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A direct algorithm of one-dimensional optimal system for the group invariant solutions

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A direct and systematic algorithm is proposed to find one-dimensional optimal system for the group invariant solutions, which is attributed to the classification of its corresponding one-dimensional Lie algebra. Since the method is based on different values of all the invariants, the process itself can both guarantee the comprehensiveness and demonstrate the inequivalence of the optimal system, with no further proof. To leave the algorithm clear, we illustrate each stage with a couple of well-known examples: the Korteweg-de Vries equation and the heat equation. Finally, we apply our method to the Novikov equation and use the found optimal system to investigate the corresponding invariant solutions. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4921229]

I. INTRODUCTION

Symmetry group theory built by Sophus Lie plays an important role in constructing explicit solutions, whether the equations are integrable or not, linear or nonlinear. For any given subgroup, an original nonlinear system can be reduced to a system with fewer independent variables, which corresponds to group invariant solutions. As Olver[1] said, since there is almost always an infinite amount of such subgroups, it is usually not feasible to list all possible group invariant solutions to the system. It is anticipated to find all those inequivalent group invariant solutions, that is to say, to give them a classification. The problem of classifying the subgroups and reduction to optimal systems takes on more importance for multidimensional partial differential equations (PDEs). Given a group that leaves a PDE invariant, one desires to minimize the search for group-invariant solutions to that of finding inequivalent branches of solutions, which leads to the concept of the optimal systems. Consequently, the problem of determining the optimal system of subgroups is reduced to the corresponding problem for subalgebras. In applications, one usually constructs the optimal system of subalgebras, from which the optimal systems of subgroup and group invariant solutions are reconstructed.

The adjoint representation of a Lie group on its Lie algebra was known to Lie. Its use in classifying group-invariant solutions appears in Ovsiannikov.[2] Ovsiannikov demonstrated the construction of the one-dimensional optimal system for the Lie algebra, using a global matrix for the adjoint transformation, and sketched the construction of higher-dimensional optimal systems with a simple example. The method has received extensive development by Patera, Winternitz, and Zassenhaus,[3,4] and many examples of optimal systems of subgroups for the important Lie groups of mathematical physics were obtained. In the investigation of the connections between Lie group and special functions, Weisner[5] first gave the classification of the symmetry algebra of the heat equation. For

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the higher-dimensional optimal systems of Lie algebra, Galas[6] also developed Ovsiannikov's idea by removing equivalent subalgebras and the problem of a nonsolvable algebra was also discussed, which is generally harder than that for a solvable algebra. Some examples of optimal systems can also be found in Ibragimov.[7,8]

Here, we are concerned with the one-dimensional optimal system of subalgebras. For the one-dimensional optimal systems, the technique of Ovsiannikov has been used until Olver gives a slightly different and elegant technique. Olver[1] constructed a table of adjoint operators to simplify a general element in Lie algebra as much as possible and applied the technique to the Korteweg-de Vries (KdV) equation and the heat equation. Since it only depends on fragments of the theory of Lie algebras, Olver's method as developed here has the feature of being very elementary. Based on Olver's method, we have also constructed many interesting and important invariant solutions[9–12] for a number of systems of PDEs in atmosphere and geometric field. However, as Olver said, although some sophisticated techniques are available for Lie algebras with additional structure, in essence this problem is attacked by the naïve approach. One knows that, if one calls a list of $\{\tilde{v}_\alpha\}_{\alpha\in\mathcal{A}}$ is a one-dimensional optimal system, it must satisfy two conditions: (1) completeness—any one-dimensional subalgebra is equivalent to some $\tilde{v}_\alpha$; (2) inequivalence—$\tilde{v}_\alpha$ and $\tilde{v}_\beta$ are inequivalent for distinct α and β. Despite these numerous results on finding the representatives of subalgebras, they did not illustrate that how these representatives are comprehensive and mutually inequivalent. Recently, Chou and Qu[13–16] offer many numerical invariants to address the inequivalence among the elements in the optimal system.

The purpose of this paper is to give a systematic method for finding an optimal system of one-dimensional Lie algebra, which can both guarantee the comprehensiveness and the inequivalence. The idea is inspired by the observation that the Killing form of the Lie algebra is an invariant for the adjoint representation.[1] Olver also points out that the detection of such an invariant is important since it places restrictions on how far one can expect to simplify the Lie algebra. In spite of the importance of the invariants for the Lie algebra, to the best of our knowledge, there are few literatures to use more common invariants except the Killing form in the process of constructing optimal system. The purpose of this paper is to introduce a direct and valid method for providing all the general invariants which are different from the numerical invariants appearing in Refs. 13–15, and then make the best use of them with the adjoint matrix to classify subalgebras. We shall demonstrate the new technique by treating three illustrative examples.

This paper is arranged as follows. In Sec. II, a direct algorithm of one-dimensional optimal system for the general symmetry algebra is proposed. Since the realization of our new algorithm builds on different invariants and the adjoint matrix, a valid method for computing all the invariants is also given in this section. To leave our algorithm clear, we would illustrate each stage with a couple of well-known examples, i.e., the KdV equation and the heat equation. In Sec. III, we apply the new algorithm to the Novikov equation and use the optimal system to find group invariant solutions. Conclusions and discussions are given in Sec. IV.

II. A DIRECT ALGORITHM OF ONE-DIMENSIONAL OPTIMAL SYSTEM

Consider an n-dimensional symmetry algebra $\mathcal{G}$ of a system of differential equations, which is generated by the vector fields $\{v_1, v_2, \ldots, v_n\}$. The corresponding symmetry group of $\mathcal{G}$ is denoted as G. Following Ovsiannikov,[2] one calls two elements $v = \sum_{i=1}^{n} a_i v_i$ and $w = \sum_{j=1}^{n} b_j v_j$ in $\mathcal{G}$ equivalent if they satisfy one of the following conditions:

(1) one can find some transformation $g \in G$ so that $Ad_g(w) = v$;

(2) there is $v = cw$ with c being constant.

Here, Ad_g is the adjoint representation of g and $Ad_g(w) = g^{-1}wg$. It needs to note that the second condition is less obvious in all the references but here it will play an important role in our method. The main tools used in our algorithm are all the invariants and the adjoint matrix. In this section, we will first give an algorithm for the general system of differential equations stage by stage and then illustrate each step with two known examples, the KdV and heat equations.

A. Calculation of the invariants

A real function ϕ on the Lie algebra $\mathcal{G}$ is called an invariant if $\phi(Ad_g(v)) = \phi(v)$ for all $v \in \mathcal{G}$ and all $g \in G$. If two vectors v and w are equivalent under the adjoint action, it is necessary that $\phi(v) = \phi(w)$ for any invariant ϕ. If we let $v = \sum_{i=1}^{n} a_i v_i$, then the invariant ϕ can be regarded as a function of $a_1, a_2, \ldots, a_n$. As Olver said, the detection of such an invariant is important since it places restrictions on how far we can expect to simplify v. However, it is a pity that people did not care more invariants except the Killing form. Now we will propose a valid method to find all the invariants of symmetry algebra and further make the best use of them to construct one-dimensional optimal system.

For an n-dimensional symmetry algebra $\mathcal{G}$, we first compute the commutation relations between all the vector fields v_i and v_j, which can be shown in a table, the entry in row i and column j representing $[v_i, v_j] = v_i v_j - v_j v_i$. Then taking any subgroup $g = e^w (w = \sum_{j=1}^{n} b_j v_j)$ to act on v, we have

$$\begin{aligned} Ad_{\exp(\epsilon w)}(v) &= e^{-\epsilon w} v e^{\epsilon w} \\ &= v - \epsilon[w, v] + \frac{1}{2!}\epsilon^2 [w, [w, v]] - \cdots \\ &= (a_1 v_1 + \cdots + a_n v_n) - \epsilon[b_1 v_1 + \cdots + b_n v_n, a_1 v_1 + \cdots + a_n v_n] \\ &\quad + O(\epsilon^2) \\ &= (a_1 v_1 + \cdots + a_n v_n) - \epsilon(\Theta_1 v_1 + \cdots + \Theta_n v_n) + O(\epsilon^2), \end{aligned} \tag{1}$$

where $\Theta_i \equiv \Theta_i(a_1, \ldots, a_n, b_1, \ldots, b_n)$ can be easily obtained from the commutator table.

Equivalently, omitting, v_i we can rewrite (1) as

$$(a_1, a_2, \ldots, a_n) \longrightarrow (a_1 - \epsilon\Theta_1, a_2 - \epsilon\Theta_2, \ldots, a_n - \epsilon\Theta_n) + O(\epsilon^2). \tag{2}$$

To determine the invariant ϕ, we expand the right hand side of (2) as

$$\begin{aligned} \phi(a_1 - \epsilon\Theta_1 + O(\epsilon^2), \ldots, a_n - \epsilon\Theta_n + O(\epsilon^2)) = \phi(a_1, a_2, \ldots, a_n) - \epsilon(\Theta_1 \frac{\partial \phi}{\partial a_1} \\ + \cdots + \Theta_n \frac{\partial \phi}{\partial a_n}) + O(\epsilon^2) \end{aligned} \tag{3}$$

and require

$$\Theta_1 \frac{\partial \phi}{\partial a_1} + \cdots + \Theta_n \frac{\partial \phi}{\partial a_n} = 0 \quad \text{for} \quad \text{any} \quad b_i. \tag{4}$$

In Eq. (4), extracting the coefficients of all b_i, $N(N \le n)$ linear differential equations of ϕ are obtained. By solving these equations, all the invariants can be found.

1. Invariants of the KdV equation

The KdV equation reads as

$$u_t + u_{xxx} + u u_x = 0, \tag{5}$$

which arises in the theory of long waves in shallow water and other physical systems in which both nonlinear and dispersive effects are relevant. Using the classical Lie group method, one can obtain the symmetry algebra of (5), i.e.,

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_x + \partial_u, \quad v_4 = x\partial_x + 3t\partial_t - 2u\partial_u. \tag{6}$$

For four-dimensional Lie algebra (6), the commutation relations are given in Table I.

Substituting $v = \sum_{i=1}^{4} a_i v_i$ and $w = \sum_{j=1}^{4} b_j v_j$ into (1), we have

$$Ad_{\exp(\epsilon w)}(v) = (a_1 v_1 + \cdots + a_4 v_4) - \epsilon(\Theta_1 v_1 + \cdots + \Theta_4 v_4) + O(\epsilon^2),$$

TABLE I. Commutator table of the KdV equation.

	v_1	v_2	v_3	v_4
v_1	0	0	0	v_1
v_2	0	0	v_1	$3v_2$
v_3	0	$-v_1$	0	$-2v_3$
v_4	$-v_1$	$-3v_2$	$2v_3$	0

with

$$\begin{aligned}\Theta_1 &= b_1a_4 + b_2a_3 - b_3a_2 - b_4a_1, & \Theta_2 &= 3b_2a_4 - 3b_4a_2,\\ \Theta_3 &= -2b_3a_4 + 2b_4a_3, & \Theta_4 &= 0.\end{aligned} \tag{7}$$

For any $b_i (i = 1 \cdots 4)$, it requires

$$\Theta_1 \frac{\partial \phi}{\partial a_1} + \cdots + \Theta_4 \frac{\partial \phi}{\partial a_4} = 0. \tag{8}$$

Extracting the coefficients of all b_i in Eq. (8), four differential equations about $\phi(a_1, a_2, a_3, a_4)$ are directly obtained as

$$\begin{cases} a_4 \dfrac{\partial \phi}{\partial a_1} = 0,\\ a_3 \dfrac{\partial \phi}{\partial a_1} + 3a_4 \dfrac{\partial \phi}{\partial a_2} = 0,\\ a_2 \dfrac{\partial \phi}{\partial a_1} + 2a_4 \dfrac{\partial \phi}{\partial a_3} = 0,\\ a_1 \dfrac{\partial \phi}{\partial a_1} + 3a_2 \dfrac{\partial \phi}{\partial a_2} - 2a_3 \dfrac{\partial \phi}{\partial a_3} = 0. \end{cases} \tag{9}$$

By solving Eq. (9), we can obtain $\phi(a_1, a_2, a_3, a_4) = F(a_4)$, where F is an arbitrary function of a_4. Here, the KdV equation has only one basic invariant a_4, and a_4 is just the Killing form given by Olver.[1]

2. Invariants of the heat equation

The equation for the conduction of heat in a one-dimensional road is written as

$$u_t = u_{xx}. \tag{10}$$

The Lie algebra of infinitesimal symmetries for this equation is spanned by six vector fields

$$\begin{aligned} &v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = u\partial_u, \quad v_4 = x\partial_x + 2t\partial_t,\\ &v_5 = 2t\partial_x - xu\partial_u, \quad v_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u, \end{aligned} \tag{11}$$

and the infinitesimal subalgebra

$$v_h = h(x,t)\partial_u,$$

where $h(x,t)$ is an arbitrary solution of the heat equation. Since the infinite-dimensional subalgebra $\langle v_h \rangle$ does not lead to group invariant solutions, it will not be considered in the classification problem.

Consider the six-dimensional symmetry algebra $\mathcal{G}$ generated by $\{v_1, v_2, \ldots, v_6\}$ in (11) and the commutator table is given in Table II. Substituting $v = \sum_{i=1}^{6} a_i v_i$ and $w = \sum_{j=1}^{6} b_j v_j$ into (1) leads to

$$Ad_{\exp(\epsilon w)}(v) = (a_1v_1 + \cdots + a_6v_6) - \epsilon(\Theta_1 v_1 + \cdots + \Theta_6 v_6) + O(\epsilon^2),$$

with

$$\begin{aligned} \Theta_1 &= -b_4a_1 - 2b_5a_2 + b_1a_4 + 2b_2a_5, & \Theta_2 &= -2b_4a_2 + 2b_2a_4,\\ \Theta_3 &= b_5a_1 + 2b_6a_2 - b_1a_5 - 2b_2a_6, & \Theta_4 &= -4b_6a_2 + 4b_2a_6,\\ \Theta_5 &= -2b_6a_1 - b_5a_4 + b_4a_5 + 2b_1a_6, & \Theta_6 &= -2b_6a_4 + 2b_4a_6. \end{aligned} \tag{12}$$

TABLE II. Commutator table of the heat equation.

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	v_1	$-v_3$	$2v_5$
v_2	0	0	0	$2v_2$	$2v_1$	$4v_4-2v_3$
v_3	0	0	0	0	0	0
v_4	$-v_1$	$-2v_2$	0	0	v_5	$2v_6$
v_5	v_3	$-2v_1$	0	$-v_5$	0	0
v_6	$-2v_5$	$2v_3-4v_4$	0	$-2v_6$	0	0

Substituting (12) into Eq. (4) and extracting the coefficients of all b_i, five linear differential equations about $\phi(a_1,a_2,\ldots,a_6)$ are directly obtained as

$$\begin{cases} a_4\dfrac{\partial\phi}{\partial a_1} - a_5\dfrac{\partial\phi}{\partial a_3} + 2a_6\dfrac{\partial\phi}{\partial a_5} = 0,\\ a_4\dfrac{\partial\phi}{\partial a_2} + a_5\dfrac{\partial\phi}{\partial a_1} + a_6(2\dfrac{\partial\phi}{\partial a_4} - \dfrac{\partial\phi}{\partial a_3}) = 0,\\ -a_1\dfrac{\partial\phi}{\partial a_1} - 2a_2\dfrac{\partial\phi}{\partial a_2} + a_5\dfrac{\partial\phi}{\partial a_5} + 2a_6\dfrac{\partial\phi}{\partial a_6} = 0,\\ a_1\dfrac{\partial\phi}{\partial a_3} - 2a_2\dfrac{\partial\phi}{\partial a_1} - a_4\dfrac{\partial\phi}{\partial a_5} = 0,\\ -a_1\dfrac{\partial\phi}{\partial a_5} + a_2(\dfrac{\partial\phi}{\partial a_3} - 2\dfrac{\partial\phi}{\partial a_4}) - a_4\dfrac{\partial\phi}{\partial a_6} = 0. \end{cases} \tag{13}$$

Solving Eq. (13), one can obtain two basic common invariants

$$\Delta_1 \equiv \phi_1(a_1,a_2,\ldots,a_6) = a_4^2 - 4a_2a_6 \tag{14}$$

and

$$\begin{aligned}\Delta_2 \equiv \phi_2(a_1,a_2,\ldots,a_6) = a_4^3 + 2a_3a_4^2 - 4a_4a_2a_6 + 2a_4a_1a_5 - 8a_2a_3a_6\\ -2a_2a_5^2 - 2a_1^2a_6.\end{aligned} \tag{15}$$

Here, Δ_1 is just the famous Killing form shown in Ref. 1 while Δ_2 is a completely new invariant of (11) which is never addressed before.

B. Calculation of the adjoint transformation matrix

The second task is the construction of the general adjoint transformation matrix A, which is the product of the matrices of the separate adjoint actions $A_1, A_2, \ldots, A_n$. For further details, one can refer to Ref. 17, in which three methods of constructing A are shown. Here, before constructing the matrix A, one is able to draw a table for convenience, where the (i,j)th entry gives $Ad_{\exp(\epsilon v_i)}(v_j)$.

First, applying the adjoint action of v_1 to $v = \sum_{i=1}^{n} a_i v_i$ and with the help of adjoint representation table, we have

$$\begin{aligned} &Ad_{\exp(\epsilon_1 v_1)}(a_1v_1 + a_2v_2 + \cdots + a_nv_n)\\ &= a_1Ad_{\exp(\epsilon_1 v_1)}v_1 + a_2Ad_{\exp(\epsilon_1 v_1)}v_2 + \cdots + a_nAd_{\exp(\epsilon_1 v_1)}v_n\\ &= R_1v_1 + R_2v_2 + \cdots + R_nv_n, \end{aligned} \tag{16}$$

with $R_i \equiv R_i(a_1,a_2,\ldots,a_n,\epsilon_1)$, $i = 1\cdots n$. To be intuitive, formula (16) can be rewritten into the following matrix form:

$$v \doteq (a_1,a_2,\ldots,a_n) \longrightarrow (R_1,R_2,\ldots,R_n) = (a_1,a_2,\ldots,a_n)A_1. \tag{17}$$

Similarly, we can construct the matrices $A_2, A_3, \ldots, A_n$ of the separate adjoint actions of $v_2, v_3, \ldots, v_n$, respectively. Then the general adjoint transformation matrix A is the product of $A_1, \ldots, A_n$ taken in any order

$$A \equiv A(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = A_1 A_2 \cdots A_n. \tag{18}$$

That is to say, applying the most general adjoint action $Ad_{\exp(\epsilon_n v_n)} \cdots Ad_{\exp(\epsilon_2 v_2)} Ad_{\exp(\epsilon_1 v_1)}$ to v, we have

$$v \doteq (a_1, a_2, \ldots, a_n) \longrightarrow (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (a_1, a_2, \ldots, a_n)A. \tag{19}$$

Remark 1: On the one hand, the orders of the product shown in (18) are not important because only the existence of the element of the group is needed in the algorithm. On the other hand, the equivalence of $\sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} \tilde{a}_i v_i$ can be shown by the following n algebraic equations with respect to $\epsilon_1, \ldots, \epsilon_n$:

$$(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (a_1, a_2, \ldots, a_n)A \tag{20}$$

$$(\text{or} \quad (a_1, a_2, \ldots, a_n) = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)A \quad). \tag{21}$$

If Eq. (20) (or Eq. (21)) has the solution, it means that $\sum_{i=1}^{n} a_i v_i$ must be equivalent to $\sum_{i=1}^{n} \tilde{a}_i v_i$ under the adjoint action.

1. *Adjoint matrix of the KdV equation*

The adjoint representation table of (6) is given in Table III. Applying the adjoint action of v_1 to

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4, \tag{22}$$

we have

$$\begin{aligned} Ad_{\exp(\epsilon_1 v_1)} v &= (a_1 - a_4 \epsilon_1) v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= (a_1, a_2, a_3, a_4) \cdot A_1 \cdot (v_1, v_2, v_3, v_4)^{\mathrm{T}}, \end{aligned} \tag{23}$$

with

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\epsilon_1 & 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

Similarly, one can obtain A_2, A_3, and A_4,

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\epsilon_2 & 0 & 1 & 0 \\ 0 & -3\epsilon_2 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \epsilon_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2\epsilon_3 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ 0 & e^{3\epsilon_4} & 0 & 0 \\ 0 & 0 & e^{-2\epsilon_4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

TABLE III. Adjoint representation table of the KdV equation.

Ad	v_1	v_2	v_3	v_4
v_1	v_1	v_2	v_3	$v_4 - \epsilon v_1$
v_2	v_1	v_2	$v_3 - \epsilon v_1$	$v_4 - 3\epsilon v_2$
v_3	v_1	$v_2 + \epsilon v_1$	v_3	$v_4 + 2\epsilon v_3$
v_4	$e^{\epsilon} v_1$	$e^{3\epsilon} v_2$	$e^{-2\epsilon} v_3$	v_4

Then the general adjoint transformation matrix A is constructed by

$$A = A_1A_2A_3A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 \\ \epsilon_3 e^{\epsilon_4} & e^{3\epsilon_4} & 0 & 0 \\ -\epsilon_2 e^{\epsilon_4} & 0 & e^{-2\epsilon_4} & 0 \\ (-\epsilon_1 - 3\epsilon_2\epsilon_3)e^{\epsilon_4} & -3\epsilon_2 e^{3\epsilon_4} & 2\epsilon_3 e^{-2\epsilon_4} & 1 \end{pmatrix}. \tag{25}$$

2. Adjoint matrix of the heat equation

For the heat equation, the adjoint representation table is given in Table IV. Applying the adjoint action of v_1 to

$$v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6, \tag{26}$$

there is

$$\begin{aligned} &Ad_{\exp(\epsilon_1 v_1)}(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6) \\ &= (a_1 - a_4\epsilon_1)v_1 + a_2v_2 + (a_3 + a_5\epsilon_1 - a_6\epsilon_1^2)v_3 + a_4v_4 \\ &\quad +(a_5 - 2\epsilon_1 a_6)v_5 + a_6v_6 \\ &= (a_1, a_2, \ldots, a_6) \cdot A_1 \cdot (v_1, v_2, \ldots, v_6)^{\mathrm{T}}. \end{aligned} \tag{27}$$

It is easy to give

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 & 1 & 0 \\ 0 & 0 & -\epsilon_1^2 & 0 & -2\epsilon_1 & 1 \end{pmatrix}. \tag{28}$$

Similarly, $A_2, \ldots, A_6$ are found to be

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\epsilon_2 & 0 & 1 & 0 & 0 \\ -2\epsilon_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4\epsilon_2^2 & 2\epsilon_2 & -4\epsilon_2 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e^{\epsilon_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\epsilon_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\epsilon_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\epsilon_4} \end{pmatrix}, \tag{29}$$

TABLE IV. The adjoint representation table of the heat equation.

Ad	v_1	v_2	v_3	v_4	v_5	v_6
v_1	v_1	v_2	v_3	$v_4-\epsilon v_1$	$v_5+\epsilon v_3$	$v_6-2\epsilon v_5-\epsilon^2 v_3$
v_2	v_1	v_2	v_3	$v_4-2\epsilon v_2$	$v_5-2\epsilon v_1$	$v_6-4\epsilon v_4+2\epsilon v_3+4\epsilon^2 v_2$
v_3	v_1	v_2	v_3	v_4	v_5	v_6
v_4	$e^{\epsilon}v_1$	$e^{2\epsilon}v_2$	v_3	v_4	$e^{-\epsilon}v_5$	$e^{-2\epsilon}v_6$
v_5	$v_1-\epsilon v_3$	$v_2+2\epsilon v_1-\epsilon^2 v_3$	v_3	$v_4+\epsilon v_5$	v_5	v_6
v_6	$v_1+2\epsilon v_5$	$v_2-2\epsilon v_3+4\epsilon v_4+4\epsilon^2 v_6$	v_3	$v_4+2\epsilon v_6$	v_5	v_6

$$A_5 = \begin{pmatrix} 1 & 0 & -\epsilon_5 & 0 & 0 & 0 \\ 2\epsilon_5 & 1 & -\epsilon_5^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \epsilon_5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 2\epsilon_6 & 0 \\ 0 & 1 & -2\epsilon_6 & 4\epsilon_6 & 0 & 4\epsilon_6^2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2\epsilon_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

with $A_3 = E$ being the identity matrix.

Hence, the general adjoint transformation matrix A can be taken as

$$\begin{aligned} A &= A_4 A_5 A_3 A_1 A_2 A_6 \\ &= \begin{pmatrix} e^{\epsilon_4} & 0 & -\epsilon_5 e^{\epsilon_4} & 0 & 2\epsilon_6 e^{\epsilon_4} & 0 \\ 2\epsilon_5 e^{\epsilon_4} & e^{2\epsilon_4} & -(\epsilon_5^2 + 2\epsilon_6)e^{2\epsilon_4} & 4\epsilon_6 e^{2\epsilon_4} & 4\epsilon_5\epsilon_6 e^{2\epsilon_4} & 4\epsilon_6^2 e^{2\epsilon_4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\epsilon_1 - 2\epsilon_2\epsilon_5 & -2\epsilon_2 & \epsilon_1\epsilon_5 + 4\epsilon_2\epsilon_6 & 1 - 8\epsilon_2\epsilon_6 & -2\epsilon_1\epsilon_6 - \epsilon_5\Xi & -2\epsilon_6\Xi \\ -2\epsilon_2 e^{-\epsilon_4} & 0 & \epsilon_1 e^{-\epsilon_4} & 0 & -\Xi e^{-\epsilon_4} & 0 \\ 4\epsilon_1\epsilon_2 e^{-2\epsilon_4} & 4\epsilon_2^2 e^{-2\epsilon_4} & -(\epsilon_1^2 + 2\epsilon_2\Xi)e^{-2\epsilon_4} & 4\epsilon_2\Xi e^{-2\epsilon_4} & 2\epsilon_1\Xi e^{-2\epsilon_4} & \Xi^2 e^{-2\epsilon_4} \end{pmatrix}, \end{aligned} \tag{31}$$

with $\Xi = 4\epsilon_2\epsilon_6 - 1$.

C. Classification of the finite-dimensional Lie algebra $\mathcal{G}$

Based on the invariants and adjoint transformation matrix A, we give out the algorithm for constructing one-dimensional optimal system of the finite-dimensional Lie algebra.

(1) The first step: scale the invariants.

If two vectors v and w are adjoint equivalent, it is necessary that $\phi(v) = \phi(w)$ for any invariant ϕ. However, if $v = cw$, where v and w are also equivalent, their corresponding invariants satisfy $\phi(v) = c'\phi(w)$ and it is usually $\phi(v) \neq \phi(w)$. To avoid the latter case, we first make a scale to the invariant by adjusting the coefficients of generators. Without loss of generality, one just needs to consider the values of the invariants to be 1, −1, and 0. To illustrate this point more clearly, we give three remarks.

Remark 2: If the degree of the invariant is odd, we obtain $\phi(v) = c^{2k+1}\phi(w)$ with $v = cw$, then the right c can be selected to transform the positive (negative) invariant into the negative (positive) one. Now, we just need to consider two cases: $\phi = 0$ and $\phi \neq 0$ (for simplicity, scaling it to 1 or −1).

Remark 3: If the degree of the invariant is even (excluding zero), there is $\phi(v) = c^{2k}\phi(w)$ with $v = cw$, then we cannot choose the right c to transform the positive (negative) invariant into the negative (positive) one. Now one needs to consider three cases: $\phi = 0$, $\phi > 0$, and $\phi < 0$. Without loss of generality, we let $\phi = 0$, $\phi = 1$, and $\phi = -1$, respectively.

Remark 4: Once one of the invariants is scaled (not zero), the other invariants (if any) cannot be adjusted.

Take the KdV equation and heat equations, for example. Since the invariant for the KdV equation is a_4, the degree of which is one, we just need to talk about $a_4 = 1$ and $a_4 = 0$. For the heat equation, it has two invariants Δ_1 and Δ_2, the degrees of which are two and three. Now one can consider four cases,

$$\{\Delta_1 = 1, \Delta_2 = c\}; \quad \{\Delta_1 = -1, \Delta_2 = c\};$$
$$\{\Delta_1 = 0, \Delta_2 = 1\}; \quad \{\Delta_1 = 0, \Delta_2 = 0\}.$$

(2) The second step: select the representative elements according to "Remark 1."

In terms of different values of the invariants given in step 1, select the corresponding representative elements in the simplest form named $\tilde{v} = \sum_{i=1}^{n} \tilde{a}_i v_i$. Then solve the adjoint transformation equation (20) (or (21)). If Eq. (20) (or Eq. (21)) has the solution with respect to $\epsilon_1, \ldots \epsilon_n$, it signifies that the selected representative element is right. If the chosen representative element makes Eq. (20) (or Eq. (21)) be incompatible, we need to adopt a new proper one. Repeat the process until all the cases in step 1 are finished. To leave it clear, this process will be illustrated with the KdV and heat equations.

1. One-dimensional optimal system of the KdV equation

Adjoint transformation equation (20) for the KdV equation is

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = (a_1, a_2, a_3, a_4)A, \tag{32}$$

where the adjoint matrix A is displayed in (25).

Case 1: $a_4 = 1$.

Select a representative element $\tilde{v} = v_4$. Substituting $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0, \tilde{a}_4 = 1$, and $a_4 = 1$ into Eq. (32), we obtain the solution

$$\epsilon_1 = a_1 - \frac{1}{3}a_2a_3, \quad \epsilon_2 = \frac{1}{3}a_2, \quad \epsilon_3 = -\frac{1}{2}a_3.$$

That is to say, all the $v_4 + a_1v_1 + a_2v_2 + a_3v_3$ are equivalent to v_4.

Case 2: $a_4 = 0$.

Substituting $a_4 = 0$ into Eq. (9) yields to a new invariant $\phi(a_1, a_2, a_3) = a_2^2a_3^3$. In terms of "*Remark 2*," there are also two cases: $a_2^2a_3^3 = 1$ and $a_2^2a_3^3 = 0$.

Case 2.1: $a_2^2a_3^3 = 1$.

Adopt two representative elements $\tilde{v} = v_2 + v_3$ and $\tilde{v} = -v_2 + v_3$.

For $a_2 > 0$ and $\tilde{v} = v_2 + v_3$, Eq. (32) with $a_2^2a_3^3 = 1$ has the solutions

$$\epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_1}{a_2}, \quad \epsilon_4 = -\frac{1}{3}\ln(a_2).$$

For $a_2 < 0$ and $\tilde{v} = -v_2 + v_3$, Eq. (32) with $a_2^2a_3^3 = 1$ has the solutions

$$\epsilon_2 = 0, \quad \epsilon_3 = -\frac{a_1}{a_2}, \quad \epsilon_4 = -\frac{1}{3}\ln(-a_2).$$

Case 2.2: $a_2^2a_3^3 = 0$.

(1) $a_3 \neq 0$ and $a_2 = 0$.

Adopt two representative elements v_3 and $-v_3$. Then $\{a_1v_1 + a_3v_3\}$ with $a_3 > 0$ is equivalent to v_3, where the solution for Eq. (32) is $\{\epsilon_2 = \frac{a_1}{a_3}, \epsilon_4 = \frac{1}{2}\ln(a_3)\}$, while $\{a_1v_1 + a_3v_3\}$ with $a_3 < 0$ is equivalent to $-v_3$, where the solution for Eq. (32) is $\{\epsilon_2 = \frac{a_1}{a_3}, \epsilon_4 = \frac{1}{2}\ln(-a_3)\}$. Essentially, v_3 and $-v_3$ are equivalent.

(2) $a_3 = 0 : a_2 \neq 0$ and $a_2 = 0$.

When $a_2 \neq 0$, similar to case (1), $a_2v_2 + a_1v_1$ is equivalent to v_2 and $-v_2$.

When $a_2 = 0$, a_1v_1 is equivalent to v_1.

Recapitulating, a one-dimensional optimal system of symmetry algebra (6) contains

$$v_4; \quad v_3 + v_2; \quad v_3 - v_2; \quad v_3; \quad v_2; \quad v_1. \tag{33}$$

The optimal system given by (33) is just the same to that found by Olver.[1]

2. One-dimensional optimal system of the heat equation

For the heat equation, the adjoint transformation equations read

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6) = (a_1, a_2, a_3, a_4, a_5, a_6)A, \tag{34}$$

where the adjoint matrix A is shown by (31).

Case 1: $\Delta_1 = a_4^2 - 4a_2a_6 = 1, \Delta_2 = c$.

Here, c is an arbitrary real constant. Under $\Delta_1 = 1$ and $\Delta_2 = c$, choose a representative element, for example, select $\tilde{v} = v_4 + \frac{c-1}{2}v_3$ (i.e., $\tilde{a}_4 = 1, \tilde{a}_3 = \frac{c-1}{2}$).

From $\Delta_1 = a_4^2 - 4a_2a_6 = 1$, one knows that a_2, a_4, a_6 cannot be all zeros simultaneously. Without loss of generality, we just consider $a_6 \neq 0$. For $a_6 = 0$ ($a_2 \neq 0$ or $a_4 \neq 0$), one can transform it into the case of $a_6 \neq 0$ by selecting the appropriate ϵ_i ($i = 1 \cdots 6$) which are shown in Eq. (34).

For $a_6 \neq 0$, the conditions $\Delta_1 = 1$ and $\Delta_2 = c$ can be expressed as

$$a_2 = \frac{a_4^2 - 1}{4a_6}, \quad a_3 = a_1^2 a_6 - a_1a_4a_5 - \frac{1}{2}a_4 + \frac{c}{2} - \frac{a_4^2 - 1}{4a_6}a_5^2, \tag{35}$$

where a_1, a_4, a_5, a_6 are arbitrary real constants. According to Eq. (34), six algebraic equations about ϵ_i are proposed. After substituting $\tilde{a}_4 = 1$, $\tilde{a}_3 = \frac{c-1}{2}$, $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_5 = \tilde{a}_6 = 0$ with (35) into these equations, one can find the solutions

$$\epsilon_1 = \frac{a_5 + 2a_1a_4a_6 - a_4^2a_5}{2a_6}e^{\epsilon_4}, \epsilon_2 = \frac{a_4 - 1}{4a_6}e^{2\epsilon_4}, \epsilon_5 = (2a_1a_6 - a_4a_5)e^{-\epsilon_4}, \epsilon_6 = -\frac{1}{2}a_6e^{-2\epsilon_4}.$$

Case 2: $\Delta_1 = a_4^2 - 4a_2a_6 \equiv -1, \Delta_2 = c$.

From $\Delta_1 = -1$, it illustrates $a_6 \neq 0$. Now the relation among a_i reads as

$$a_2 = \frac{a_4^2 + 1}{4a_6}, \quad a_3 = -a_1^2a_6 + a_1a_4a_5 - \frac{1}{2}a_4 - \frac{c}{2} - \frac{a_4^2 + 1}{4a_6}a_5^2. \tag{36}$$

When $a_6 > 0$ and $a_6 < 0$, take the representative element $\tilde{v} = \frac{1}{2}(v_2 + v_6 - cv_3)$ and $\tilde{v} = \frac{1}{2}(-v_2 - v_6 - cv_3)$, respectively. Then Eq. (34) with (36) is separately proved solvable with the solutions

$$\epsilon_1 = -\frac{\sqrt{2}}{2}\frac{2a_1a_4a_6 - a_5 - a_4^2a_5}{\sqrt{a_6}}, \epsilon_2 = \frac{1}{2}a_4, \epsilon_4 = \frac{1}{2}\ln(2a_6), \epsilon_5 = -\frac{\sqrt{2}}{2}\frac{2a_1a_6 - a_4a_5}{\sqrt{a_6}}, \epsilon_6 = 0$$

and

$$\epsilon_1 = -\frac{1}{2}\frac{\sqrt{-2a_6}(2a_1a_4a_6 - a_5 - a_4^2a_5)}{a_6}, \epsilon_2 = -\frac{1}{2}a_4, \epsilon_4 = \frac{1}{2}\ln(-2a_6),$$
$$\epsilon_5 = \frac{1}{2}\frac{\sqrt{-2a_6}(2a_1a_6 - a_4a_5)}{a_6}, \epsilon_6 = 0.$$

In this case, general one-dimensional Lie algebra (26) is equivalent to $v_2 + v_6 + \beta v_3$ with β being arbitrary.

Case 3: $\Delta_1 = 0$.

Notice that Δ_2 itself is an odd polynomial with respect to a_i, so one just needs consider $\Delta_2 = 1$ and $\Delta_2 = 0$ according to "*Remark 2*."

Case 3.1: $\Delta_1 = 0, \Delta_2 = 1$.

Due to $\Delta_2 = 1$, one knows that not all a_2, a_4, and a_6 are zeros. Without loss of generality, we let $a_6 \neq 0$. Select a representative element $\tilde{v} = -v_2 - \frac{\sqrt{2}}{2}v_5$.

When $a_6 \neq 0$, there must be $a_6 < 0$ for the identity $2a_6 = -(2a_1a_6 - a_4a_5)^2$ solved by $\Delta_1 = 0$ and $\Delta_2 = 1$. Under the restriction of invariants, we have

$$a_2 = \frac{a_4^2}{4a_6}, \quad a_1 = \frac{a_4a_5 \pm \sqrt{-2a_6}}{2a_6}. \tag{37}$$

Then after choosing

$$\epsilon_1 = \frac{\sqrt{2}}{4} \cdot \frac{e^{\epsilon_4}}{a_6}(\sqrt{2}a_5 + e^{\epsilon_4} + \sqrt{2}a_4\epsilon_5e^{\epsilon_4}), \quad \epsilon_2 = \frac{e^{\epsilon_4}}{4a_6}(\mp 2\sqrt{-a_6} + a_4e^{\epsilon_4}),$$
$$\epsilon_5 = \pm\frac{\sqrt{2}}{8} \cdot \frac{e^{-\epsilon_4}}{\sqrt{-a_6}}(4a_4a_6 + 2a_5^2 + 8a_3a_6 + e^{2\epsilon_4}), \quad \epsilon_6 = \pm\frac{1}{2}\sqrt{-a_6}e^{-\epsilon_4},$$

one can transform (26) with (37) into $\tilde{v} = -v_2 - \frac{\sqrt{2}}{2}v_5$.

Case 3.2: $\Delta_1 = 0, \Delta_2 = 0$.

Case 3.2.1: Not all a_2, a_4, and a_6 are zeros.

For the same reason, we just need consider $a_6 \neq 0$. Substituting $\Delta_1 = 0$ and $\Delta_2 = 0$ into Eq. (13), we obtain a new invariant

$$\Delta_3 = 4a_3 + 2a_4 + \frac{a_5^2}{a_6}. \tag{38}$$

Then there are two cases, depending on the sign of the invariant Δ_3,

(1) $\Delta_3 = 1$. In terms of $\{\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 1\}$, we have

$$a_1 = \frac{a_4 a_5}{2a_6}, \quad a_2 = \frac{a_4^2}{4a_6}, \quad a_3 = \frac{a_6 - a_5^2 - 2a_4 a_6}{4a_6}. \tag{39}$$

When $a_6 > 0$, choose the representative element $\tilde{v} = \frac{1}{4}v_3 + v_6$, then Eq. (34) with (39) has the solutions

$$\epsilon_1 = \frac{a_5\sqrt{a_6} + a_4 a_6 \epsilon_5}{2a_6}, \quad \epsilon_2 = \frac{a_4}{4}, \quad \epsilon_4 = \frac{1}{2}\ln a_6. \tag{40}$$

When $a_6 < 0$, the representative element is taken as $\tilde{v} = \frac{1}{4}v_3 - v_6$. It is easy to see that Eq. (34) holds with

$$\epsilon_1 = \frac{a_5\sqrt{-a_6} - a_4 a_6 \epsilon_5}{2a_6}, \quad \epsilon_2 = -\frac{a_4}{4}, \quad \epsilon_4 = \frac{1}{2}\ln(-a_6). \tag{41}$$

Further, it is noted that $\frac{1}{4}v_3 + v_6$ and $\frac{1}{4}v_3 - v_6$ are inequivalent.

(2) $\Delta_3 = 0$. Now we have

$$a_1 = \frac{a_4 a_5}{2a_6}, \quad a_2 = \frac{a_4^2}{4a_6}. \tag{42}$$

It can be easily proved that via same adjoint transformations (40) and (41), Lie algebra (26) is converted into v_6 and $-v_6$, respectively.

Case 3.2.2: $a_2 = a_4 = a_6 = 0$. Substituting $a_2 = a_4 = a_6 = 0$ into (34), we find that it can also be divided into two cases:

(1) Not all a_1 and a_5 are zeros. Here, we suppose $a_5 \neq 0$.

In this case, choose a representative element v_1. One can see that Eq. (34) with $a_2 = a_4 = a_6 = 0$ has a solutions

$$\epsilon_1 = \frac{e^{\epsilon_4}(e^{\epsilon_4}\epsilon_5 a_1 - a_3)}{a_5}, \quad \epsilon_2 = \frac{1}{2}\frac{e^{\epsilon_4}(e^{\epsilon_4}a_1 - 1)}{a_5}, \quad \epsilon_6 = -\frac{1}{2}a_5 e^{-\epsilon_4}.$$

(2) $a_1 = a_5 = 0$. Now we have $a_1 = a_2 = a_4 = a_5 = a_6 = 0$ and general Lie algebra (26) becomes v_3.

In summary, an optimal system of one-dimensional subalgebras of the heat equation is found to be those spanned by

$$\begin{aligned} &\omega_1(\alpha) = v_4 + \alpha v_3 \quad (\alpha \in \mathbb{R}), \quad \omega_2(\beta) = v_2 + v_6 + \beta v_3 \quad (\beta \in \mathbb{R}), \quad \omega_3 = v_2 + \frac{\sqrt{2}}{2}v_5, \\ &\omega_4 = \frac{1}{4}v_3 + v_6, \quad \omega_5 = \frac{1}{4}v_3 - v_6, \quad \omega_6 = v_6, \quad \omega_7 = v_1, \quad \omega_8 = v_3. \end{aligned} \tag{43}$$

Resulting optimal system (43) of the heat equation is really optimal and completely equivalent to that given in Ref. 13, which is a further reduction to the result of Olver.[1] In Ref. 1, Olver uses the Killing form (that is, Δ_1 here) to classify the one-dimensional subalgebras and construct the following optimal system:

$$\begin{aligned} &(a) v_4 + a v_3; \quad (b) v_2 + v_6 + a v_3; \quad (c1) v_2 - v_5; \\ &(c2) v_2 + v_5; \quad (d) v_2 + a v_3; \quad (e) v_1; \quad (f) v_3. \end{aligned} \tag{44}$$

Obviously, the differences between (44) and (43) lie in cases (c1), (c2), and (d). In fact, cases (c1) and (c2), and ω_3 are all equivalent and this equivalence can be reflected by our adjoint transformation equation (34),

$$v_2 - v_5 \sim v_2 + 2v_4 - v_5 + v_6 \quad (\epsilon_1 = -1, \epsilon_6 = \frac{1}{2}, \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0) \tag{45}$$

$$\sim v_2 + v_5 \quad (\epsilon_1 = -2, \epsilon_2 = \epsilon_5 = 1, \epsilon_3 = \epsilon_4 = 0, \epsilon_6 = \frac{1}{2}) \tag{46}$$

$$\sim \frac{1}{\sqrt[3]{2}}(v_2 + v_5) \tag{47}$$

$$\sim v_2 + \frac{\sqrt{2}}{2} v_5 \quad (\epsilon_4 = \frac{1}{6}\ln 2, \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_5 = \epsilon_6 = 0). \tag{48}$$

For case (d), one can use $Ad(\exp(\epsilon v_3))$ for suitable ϵ to turn $v_2 + a v_3$ into either $\frac{1}{4}v_3 + v_2$, $\frac{1}{4}v_3 - v_2$, or v_2, which is shown, respectively, equivalent to ω_4, ω_5, and ω_6 of (43) for the solution of Eq. (34),

$$\epsilon_2 = \epsilon_6 = \frac{1}{2}, \epsilon_1 = \epsilon_3 = \epsilon_4 = \epsilon_5 = 0.$$

Remark 5: The key point of our new method is to solve some algebraic equations which are embedded in (20) (or (21)) and it can easily be carried out by Maple.

III. ONE-DIMENSIONAL OPTIMAL SYSTEM AND GROUP INVARIANT SOLUTIONS OF THE NOVIKOV EQUATION

In this section, we will apply the new method to the Novikov equation

$$u_t - u_{txx} + 4u^2 u_x - 3u u_x u_{xx} - u^2 u_{xxx} = 0, \tag{49}$$

which was discovered by Novikov in a recent communication[18] and can be considered as a type of generalization of the known Camassa-Holm equation. Equation (49) has attracted attention of different researchers; see Refs. 19–26 and references therein. In Ref. 26, the authors give out a five-dimensional Lie algebra of Eq. (49), which is spanned by the following basis:

$$\begin{aligned} &v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = e^{2x}\partial_x + e^{2x}u\partial_u, \\ &v_4 = e^{-2x}\partial_x - e^{-2x}u\partial_u, \qquad v_5 = -2t\partial_t + u\partial_u. \end{aligned} \tag{50}$$

Now we use the new algorithm to construct the one-dimensional optimal system of five-dimensional Lie algebra (50), which has not been found so far, and use the optimal system to find corresponding group invariant solutions.

A. Construction of one-dimensional optimal system

To begin with, one can easily propose the commutator and adjoint representation relations of (50), which are shown by Tables V and VI.

TABLE V. Commutator table of the Novikov equation.

	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	$-2v_1$
v_2	0	0	$2v_3$	$-2v_4$	0
v_3	0	$-2v_3$	0	$-4v_2$	0
v_4	0	$2v_4$	$4v_2$	0	0
v_5	$2v_1$	0	0	0	0

TABLE VI. Adjoint representation table of the Novikov equation.

Ad	v_1	v_2	v_3	v_4	v_5
v_1	v_1	v_2	v_3	v_4	$v_5+2\epsilon v_1$
v_2	v_1	v_2	$e^{-2\epsilon}v_3$	$e^{2\epsilon}v_4$	v_5
v_3	v_1	$v_2+2\epsilon v_3$	v_3	$v_4+4\epsilon v_2+4\epsilon^2 v_3$	v_5
v_4	v_1	$v_2-2\epsilon v_4$	$v_3-4\epsilon v_2+4\epsilon^2 v_4$	v_4	v_5
v_5	$e^{-2\epsilon}v_1$	v_2	v_3	v_4	v_5

Applying $w = \sum_{j=1}^{5} b_j v_j$ to $v = \sum_{i=1}^{5} a_i v_i$, we have

$$Ad_{\exp(\epsilon w)}(v) = (a_1 v_1 + \cdots + a_5 v_5) - \epsilon(\Theta_1 v_1 + \cdots + \Theta_5 v_5) + O(\epsilon^2),$$

with

$$\begin{aligned}&\Theta_1 = 2a_1b_5 - 2b_1a_5, \quad \Theta_2 = 4a_3b_4 - 4b_3a_4,\\ &\Theta_3 = -2a_2b_3 + 2b_2a_3, \quad \Theta_4 = -2a_4b_2 + 2b_4a_2, \quad \Theta_5 = 0.\end{aligned} \tag{51}$$

By dealing with

$$\phi(a_1 - \epsilon\Theta_1 + O(\epsilon^2), \ldots, a_5 - \epsilon\Theta_5 + O(\epsilon^2)) = \phi(a_1, \ldots, a_5) - \epsilon(\Theta_1 \frac{\partial\phi}{\partial a_1} + \cdots + \Theta_5 \frac{\partial\phi}{\partial a_5}) + O(\epsilon^2), \tag{52}$$

two basic invariants of (50) are obtained as

$$\Delta_1 = a_5, \qquad \Delta_2 = a_2^2 - 4a_3a_4. \tag{53}$$

Then, the matrices of the separate adjoint actions $A_1, \ldots A_5$ are found to be

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 2\epsilon_1 & 0 & 0 & 0 & 1\end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & e^{-2\epsilon_2} & 0 & 0\\ 0 & 0 & 0 & e^{2\epsilon_2} & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 2\epsilon_3 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 4\epsilon_3 & 4\epsilon_3^2 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -2\epsilon_4 & 0\\ 0 & -4\epsilon_4 & 1 & 4\epsilon_4^2 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}, \quad A_5 = \begin{pmatrix} e^{-2\epsilon_5} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

The matrix A is the product of all these taken in any order

$$\begin{aligned}A &= A_1A_2A_3A_4A_5\\ &= \begin{pmatrix} e^{-2\epsilon_5} & 0 & 0 & 0 & 0\\ 0 & 1-8\epsilon_3\epsilon_4 & 2\epsilon_3 & 2\epsilon_4(4\epsilon_3\epsilon_4-1) & 0\\ 0 & -4\epsilon_4 e^{-2\epsilon_2} & e^{-2\epsilon_2} & 4\epsilon_4^2 e^{-2\epsilon_2} & 0\\ 0 & 4\epsilon_3 e^{2\epsilon_2}(1-4\epsilon_3\epsilon_4) & 4\epsilon_3^2 e^{2\epsilon_2} & e^{2\epsilon_2}(1-4\epsilon_3\epsilon_4)^2 & 0\\ 2\epsilon_1 e^{-2\epsilon_5} & 0 & 0 & 0 & 1\end{pmatrix},\end{aligned} \tag{54}$$

which determines the adjoint transformation equations in the form of

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5) = (a_1, a_2, a_3, a_4, a_5)A. \tag{55}$$

According to the degree of invariant (53), we concentrate on four cases,

$$\begin{aligned}&\{\Delta_1 = 1, \Delta_2 = c\}; \quad \{\Delta_1 = 0, \Delta_2 = 1\};\\ &\{\Delta_1 = 0, \Delta_2 = -1\}; \quad \{\Delta_1 = 0, \Delta_2 = 0\}.\end{aligned}$$

Case 1: $\Delta_1 = 1, \Delta_2 = c$.

(1) Not all a_2, a_3, and a_4 are zeros. Without loss of generality, take $a_3 \neq 0$.

When $a_3 > 0$, select a representative element $\tilde{v} = v_3 - \frac{c}{4}v_4 + v_5$ and Eq. (55) has the solutions

$$\epsilon_1 = -\frac{1}{2}a_1, \quad \epsilon_2 = \frac{1}{2}\ln(a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = \frac{1}{4}a_2.$$

When $a_3 < 0$, select a representative element $\tilde{v} = -v_3 + \frac{c}{4}v_4 + v_5$ and Eq. (55) has the solutions

$$\epsilon_1 = -\frac{1}{2}a_1, \quad \epsilon_2 = \frac{1}{2}\ln(-a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = -\frac{1}{4}a_2.$$

Further, when $c > 0$, we can verify that $v_3 - \frac{c}{4}v_4 + v_5$ is equivalent to $-v_3 + \frac{c}{4}v_4 + v_5$ for the solutions

$$\epsilon_1 = \epsilon_2 = 0, \quad \epsilon_3 = \sqrt{\frac{c}{2}}, \quad \epsilon_4 = \frac{1}{4}c\sqrt{\frac{c}{2}};$$

But unluckily, when $c \leq 0$, the vector $v_3 - \frac{c}{4}v_4 + v_5$ is inequivalent to $-v_3 + \frac{c}{4}v_4 + v_5$.

In this case, the general one-dimensional Lie algebra is equivalent to $v_3 - \frac{c}{4}v_4 + v_5$ (c being arbitrary constant) and $-v_3 + \frac{c}{4}v_4 + v_5\,(c \leq 0)$.

(2) $a_2 = a_3 = a_4 = 0$.

Now the general algebra becomes $a_1v_1 + v_5$ and it can be easily converted into v_5 for the solution $\{\epsilon_1 = -\frac{1}{2}a_1\}$ of Eq. (55).

Case 2: $\Delta_1 = 0, \Delta_2 = 1$.

Because of $\Delta_2 = 1$, a_2, a_3, and a_4 cannot be all zeros. Here, we also let $a_3 \neq 0$.

(1) $a_1 = 0$.

When $a_3 > 0$, choose $\tilde{v} = v_3 - \frac{1}{4}v_4$ and Eq. (55) easily holds for

$$\epsilon_2 = \frac{1}{2}\ln a_3, \quad \epsilon_3 = 0, \quad \epsilon_4 = \frac{1}{4}a_2.$$

When $a_3 < 0$, choose $\tilde{v} = -v_3 + \frac{1}{4}v_4$ and Eq. (55) easily holds for

$$\epsilon_2 = \frac{1}{2}\ln(-a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = -\frac{1}{4}a_2.$$

It is very clear that $v_3 - \frac{1}{4}v_4$ and $-v_3 + \frac{1}{4}v_4$ are equivalent to each other.

(2) $a_1 \neq 0$.

For $a_1 > 0$, the general algebra with $a_3 > 0$ can be converted into $\tilde{v} = v_1 + v_3 - \frac{1}{4}v_4$ with

$$\epsilon_2 = \frac{1}{2}\ln a_3, \quad \epsilon_3 = 0, \quad \epsilon_4 = \frac{1}{4}a_2, \quad \epsilon_5 = \frac{1}{2}\ln a_1.$$

Meanwhile, the case of $a_3 < 0$ is equivalent to $\tilde{v} = v_1 - v_3 + \frac{1}{4}v_4$ for

$$\epsilon_2 = \frac{1}{2}\ln(-a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = -\frac{1}{4}a_2, \quad \epsilon_5 = \frac{1}{2}\ln a_1.$$

Luckily, $v_1 + v_3 - \frac{1}{4}v_4$ can be transformed into $v_1 - v_3 + \frac{1}{4}v_4$ for the solutions

$$\epsilon_2 = \epsilon_5 = 0, \quad \epsilon_3 = \sqrt{2}, \quad \epsilon_4 = \frac{1}{4}\sqrt{2}.$$

In the same way, the case of $a_1 < 0$ is equivalent to $-v_1 + v_3 - \frac{1}{4}v_4$ (or $-v_1 - v_3 + \frac{1}{4}v_4$), which is just the opposite situation of $a_1 > 0$.

Hence, all the general algebra with $a_1 \neq 0$ can be equivalent to $v_1 + v_3 - \frac{1}{4}v_4$.

Case 3: $\Delta_1 = 0, \Delta_2 = -1$.

In this case, the selection of the representative elements is exactly similar to "case 2," so the process is not repeated here and we just provide the final results,

$$v_3 + \frac{1}{4}v_4; \quad v_1 + v_3 + \frac{1}{4}v_4; \quad v_1 - v_3 - \frac{1}{4}v_4.$$

Case 4: $\Delta_1 = 0, \Delta_2 = 0$.

(1) Not all a_2, a_3, and a_4 are zeros. Without loss of generality, let $a_3 \neq 0$. There are three cases to be considered: $a_1 = 0$, $a_1 > 0$, and $a_1 < 0$.

When $a_1 = 0$, adopt $\tilde{v} = v_3$ (and $\tilde{v} = -v_3$). Then Eq. (55) is solvable with

$$\epsilon_2 = \frac{1}{2}\ln a_3, \quad \epsilon_3 = 0, \quad \epsilon_4 = \frac{1}{4}a_2$$
$$(\text{and} \quad \epsilon_2 = \frac{1}{2}\ln(-a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = -\frac{1}{4}a_2).$$

When $a_1 > 0$, select two inequivalent representative elements $\tilde{v} = v_1 + v_3$ and $\tilde{v} = v_1 - v_3$. Correspondingly, Eq. (55) has the solutions

$$\epsilon_2 = \frac{1}{2}\ln a_3, \quad \epsilon_3 = 0, \quad \epsilon_4 = \frac{1}{4}a_2, \quad \epsilon_5 = \frac{1}{2}\ln a_1,$$

and

$$\epsilon_2 = \frac{1}{2}\ln(-a_3), \quad \epsilon_3 = 0, \quad \epsilon_4 = -\frac{1}{4}a_2, \quad \epsilon_5 = \frac{1}{2}\ln a_1.$$

When $a_1 < 0$, one can choose $\tilde{v} = -v_1 + v_3$ and $\tilde{v} = -v_1 - v_3$, which are obviously equivalent to the results of $a_1 > 0$.

(2) $a_2 = a_3 = a_4 = 0$. It leaves only the vector v_1.

In summary, the one-dimensional optimal system of five-dimensional Lie algebra (50) is found to be

$$\begin{aligned} &r_1 = v_3 - \frac{1}{4}\alpha v_4 + v_5 \quad (\alpha \in \mathbb{R}), \quad r_2 = -v_3 + \frac{1}{4}\alpha v_4 + v_5 \quad (\alpha \le 0),\\ &r_3 = v_5, \quad r_4 = v_3 - \frac{1}{4}v_4, \quad r_5 = v_3 + \frac{1}{4}v_4, \quad r_6 = v_1 + v_3 - \frac{1}{4}v_4,\\ &r_7 = v_1 + v_3 + \frac{1}{4}v_4, \quad r_8 = v_1 - v_3 - \frac{1}{4}v_4, \quad r_9 = v_1 + v_3,\\ &r_{10} = v_1 - v_3, \quad r_{11} = v_3, \quad r_{12} = v_1. \end{aligned} \tag{56}$$

B. Invariant solutions

By virtue of the one-dimensional optimal system, one can reduce the Novikov equation to kinds of ordinary differential equations (ODEs) which may further generate inequivalent group invariant solutions. The corresponding invariant solutions of v_1, v_3, and v_5 have been investigated in Ref. 26 and the rest classical solutions corresponding to optimal system (56) are all presented in this section.

(1) For r_1 and r_2, we have the following three cases according to the sign of α.

When $\alpha > 0$ in r_1, make the transformation

$$u = e^{-x}(2e^{2x} - \sqrt{\alpha})t^{\frac{1}{2}(\sqrt{\alpha}-1)}\psi(X), \quad X = \frac{\sqrt{\alpha} + 2e^{2x}}{\sqrt{\alpha} - 2e^{2x}} \cdot t^{-\sqrt{\alpha}}, \tag{57}$$

and it leads Novikov equation (49) to

$$48\sqrt{\alpha}\psi\psi'\psi'' - (3\sqrt{\alpha} + 1)\psi'' + 16\sqrt{\alpha}\psi^2\psi''' - 2\sqrt{\alpha}X\psi''' = 0. \tag{58}$$

When $\alpha < 0$ in r_1 and r_2 , substituting

$$u = \sqrt{1 + \delta^2 e^{4x}}e^{-x \pm \frac{\delta}{2}\arctan(\delta e^{2x})}\psi(X), \quad X = \ln(t) \pm \delta\arctan(\delta e^{2x}) \tag{59}$$

into Eq. (49), we obtain

$$\begin{aligned} &4e^X\psi[6(\psi')^2 + 6\psi'\psi'' + (6 - 2\alpha)\psi\psi' + 6\psi\psi'' + 2\psi\psi'''\\ &+(1 - \alpha)\psi^2] \pm \frac{1}{4}(\alpha^2 - \alpha)\psi' \mp \alpha(\psi'' + \psi''') = 0. \end{aligned} \tag{60}$$

Here, we denote $\alpha = -\frac{4}{\delta^2}$ in (59) for simplicity.

When $\alpha = 0$ in r_1 and r_2, applying

$$u = \frac{e^x}{\sqrt{t}}\psi(X), \quad X = e^{-2x} \mp \ln t \tag{61}$$

to Eq. (49), we have

$$12\psi\psi'\psi'' + \psi'' \pm 2\psi''' + 4\psi^2\psi''' = 0. \tag{62}$$

(2) For r_4 and r_5, one can easily give out the corresponding invariant solutions

$$u = c_0 e^{-x}\sqrt{4e^{4x} \mp 1}. \tag{63}$$

(3) For r_6, we take

$$u = \frac{\sqrt{2}}{4}e^{-x+t}(2e^{2x}+1)\psi(X), \quad X = \frac{2e^{2x}-1}{2e^{2x}+1}\cdot e^{-2t} \tag{64}$$

into Eq. (49) and it yields

$$3\psi\psi'\psi'' - 3\psi'' - 2X\psi''' + \psi^2\psi''' = 0. \tag{65}$$

(4) For r_7 and r_8, let

$$u = e^{-x}\sqrt{1+4e^{4x}}\psi(X), \quad X = \arctan(2e^{2x}) \mp t \tag{66}$$

and we have

$$12\psi\psi'\psi'' \mp \psi' + 4\psi^2\psi''' + 16\psi^2\psi' - \psi''' = 0. \tag{67}$$

A special solution to Eq. (67) is $\psi = \sin(X + c_0)$, which leads the invariant solutions to the Novikov equation,

$$u = (2c_1 e^x \pm c_2 e^{-x})\cos(t) + (2c_2 e^x \mp c_2 e^{-x})\sin(t). \tag{68}$$

(5) For r_9 and r_{10}, we have

$$u = e^x\psi(X), \quad X = e^{-2x} \pm 2t, \tag{69}$$

which can transform Eq. (49) into

$$3\psi\psi'\psi'' + \psi^2\psi''' \mp \psi''' = 0. \tag{70}$$

Different types of ODEs shown by (58), (60), (62), (65), (67), and (70) are successfully proposed and these abundant reduced equations with their solutions remain to be investigated thoroughly.

IV. CONCLUSION AND DISCUSSIONS

Group invariant solutions have been used to great effect in the description of the asymptotic behavior of much more general solutions to systems of PDEs. These group invariant solutions are characterized by their invariance under some symmetry group of the PDEs. Since there are almost always an infinite number of different symmetry groups, one might employ to find group invariant solutions; a means of determining which groups give fundamentally different types of invariant solutions is essential for gaining a complete understanding of the solutions which might be available. This classification problem can be solved by looking at the adjoint representation of the symmetry group on its Lie algebra, which first used by Ovsiannikov. The one-dimensional classification of the symmetry algebras of the KdV equation and the heat equation are demonstrated by Olver with an easy-to-operate method in detail, which only depends on the fragments of the theory of Lie algebras. However, as Olver said, in essence, this problem is attacked by the naïve approach of taking a general element in Lie algebra and subjecting it to various adjoint transformations so as to "simplify" it as much as possible. To make up this problem and ensure the comprehensiveness with inequivalence, we develop a direct and systemic algorithm for the one-dimensional optimal system. The new approach is very natural and every element in the optimal system can be found step by step.

Our method introduced in this paper, which is essentially new, only depends on the commutator and adjoint representative relations among the generators of Lie algebras. The main work includes the following.

(1) A valid method is proposed to compute all the general invariants of the one-dimensional Lie algebra, which include the well-known Killing form.

(2) A criterion is introduced to scale the invariants, which appears in "*Remark 2*," "*Remark 3*," and "*Remark 4*."

(3) For two one-dimensional subalgebras v and $\tilde{v}$, we introduce an algebraic system of Eq. (20) (or (21)) to determine their equivalences in the sense of adjoint transformation.

(4) Based on all the scaled invariants, we put forward a direct and effective algorithm to construct one-dimensional optimal system. With the new approach, every element in the optimal system can be found step by step.

Since all the representative elements are attached to different values of the invariants, it ensures the optimality of the optimal system. From the process of the operation in our method, one can easily see that how these representatives are mutually inequivalent. It shows that the designed algorithm in this paper essentially starts from a finite dimensional symmetry Lie (sub)algebra of the system of differential equations rather than the system itself. Hence, although the given examples in this paper are all in the form of single partial differential equation, the method can also be applied to ODEs and systems of differential equations. Due to the one-dimensional optimal system of the symmetry Lie algebra, the original $(1+1)$-dimensional system of differential equations would be reduced to inequivalent ODEs and then the corresponding group invariant solutions can be recovered. Furthermore, how to apply all the invariants to construct r-parameter ($r \geq 2$) optimal systems is in our consideration. Since the algorithm is very systemic, we believe that it will provide a very good manner for the mechanization.

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