation which together provide an invertible static feedback transformation in the usual sense.*

6. The single-input case. It was proved in [7, 6] that a single-input system which is "dynamic feedback linearizable" is "static feedback linearizable". The meaning of dynamic feedback linearizable was weaker that being equivalent to a linear system as meant here: "exogenous" feedbacks (see [21]) were allowed in [7] as well as singular (feedbacks which may change the number of inputs for example). The following Theorem 3 may be viewed as a generalization of this result to non-linearizable systems, but with a more restrictive dynamic equivalence.

It is known that the only transformations on an infinite jet bundle with only one "dependent variable" which preserves the contact structure (Lie-Bäcklund transformation in [1], $\mathcal{C}$-transformation in [20]) are infinite prolongations of transformations on first jets (Lie transformation according to [20]), see for instance [20, Theorems 6.3.7 and 4.4.5]. The following result is similar in spirit. We give the full proof, a little long but elementary: it basically consists in counting the dimensions carefully, it is complicated by the fact that we do not make any a priori regularity assumption (for instance, the functions χ_i and ψ_i defining the diffeomorphism are not assumed to depend on a locally constant number of derivatives of u).

THEOREM 3. *Let* F *and* $\widetilde{F}$ *be two systems on* $\mathcal{M}_\infty^{1,n}$ (*i.e. two single input systems with the same number of states*). *Any* (*local/global*) *diffeomorphism* φ *such that* $\widetilde{F} = \varphi_* F$ *is static. Hence they are* (*locally/globally*) *equivalent if and only if they are* (*locally/globally*) *static equivalent.*

Proof. The second statement is a straightforward consequence of the first one. Let us consider a diffeomorphism φ such that $\widetilde{F} = \varphi_* F$ and prove that φ is static. Suppose that, in coordinates, φ and φ^{-1} are given by $\varphi(x,\mathcal{U}) = (z,\mathcal{V})$ and $\varphi^{-1}(z,\mathcal{V}) = (x,\mathcal{U})$ with:

$$
\begin{aligned}
z &= \chi_{-1}(x,\mathcal{U}), & x &= \psi_{-1}(z,\mathcal{V}),\\
v &= \chi_0(x,\mathcal{U}), & u &= \psi_0(z,\mathcal{V}),\\
&\vdots & &\vdots \\
v^{(j)} &= \chi_j(x,\mathcal{U}), & u^{(j)} &= \psi_j(z,\mathcal{V}),\\
&\vdots & &\vdots
\end{aligned}
\tag{31}
$$

Since $\widetilde{F} = \varphi_* F$, we have

$$L_F \chi_{-1}(x,\mathcal{U}) = \widetilde{f}(\chi_{-1}(x,\mathcal{U}), \chi_0(x,\mathcal{U})),$$
$$L_F \chi_j(x,\mathcal{U}) = \chi_{j+1}(x,\mathcal{U}) \qquad \text{for } j \geq 0. \tag{32}$$

Let $\mathcal{X}$ be an arbitrary point of the domain where φ is defined. From the definition of a diffeomorphism, there is an integer $J \geq -1$ and a neighborhood U of $\mathcal{X}$ (J is $\delta(\pi_0 \circ \varphi)(\mathcal{X})$ if U is small enough) such that χ_{-1} and χ_0 depend only on $x, u, \dot{u}, \ldots, u^{(J)}$ on U and $\frac{\partial \chi_{-1}}{\partial u^{(J)}}$ and $\frac{\partial \chi_0}{\partial u^{(J)}}$ are not both identically zero on U (one might take any open set where φ is defined—$\mathcal{M}_\infty^{m,1}$ in the global case—instead of U, but this might cause J to be infinite).

If J were -1, then χ_{-1} and χ_0 would both depend only on x, but the dimension of x is n and the dimension of (χ_{-1}, χ_0) is $n+1$: there would be a function such that $h(\chi_{-1}, \chi_0)$ would be zero on U and this would prevent φ from being a diffeomorphism; hence $J \geq 0$.

The first equation in (32), and the second one for $j = 0$, imply:

$$\frac{\partial \chi_{-1}}{\partial x} f(x,u) + \frac{\partial \chi_{-1}}{\partial u} \dot{u} + \ldots + \frac{\partial \chi_{-1}}{\partial u^{(J)}} u^{(J+1)} = \widetilde{f}(\chi_{-1}(x \ldots u^{(J)}), \chi_0(x \ldots u^{(J)})),$$
$$\frac{\partial \chi_0}{\partial x} f(x,u) + \frac{\partial \chi_0}{\partial u} \dot{u} + \ldots + \frac{\partial \chi_0}{\partial u^{(J)}} u^{(J+1)} = \chi_1(x,\mathcal{U}).$$

By taking the derivative with respect to $u^{(J+1)}$ of the first equation and with respect to $u^{(j)}$ for $j \geq J+2$ of the second equation,

$$\frac{\partial \chi_{-1}}{\partial u^{(J)}} = 0 \quad \text{and} \quad 0 = \frac{\partial \chi_1}{\partial u^{(j)}} \quad \text{for } j \geq J+2. \tag{33}$$

This implies that χ_{-1} is a function of $x, u, \ldots, u^{(J-1)}$ (x if $J=0$) only, χ_0 is a function of $x, u, \ldots, u^{(J-1)}, u^{(J)}$ only (by definition of J), and χ_1 of x, $u, \ldots, u^{(J-1)}$, $u^{(J)}$, $u^{(J+1)}$ only. It is then easy to deduce by induction from the second relation in (32) that for all $j \geq 0$, χ_j is a function of $x, u, \ldots, u^{(J+j+1)}$ on this neighborhood with

$$\frac{\partial \chi_j}{\partial u^{(J+j)}} = \frac{\partial \chi_0}{\partial u^{(J)}}, \qquad j \geq 0. \tag{34}$$

From the first relation in (33) and the definition of J, $\frac{\partial \chi_0}{\partial u^{(J)}}$ is not identically zero on U. Hence, there is a point $\overline{\mathcal{X}} = (\overline{x}, \overline{u}, \dot{\overline{u}}, \ldots) \in U$ such that $\frac{\partial \chi_0}{\partial u^{(J)}}(\overline{\mathcal{X}}) = \frac{\partial \chi_0}{\partial u^{(J)}}(\overline{x}, \overline{u}, \ldots, \overline{u}^{(J)}) \neq 0$. Let K be $\delta(\pi_0 \circ \varphi^{-1})(\overline{\mathcal{X}})$—note that it might be smaller than $\delta(\pi_0 \circ \varphi^{-1})(\mathcal{X})$—i.e. ψ_{-1} and ψ_0 locally depend only on $z, v, \ldots, v^{(K)}$, and $\frac{\partial \psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial \psi_0}{\partial v^{(K)}}$ are not both identically zero on any neighborhood of $\overline{\mathcal{X}}$. This implies, since $\frac{\partial \chi_0}{\partial u^{(J)}}$ is nonzero at $\overline{\mathcal{X}}$, that there is a neighborhood $\overline{U}$ of $\overline{\mathcal{X}}$ such that, on $\overline{U}$, $\frac{\partial \chi_0}{\partial u^{(J)}}$ does not vanish, ψ_{-1} and ψ_0 depend only on $z, v, \ldots, v^{(K)}$ and $\frac{\partial \psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial \psi_0}{\partial v^{(K)}}$ are not both identically zero. We have, on $\overline{U}$,

$$
(35)\quad
\begin{aligned}
x &= \psi_{-1}(\chi_{-1}(x,u,\dots,u^{(J-1)}),\chi_0(x,u,\dots,u^{(J)}),\dots\\
&\qquad\qquad\qquad\dots,\chi_K(x,u,\dots,u^{(J+K)})),\\
u &= \psi_0(\chi_{-1}(x,u,\dots,u^{(J-1)}),\chi_0(x,u,\dots,u^{(J)}),\dots\\
&\qquad\qquad\qquad\dots,\chi_K(x,u,\dots,u^{(J+K)})).
\end{aligned}
$$

K cannot, for the same dimensional reasons as J, be equal to -1, hence $K \geq 0$. Now, suppose that $J \geq 1$. Then $J+K \geq 1$, and taking the derivative of both identities in (35) with respect to $u^{(J+K)}$ therefore yields

$$
(36)\quad \frac{\partial\psi_{-1}}{\partial v^{(K)}}\frac{\partial\chi_K}{\partial u^{(J+K)}} = \frac{\partial\psi_0}{\partial v^{(K)}}\frac{\partial\chi_K}{\partial u^{(J+K)}} = 0
$$

identically on $\overline{U}$. This is impossible because on the one hand $\frac{\partial\chi_K}{\partial u^{(J+K)}}$ does not vanish because of (34) and on the other hand K has been defined so that $\frac{\partial\psi_{-1}}{\partial v^{(K)}}$ and $\frac{\partial\psi_0}{\partial v^{(K)}}$ are not both identically zero on $\overline{U}$. Hence $J \geq 1$ is impossible.

We have proved that $J = 0$. Hence χ_j depends only on $x, u \dots u^{(j)}$ (x for $j = -1$) for all $j \geq -1$ (see above) and $\frac{\partial\chi_j}{\partial u^{(j)}}$ is, for all j, nonsingular at all points (consequence of the smooth invertibility of φ). This is the definition of a static diffeomorphism. ∎

7. Dynamic linearization. A *controllable linear system* is a system of the form (19) where the function f is linear, i.e. $f(x,u) = Ax + Bu$ with A and B constant matrices, and (Kalman rank condition) the rank of the columns of B, AB, A^2B is n.

There is a canonical form under static feedback, known as Brunovský canonical form [5] for these systems: they may be transformed via a static diffeomorphism (from $\mathcal{M}_\infty^{m,n}$ to itself) to a linear system where A and B have the form of some "chains of integrators" of "length" $r_1,\dots,r_m$; the diffeomorphism $\Upsilon_{n,(-r_1,\dots,-r_m)}$ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,0}$ (see (15)) which "cuts off" all these integrators then transforms this system into C (see (20)):

Proposition 4 ([5]). *A controllable linear system with m inputs is globally equivalent to the canonical system C on $\mathcal{M}_\infty^{m,0}$.*

We wish to call a system which is equivalent to a *controllable linear* system *dynamic linearizable.* From the above proposition, this may equivalently be stated as:

Definition 3. A system is (*locally/globally*) *dynamic linearizable* if and only if it is (locally/globally) equivalent to the canonical linear system C on $\mathcal{M}_\infty^{0,m}$.

Of course this concept is the same as in [21, "analytic approach"] since the equivalence is the same. In [12, 13, 21], the notion of *linearizing outputs* or *flat outputs* is used to define *flat control systems* as these which admit such outputs. It is proved that flatness coincides with equivalence by endogenous feedback to a controllable linear system. In [18, 19] a system is called free if the differential algebra $(\mathcal{C}^\infty(\mathcal{M}_\infty^{m,n}), L_F)$ is free; the linearizing outputs we define below are free generators of this differential algebra. The following theorem in a sense re-states the result "flat $\Leftrightarrow$ linearizable by endogenous feedback".

THEOREM 4 (linearizing outputs). *A system F on $\mathcal{M}_\infty^{m,n}$ is locally dynamic linearizable at a point $\mathcal{X}$ if and only if there exist m smooth functions $h_1, \dots, h_m$ from a neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\le k\le m, 0\le j}$ is a system local of coordinates at $\mathcal{X}$. It is globally dynamic linearizable and only if there exist m smooth functions $h_1, \dots, h_m$ from $\mathcal{M}_\infty^{m,n}$ to $\mathbb{R}$ such that $(L_F^j h_k)_{1\le k\le m, 0\le j}$ is a global system of coordinates. These functions are called linearizing outputs.*

Proof. If F is dynamic linearizable, there exists a (local/global) diffeomorphism φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{m,0}$ such that $C = \varphi_* F$. Define h_k by $h_k = v_k^{(j)} \circ \varphi$ with $v_k^{(j)}$ the canonical coordinates on $\mathcal{M}_\infty^{0,m}$. Since $v_k^{(j)} = L_C^j v_k$ (the jth Lie derivative of v_k along C) and $C = \varphi_* F$, we have

$$L_F^j h_k = L_F^j (v_k \circ \varphi) = \left(L_{\varphi_* F}^j v_k\right) \circ \varphi = v_k^{(j)} \circ \varphi,$$

so that, since φ is a diffeomorphism and $(v_k^{(j)})_{1\le k\le m, 0\le j}$ is a system of coordinates on $\mathcal{M}_\infty^{m,0}$, $(v_k^{(j)} \circ \varphi)_{1\le k\le m, 0\le j}$ is a system of coordinates on $\mathcal{M}_\infty^{m,n}$. Conversely, if there exist m functions $h_1, \dots, h_m$ enjoying this property, then one may define the diffeomorphism φ mapping a point $(x, \mathcal{U})$ of $\mathcal{M}_\infty^{m,n}$ to the point of $\mathcal{M}_\infty^{0,m}$ whose coordinate $v_k^{(j)}$ is $L_F^j h_k(x, \mathcal{U})$. It is clear that $\varphi_* F = C$. ∎

Of course, this is far from being a solution to dynamic feedback linearization since one has to determine if linearizing outputs exist, which is not an easy task; see [3] for bibliography and a discussion of this topic. Let us give a rather convenient way of tackling this problem by transforming it into its "infinitesimal" version. Recall that a pfaffian system is a family of differential forms of degree 1 with constant rank; any family of forms generating the same module (or co-distribution) defines the same pfaffian system. The infinitesimal version of linearizing outputs is:

DEFINITION 4. A pfaffian system $(\omega_1, \dots, \omega_m)$ is called a *linearizing pfaffian system* at a point $\mathcal{X}$ if and only if, for a certain neighborhood U of $\mathcal{X}$, the restriction to U of the forms $L_F^j \omega_k$, $j \ge 0$, $1 \le k \le m$ form a basis of the $\mathcal{C}^\infty(U)$-module $\Lambda^1(U)$ of all differential forms on U.

We have three comments on this definition. Firstly, this is a property of the pfaffian system $(\omega_1, \dots, \omega_m)$ rather than the m-uple of 1-forms since it is not changed when changing the collection of forms $\omega_1, \dots, \omega_m$ into another collection which span the same module. Secondly, one may prove than the rank of such a pfaffian system *must* be m (see the proof of proposition 2). Finally, one should not be misled by the terminology: *existence of a linearizing pfaffian system does not imply linearizability*:

Theorem 5. *A system F on $\mathcal{M}_\infty^{m,n}$ is locally dynamic linearizable at point $\mathcal{X}$ if and only if there exists, on a neighborhood of $\mathcal{X}$, a linearizing pfaffian system $(\omega_1, \ldots, \omega_m)$ which is locally completely integrable.*

By locally completely integrable, we mean the classical Frobenius condition $\mathrm{d}\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_m = 0$; note that the condition that $(L_F^j \omega_k)_{1\le k\le m, 0\le j}$ be a basis of $\Lambda^1(U)$ implies that the rank at all point of $(\omega_1, \ldots, \omega_m)$ is m, and is therefore constant.

Proof. The condition is obviously necessary from Theorem 4 by taking $\omega_k = \mathrm{d}h_k$. Conversely, one may apply the finite-dimensional Frobenius theorem to $(\omega_1, \ldots, \omega_m)$ because they depend on a finite number of variables, and, as noticed above, they have constant rank m: there exists m functions $h_1 \ldots h_m$ (of the same number of variables than these appearing in $\omega_1 \ldots \omega_m$) such that $\mathrm{d}h_1, \ldots, \mathrm{d}h_m$ span the same co-distribution than $\omega_1, \ldots, \omega_m$; this implies that $(L_F^j \mathrm{d}h_k)_{1\le k\le m, 0\le j}$ is also a basis of $\Lambda^1(U)$. Define the map $\varphi : U \to \mathcal{M}_\infty^{m,0}$ as assigning to a point $(x,\mathcal{U})$ of $\mathcal{M}_\infty^{m,n}$ to the point of $\mathcal{M}_\infty^{0,m}$ whose coordinate $v_k^{(j)}$ is $L_F^j h_k(x,\mathcal{U})$. It is clear that for all functions $\tilde{h} \in \mathcal{C}^\infty(\mathcal{M}_\infty^{0,m})$, $(L_C \tilde{h})\circ\varphi = L_\varphi(\tilde{h}\circ\varphi)$, so that theorem 3 implies that φ is a local diffeomorphism. ■

This result is more interesting in the light of the fact that a controllable system admits a linearizing pfaffian system at "almost all" points. This is further developed in [3], which presents results similar to [2, 24] in the present geometric framework.

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