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A direct bijection for the Harer–Zagier formula

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Abstract

We give a combinatorial proof of Harer and Zagier's formula for the disjoint cycle distribution of a long cycle multiplied by an involution with no fixed points, in the symmetric group on a set of even cardinality. The main result of this paper is a direct bijection of a set $\mathcal{B}_{p,k}$, the enumeration of which is equivalent to the Harer–Zagier formula. The elements of $\mathcal{B}_{p,k}$ are of the form (μ, π), where μ is a pairing on $\{1, \ldots, 2p\}$, π is a partition into k blocks of the same set, and a certain relation holds between μ and π. (The set partitions π that can appear in $\mathcal{B}_{p,k}$ are called "shift-symmetric", for reasons that are explained in the paper.) The direct bijection for $\mathcal{B}_{p,k}$ identifies it with a set of objects of the form (ρ, t), where ρ is a pairing on a $2(p-k+1)$-subset of $\{1, \ldots, 2p\}$ (a "partial pairing"), and t is an ordered tree with k vertices. If we specialize to the extreme case when $p = k - 1$, then ρ is empty, and our bijection reduces to a well-known tree bijection.

MSC: Primary, 05A15; Secondary, 05C05

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1. Introduction

We begin by reviewing some standard terminology, which will be used throughout the paper.

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1.1. Notation

(a) *Pairings*: Let $[n] = \{1, \ldots, n\}$, and $\mathcal{S}_n$ be the set of permutations of $[n]$, for $n \geqslant 0$. For $\delta \in \mathcal{S}_n$ and $\kappa \subseteq [n]$, we write $\delta(\kappa)$ for the set $\{\delta(i) : i \in \kappa\}$. Let $\mathcal{P}_p$ be the set of *pairings* on $[2p]$, which are partitions of the set $[2p]$ into disjoint subsets of size 2, for $p \geqslant 0$ (we refer to the single element of $\mathcal{P}_0$ as the *empty* pairing). For $\mu \in \mathcal{P}_p$, we thus have $\mu = \{\{m_{11}, m_{12}\}, \ldots, \{m_{p1}, m_{p2}\}\}$, where $m_{i1} < m_{i2},\ i = 1, \ldots, p$, and $m_{11} < \cdots < m_{p1}$. Where the context is appropriate, we shall also regard $\mathcal{P}_p$ as the conjugacy class of involutions with no fixed points in $\mathcal{S}_{2p}$, and in this context, we regard μ above as having disjoint cycle representation $(m_{11}\, m_{12}) \cdots (m_{p1}\, m_{p2})$, and write $\mu(m_{i1}) = m_{i2},\ \mu(m_{i2}) = m_{i1},\ i = 1, \ldots, p$. Of course, the number of pairings in $\mathcal{P}_p$ is $(2p-1)!! = \prod_{j=1}^{p}(2j-1)$, with the empty product convention that $(-1)!! = 1$.

(b) *Partial pairings*: A *partial* pairing on $[2p]$ is a pairing on a set $\alpha \subseteq [2p]$ of even cardinality. If $|\alpha| = 2k$, then we also call it a k-partial pairing. For each of these partial pairings μ, we call α the *support*, and denote this by $\mathrm{supp}(\mu) = \alpha$. For $p \geqslant k \geqslant 0$, let $\mathcal{R}_{p,k}$ be the set of k-partial pairings on $[2p]$. For each α of size $2k$, there are $(2k-1)!!$ pairings on α, so the number of k-partial pairings on $[2p]$ is given by

$$|\mathcal{R}_{p,k}| = \binom{2p}{2k}(2k-1)!! \tag{1}$$

(c) *Ordered trees*: An *ordered* tree is a tree with a root vertex, which is adjacent to an ordered list of vertices (called the *descendants* of the root vertex), each of which is itself, recursively, the root vertex of an ordered tree. The latter ordered trees are called ordered *subtrees* of the ordered tree. Let $\mathcal{T}_k$ denote the set of ordered trees on k vertices, for $k \geqslant 1$. It is well known (see, e.g., [9, p. 60]), that the number of ordered trees on k vertices is given by

$$|\mathcal{T}_k| = \frac{1}{k}\binom{2k-2}{k-1}, \tag{2}$$

which is a *Catalan* number. If we draw an ordered tree, the root vertex is placed at the bottom, with descendants above, ordered from left to right; thus we shall refer to the order of descendants as left to right order. The *level* of a vertex in an ordered tree is a nonnegative integer defined recursively as follows: the root vertex has level 0, and if vertex v is a descendant of vertex u, then the level of v is one greater than the level of u. (Equivalently, the level of a vertex v in an ordered tree is the edge-length of the unique path in the tree from the root vertex to v.) If u and v are descendants of a vertex, with u to the left of v, then all vertices in the subtree rooted at u are to the left of all vertices in the subtree rooted at v. For each $i \geqslant 1$, this totally orders the vertices at level i from left to right.

(d) Labellings of ordered trees: A *labelled* ordered tree on k vertices is an ordered tree on k vertices, each of which is assigned a unique label from $[k]$. *Reverse-labelling* is the canonical labelling in which the root vertex is labelled k, then the vertices at level 1 are labelled in decreasing order from right to left, beginning with $k-1$, followed by the vertices at level 2, decreasing from right to left, repeating until the leftmost vertex at the highest level in the tree is labelled 1. For $t \in \mathcal{T}_k$, we shall use t' to denote the tree obtained by reverse-labelling t.

Let us now describe the formula in the title of the paper. For $p \geqslant 1$, we consider the *shift* permutation γ in $\mathcal{S}_{2p}$, which has disjoint cycle representation $\gamma = (1\,2\ldots 2p)$. Let $\mathcal{A}_p = \{\mu\gamma^{-1} : \mu \in \mathcal{P}_p\}$, and $a_{p,k}$ be the number of permutations in $\mathcal{A}_p$ with k cycles in the disjoint cycle representation, for $k \geqslant 1$. Harer and Zagier [2] obtained the following result.

Theorem 1.1 (*Harer and Zagier [2]*). *For* $p \geqslant 1$,

$$\sum_{k\geqslant 1} a_{p,k} x^k = (2p-1)!! \sum_{k\geqslant 1} 2^{k-1} \binom{p}{k-1} \binom{x}{k}.$$

Other proofs of Theorem 1.1 have been given by Itzykson and Zuber [3], Jackson [4], Kerov [5], Kontsevich [6], Lass [7], Penner [8] and Zagier [10] (see also the survey by Zvonkin [11], and the discussion in Section 4 of the paper by Haagerup and Thorbjornsen [1]). Despite the elementary statement of the theorem, the proofs are not easy, and as a rule, they move out of the realm of enumerative combinatorics. A notable exception to this rule is the paper of Lass [7]. The method of Lass is purely combinatorial, and relies on an ingenious application of the BEST Theorem, which enumerates Eulerian tours in a multigraph as the product of two factors, one explicit, and the other giving the number of spanning arborescences of the multigraph. He then uses Cayley's result for counting labelled trees by degree to obtain the result, overcounting by a factor of $k!$. While this combinatorial proof implies a bijection, it does not specify a direct bijection.

In this paper, we present a direct bijection for a set of objects, the enumeration of which is equivalent to the Harer–Zagier formula. These objects are introduced in the next definition.

Definition 1.2. Let $\mathcal{B}_{p,k}$ be the set of ordered pairs (μ, π), where $\mu \in \mathcal{P}_p$ and π is a partition of $[2p]$ into k nonempty, unordered sets (called the *blocks* of the partition), satisfying the condition

$$\mu(i),\ \gamma(i) \text{ are in the same block of } \pi \text{ for all } i \in [2p]. \tag{3}$$

Let $b_{p,k}$ be the number of elements in $\mathcal{B}_{p,k}$. We call a partition π for which there exists μ with $(\mu, \pi) \in \mathcal{B}_{p,k}$ a *shift-symmetric* partition, for reasons that are explained in Section 4.

It is immediate to see that the numbers $b_{p,k}$ of Definition 1.2 give an alternative way of looking at the left-hand side of the Harer–Zagier formula, as specified in the following result.

Proposition 1.3. *For* $p \geqslant 1$,

$$\sum_{k\geqslant 1} a_{p,k} x^k = \sum_{k\geqslant 1} b_{p,k} (x)_k,$$

where $(x)_k := \prod_{j=0}^{k-1} (x-j)$, $k \geqslant 1$ *is the falling factorial.*

Proof. If $(\mu, \pi) \in \mathcal{B}_{p,k}$, then condition (3) is equivalent to $\mu\gamma^{-1}(j)$ and j belonging to the same block of π for all $j \in [2p]$ (by replacing i above by $\gamma^{-1}(j)$). But this means that, for each μ, the blocks of π are unions of disjoint cycles of $\mu\gamma^{-1}$. Thus

$$b_{p,k} = \sum_{m \geqslant k} S(m,k) a_{p,m}, \tag{4}$$

where $S(m,k)$, the Stirling number of the second kind, gives the number of partitions of an m-set into k nonempty, unordered subsets. But $\sum_{k=1}^{m} S(m,k)(x)_k = x^m$, from, e.g. Stanton and White [9, p. 78], so multiplying (4) by $(x)_k$ and summing over $k \geqslant 1$ gives the result. □

Now, comparing the expressions given in Theorem 1.1 and Proposition 1.3, we obtain

$$\begin{aligned} b_{p,k} &= (2p-1)!!\, 2^{k-1} \binom{p}{k-1} \frac{1}{k!} \\ &= \binom{2p}{2p-2k+2} (2p-2k+1)!!\, \frac{1}{k} \binom{2k-2}{k-1}, \end{aligned} \tag{5}$$

where, for the second equality, we have simply manipulated the factors in the quotient. But, considering (1) and (2), we find that the latter expression for $b_{p,k}$ is equivalent to

$$|\mathcal{B}_{p,k}| = |\mathcal{R}_{p,p-k+1}| \cdot |\mathcal{T}_k| \tag{6}$$

for $p \geqslant k-1 \geqslant 0$. In this paper, we shall give a combinatorial proof of Theorem 1.1, directly proving (6) by giving a direct bijection between $\mathcal{B}_{p,k}$ and $\mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$.

Theorem 1.4. *For $p \geqslant k-1 \geqslant 0$, there exists a direct bijection*

$$\psi_{p,k} : \mathcal{B}_{p,k} \to \mathcal{R}_{p,p-k+1} \times \mathcal{T}_k.$$

Theorem 1.4 is our main result. The construction of the bijection $\psi_{p,k}$ is given in Section 2. Described very succinctly, the ideas behind the construction of $\psi_{p,k}$ are as follows. Given $(\mu, \pi) \in \mathcal{B}_{p,k}$, let the blocks of π be denoted by $\pi_1, \ldots, \pi_k$, where π_k is the block containing the number 1. We let $m_1, \ldots, m_{k-1}$ be the maximum elements in $\pi_1, \ldots, \pi_{k-1}$, respectively, indexed so that $m_1 < \cdots < m_{k-1}$. Then, by considering the blocks in which the "mates", $\mu(m_1), \ldots, \mu(m_{k-1})$ occur, we determine a tree $t \in \mathcal{T}_k$—see Proposition 2.1, and the Notation following it. Also, from these maximum elements and their mates, we determine two partial pairings: $\mu_2 = \{\{m_1, \mu(m_1)\}, \ldots, \{m_{k-1}, \mu(m_{k-1})\}\}$, and $\mu_1 = \mu \setminus \mu_2$, where $\mu_1 \in \mathcal{R}_{p,p-k+1}$.

It would be too much to hope that $(\mu, \pi) \mapsto (\mu_1, t)$ is our desired bijection, but it turns out that we are only one canonical relabelling away from the bijection $\psi_{p,k}$ in Theorem 1.4. More precisely, in the Notation following Proposition 2.1, we create a relabelling permutation $\sigma \in \mathcal{S}_{2p}$ from (μ, π), so that $\psi_{p,k}$ is described by the map $(\mu, \pi) \mapsto (\sigma(\mu_1), t)$. The proof that this is bijective is given in Section 3.

The paper concludes in Section 4 with some remarks about related results.

2. A mapping for shift-symmetric paired partitions

We consider $(\mu, \pi) \in \mathcal{B}_{p,k}$, and construct various objects associated with (μ, π). First, let the blocks of π be given by $\pi_1, \ldots, \pi_k$, indexed as follows: π_k is the block containing the element 1, and the remaining blocks are indexed according to the order of their maximum elements, by

$$\max(\pi_1) < \cdots < \max(\pi_{k-1})$$

and we let $m_i = \max(\pi_i)$ for $i = 1, \ldots, k-1$. Define $\phi : [k-1] \to [k]$ by $\phi(i) = j$ when $\mu(m_i) \in \pi_j$, for $i = 1, \ldots, k-1$.

Proposition 2.1. *For $(\mu, \pi) \in \mathcal{B}_{p,k}$, and in the notation above,*

(i) *for $i = 1, \ldots, k-1$, we have $i < \phi(i)$,*
(ii) *for $i = 1, \ldots, k-1$, we have $\mu(m_i) \neq m_j$ for any $j = 1, \ldots, k-1$.*

Proof. (i) For $i = 1, \ldots, k-1$, we consider two cases for m_i:

- if $m_i = 2p$ (this can only happen when $i = k-1$), then $\gamma(m_i) = 1 \in \pi_k$, and condition (3) implies that $\mu(m_i) \in \pi_k$. Thus $\phi(i) = k$, and the result is true in this case;
- if $m_i < 2p$, then $\gamma(m_i) = m_i + 1 \in \pi_j$ where $\max(\pi_j) = m_j \geqslant m_i + 1 > m_i$, and so from the indexing convention for the blocks of π, we have $i < j$ (for both the possible choices $j = k$ and $j < k$). But again the μ, π condition implies that $\mu(m_i) \in \pi_j$, so $\phi(i) = j$, and the result is true in this case also.

(ii) If $\mu(m_i) = m_j \in \pi_j$, then $\phi(i) = j$, so from part (i) we have $i < j$. But we also have $\mu(m_j) = m_i \in \pi_i$, so $\phi(j) = i$, and from part (i) we have $j < i$, a contradiction, and the result follows. $\square$

For the same (μ, π) considered at the beginning of the section, we now construct three objects, which will appear in our direct bijection. The tree in Notation (b) below is analogous to the tree used in Lass [7], with the difference that here we have an ordered tree.

2.1. Notation

(a) *Partial pairings*: Split the pairing μ into two partial pairings μ_1 and μ_2, where we let $\mu_2 = \{\{m_1, \mu(m_1)\}, \ldots, \{m_{k-1}, \mu(m_{k-1})\}\}$, and $\mu_1 = \mu \setminus \mu_2$. Note that μ_2 is well-defined as a $(k-1)$-partial pairing, from Proposition 2.1(ii), and thus μ_1 is a $(p-k+1)$-partial pairing.

(b) *Ordered tree*: From ϕ, create a labelled ordered tree T on vertex-set $[k]$, as follows: the root is k, and for every $i = 1, \ldots, k-1$, i is a descendant of $\phi(i)$; if vertices i, j are both descendants of vertex v, then i is to the left of j when $\mu(m_i) < \mu(m_j)$ (otherwise, j is to the left of i and $\mu(m_j) < \mu(m_i)$). The fact that this is well-defined follows immediately from Proposition 2.1(i), which implies that for every $i = 1, \ldots, k$, the increasing sequence $i, \phi(i), \phi(\phi(i)), \ldots$ will uniquely terminate at k, thus specifying the unique path from vertex i to the root vertex k in T.

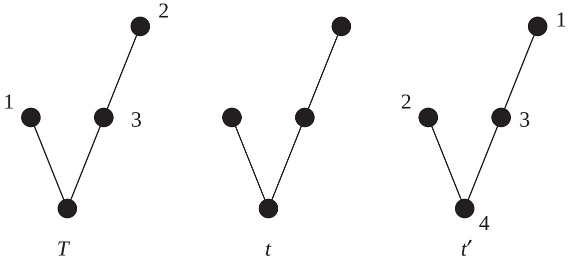

Fig. 1. Three trees T, t, t'.

Then we remove the labels from T to obtain the ordered tree $t \in \mathcal{T}_k$, and consider the reverse-labelled tree t'. Thus, T and t' give two, possibly different, labellings of t. Suppose that the vertex of t labelled i in T is labelled j in t'. Then define $\pi^{(j)} = \pi_i$, and repeat for each $i = 1, \ldots, k$. Now $\pi^{(1)}, \ldots, \pi^{(k)}$ gives a different indexing of the blocks of π, in which $\pi^{(k)} = \pi_k$, and we define $m^{(i)}$ to be the maximum element of $\pi^{(i)}$ for $i = 1, \ldots, k-1$.

(c) *Relabelling permutation*: Let $\omega^{(i)}$ be the string obtained by writing the elements of $\pi^{(i)}$ in increasing order, for $i = 1, \ldots, k$, and let $\omega = \omega^{(1)} \cdots \omega^{(k)}$, the concatenation of $\omega^{(1)}, \ldots, \omega^{(k)}$. Then ω contains element j exactly once for each $j \in [2p]$, since $\pi^{(1)}, \ldots, \pi^{(k)}$ are the blocks of π, a set partition of $[2p]$. Thus we define $\sigma \in \mathcal{S}_{2p}$ by specifying that ω is the second line in the two-line representation of σ^{-1}. Finally, we consider $\sigma(\mu_1)$, to mean that each pair $\{i, j\}$ in μ_1 becomes pair $\{\sigma(i), \sigma(j)\}$ in $\sigma(\mu_1)$. Since μ_1 is a $(p-k+1)$-partial pairing, then $\sigma(\mu_1)$ is also a $(p-k+1)$-partial pairing.

Example 2.2. In the case $p = 9$ and $k = 4$, consider $(\mu, \pi) \in \mathcal{B}_{9,4}$, with

$$\mu = \{\{1, 18\}, \{2, 7\}, \{3, 6\}, \{4, 5\}, \{8, 13\}, \{9, 12\}, \{10, 11\}, \{14, 17\}, \{15, 16\}\},$$
$$\pi = \{\pi_1, \pi_2, \pi_3, \pi_4\},$$

where $\pi_4 = \{1, 2, 8, 14, 18\}$, and $\pi_1 = \{3, 4, 6, 7\}$, $\pi_2 = \{5, 11, 16\}$, $\pi_3 = \{9, 10, 12, 13, 15, 17\}$. Note that the indexing has already been assigned, and indeed $1 \in \pi_4$, $m_1 < m_2 < m_3$, where $m_1 = \max(\pi_1) = 7$, $m_2 = \max(\pi_2) = 16$, $m_3 = \max(\pi_3) = 17$. (The condition that $\mu(i)$ and $\gamma(i)$ are in the same block of π for all $i \in [18]$ requires more detailed checking: e.g., $\gamma(1) = 2$, $\mu(1) = 18$, and $2, 18 \in \pi_4$; $\gamma(2) = 3$, $\mu(2) = 7$, and $3, 7 \in \pi_1$; $\gamma(4) = 5$, $\mu(4) = 5$, and nothing to check here, etc.)

From these m_i's we now determine ϕ, by $\phi(1) = \phi(3) = 4$, $\phi(2) = 3$. Thus $\mu_2 = \{\{2, 7\}, \{15, 16\}, \{14, 17\}\}$, so $\mu_1 = \{\{1, 18\}, \{3, 6\}, \{4, 5\}, \{8, 13\}, \{9, 12\}, \{10, 11\}\}$. Next we determine the trees T, t, t', given in Fig. 1 (in T, vertices 1 and 3 are both descendants of 4, with 1 to the left of 3 because $\mu(m_1) = 2 < 14 = \mu(m_3)$). Therefore, we have $\pi^{(1)} = \pi_2$, $\pi^{(2)} = \pi_1$, $\pi^{(3)} = \pi_3$, $\pi^{(4)} = \pi_4$, and $m^{(1)} = 16$, $m^{(2)} = 7$, $m^{(3)} = 17$, so $\omega^{(1)} = 5\,11\,16$, $\omega^{(2)} = 3\,4\,6\,7$, $\omega^{(3)} = 9\,10\,12\,13\,15\,17$, $\omega^{(4)} = 1\,2\,8\,14\,18$, and $\omega = 5\,11\,16\,3\,4\,6\,7\,9\,10\,12\,13\,15\,17\,1\,2\,8\,14\,18$, so the two-line representation of σ^{-1} is given by

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
5	11	16	3	4	6	7	9	10	12	13	15	17	1	2	8	14	18

The placement of the vertical lines in this two-line representation will be referred to again in Section 3. (Alternatively, $\sigma = (1\,14\,17\,13\,11\,2\,15\,12\,10\,9\,8\,16\,3\,4\,5)(6)(7)$ in disjoint cycle notation.) Thus, we have $\sigma(\mu_1) = \{\{1, 5\}, \{2, 9\}, \{4, 6\}, \{8, 10\}, \{11, 16\}, \{14, 18\}\}$. Also note that $\sigma(\mu_2) = \{\{3, 12\}, \{7, 15\}, \{13, 17\}\}$, and $\sigma(\pi^{(1)}) = \{1, 2, 3\}$, $\sigma(\pi^{(2)}) = \{4, 5, 6, 7\}$, $\sigma(\pi^{(3)}) = \{8, 9, 10, 11, 12, 13\}$, $\sigma(\pi^{(4)}) = \{14, 15, 16, 17, 18\}$.

Now, among the objects that we have constructed from (μ, π) are the $(p-k+1)$-partial pairing $\sigma(\mu_1)$ and the ordered tree t, so the following mapping $\psi_{p,k}$ is well defined.

Definition 2.3. For $p \geqslant k-1 \geqslant 0$, let

$$\psi_{p,k} : \mathcal{B}_{p,k} \to \mathcal{R}_{p,p-k+1} \times \mathcal{T}_k : (\mu, \pi) \mapsto (\sigma(\mu_1), t)$$

We claim that $\psi_{p,k}$ is actually a bijection, and prove this in Section 3.

3. Proof that the mapping is bijective

We begin with an observation about the relationship between μ_2 and t in the construction of $\psi_{p,k}(\mu, \pi)$ (where (μ, π) is a given element of $\mathcal{B}_{p,k}$ and where we use notation consistent with Section 2). Let us denote

$$\beta := \text{supp}(\mu_2) = \bigcup_{i=1}^{k-1} \{m^{(i)}, \mu(m^{(i)})\}$$

and $\beta^{(i)} := \beta \cap \pi^{(i)}$, for $i = 1, \ldots, k$. On the other hand, recall that t' is the reverse-labelling of the ordered tree t, and let us denote by $d^{(i)}$ the number of descendants of the vertex i of t', for $i = 1, \ldots, k$. Our observation is that we have $|\beta^{(i)}| = d^{(i)} + 1$ for $i = 1, \ldots, k-1$, and $|\beta^{(k)}| = d^{(k)}$. In the case $i < k$, for instance, this is because $\beta^{(i)}$ consists of $m^{(i)}$ and of $d^{(i)}$ elements of the form $\mu(m^{(j)})$ with j a descendant of i in t'. Moreover, we observe that the ordering of the $d^{(i)}+1$ elements of $\beta^{(i)}$ can be read from the tree t'—the largest element of $\beta^{(i)}$ is $m^{(i)}$, and the remaining elements of the form $\mu(m^{(j)})$ are ordered exactly in the same way as the corresponding j's are ordered as descendants of i in t'. (This is for $i < k$. The ordering of the $d^{(k)}$ elements of $\beta^{(k)}$ is, of course, read from t' in a similar manner.)

The above observation about the $\beta^{(i)}$'s and $d^{(i)}$'s has the following consequence: Consider the sequence ω (defined as in the part (c) of the Notation in Section 2). If we know β (as a set) and t, and if we also know in what order the elements of β appear in the sequence ω, then we can deduce the precise structure of β—i.e. what $m^{(i)}$ is and what $\mu(m^{(i)})$ is, for every $1 \leqslant i \leqslant k-1$.

Let us illustrate the observation (and its consequence) in the concrete situation of Example 2.2. There $\beta = \{2, 7, 14, 15, 16, 17\}$. Suppose we know β but we do not remember which elements of β were $m^{(1)}, m^{(2)}, m^{(3)}$ and $\mu(m^{(1)}), \mu(m^{(2)}), \mu(m^{(3)})$. Suppose on the other hand that we also remember t (hence we know t') and the fact that the elements of β appear in ω in the following order:

$$(*) \qquad \cdots 16 \cdots 7 \cdots 15 \cdots 17 \cdots 2 \cdots 14 \cdots$$

By looking at t' we see that $\mu(m^{(1)}) \in \pi^{(3)}$ and that $\mu(m^{(2)}), \mu(m^{(3)}) \in \pi^{(4)}$; hence the 6 elements of β have to be distributed between the blocks of π as follows:

$$m^{(1)} \in \pi^{(1)}; \quad m^{(2)} \in \pi^{(2)}; \quad \mu(m^{(1)}), m^{(3)} \in \pi^{(3)}; \quad \mu(m^{(2)}), \mu(m^{(3)}) \in \pi^{(4)}.$$

But then the order in which the elements of β appear in ω must be:

$$(**) \qquad \cdots m^{(1)} \cdots m^{(2)} \cdots \mu(m^{(1)}) \cdots m^{(3)} \cdots \mu(m^{(2)}) \cdots \mu(m^{(3)}) \cdots$$

(The fact that $\mu(m^{(2)})$ precedes $\mu(m^{(3)})$ in $(**)$ is inferred from the fact that 2 is to the left of 3, as descendants of 4 in t'.) By comparing $(*)$ against $(**)$ we can thus determine what $m^{(i)}$ is and what $\mu(m^{(i)})$ is, for every $1 \leqslant i \leqslant 3$.

Proposition 3.1. *For $p \geqslant k-1 \geqslant 0$, $\psi_{p,k} : \mathcal{B}_{p,k} \to \mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$ is an injection.*

Proof. Suppose we are given $(\sigma(\mu_1), t)$ arising as $\psi_{p,k}(\mu, \pi)$ for some $(\mu, \pi) \in \mathcal{B}_{p,k}$. Our goal is to prove that (μ, π) can be uniquely recovered from $(\sigma(\mu_1), t)$.

The given data determines in particular the set $\alpha := [2p] \setminus \mathrm{supp}(\sigma(\mu_1))$. Clearly, we have $\alpha = \sigma(\beta)$ where β is as in the discussion which preceded the statement of the proposition. At this stage of the proof we do not know what β is; but observe that the knowledge of α tells us in what positions the elements of β appear when we form the (also unknown at the moment) sequence ω—this is just because ω is the second line in the two-line representation of σ^{-1}. Since we know the ordered tree t, the discussion which preceded the proposition can then be put to work: while we still would not know $m^{(1)}, \ldots, m^{(k-1)}$ and $\mu(m^{(1)}), \ldots, \mu(m^{(k-1)})$, we will nevertheless deduce in what order these numbers appear in the sequence ω, and hence under which elements of α they appear. In other words: the upshot of the discussion presented before the proposition will determine explicitly $\sigma(m^{(1)}), \ldots, \sigma(m^{(k-1)})$ and $\sigma(\,\mu(m^{(1)})\,), \ldots, \sigma(\,\mu(m^{(k-1)})\,)$.

Now, the numbers $\sigma(m^{(1)}), \ldots, \sigma(m^{(k-1)})$ mark the placement of the vertical bars in the two-line representation of σ^{-1}. So if we know them, then we know where the vertical bars are, and consequently we know the partition $\sigma(\pi) := \{\sigma(\pi^{(1)}), \ldots, \sigma(\pi^{(k)})\}$. (The blocks of the latter partition are intervals, $\sigma(\pi^{(1)}) = [\sigma(m^{(1)})]$, $\sigma(\pi^{(i)}) = [\sigma(m^{(i)})] \setminus [\sigma(m^{(i-1)})]$ for $2 \leqslant i \leqslant k-1$, and $\sigma(\pi^{(k)}) = [2p] \setminus [\sigma(m^{(k-1)})]$.) Also, note that at this stage of the proof we know how to complete the given partial pairing $\sigma(\mu_1)$ to the pairing $\sigma(\mu)$ of $[2p]$ (since the missing pairs in $\sigma(\mu)$ were $\{\sigma(m^{(i)}), \sigma(\,\mu(m^{(i)})\,)\}$, for $1 \leqslant i \leqslant k-1$).

To this stage, we have proved that $(\sigma(\mu), \sigma(\pi))$ can be uniquely recovered from the given data $(\sigma(\mu_1), t)$. In order to finish the proof that $\psi_{p,k}$ is injective, we now prove that σ^{-1} can be uniquely recovered from $(\sigma(\mu), \sigma(\pi))$, when one considers the relationship between μ, π, γ. Applying (the unknown relabelling permutation) σ to the μ, π condition (3) gives the following condition: for $i = 1, \ldots, 2p-1$, then if $\sigma(i) = c$ and $\sigma(\mu(i)) = d \in \sigma(\pi^{(j)})$, we have $\sigma(i+1) = \sigma(\gamma(i)) \in \sigma(\pi^{(j)})$. But, applying σ to our indexing convention for the blocks of π, we have $\sigma(1) \in \sigma(\pi^{(k)})$. Moreover, since the symbols in $\omega^{(j)}$ increase from left to right for each $j = 1, \ldots, k$, we know that 1 is the left-most element of $\omega^{(k)}$, and that if $i+1 \in \pi^{(j)}$, then to the left of $i+1$ in $\omega^{(j)}$ are precisely the elements of $[i] \cap \pi^{(j)}$. We claim that this gives enough information to uniquely determine ω (equivalently, σ or σ^{-1}). We describe exactly how to do so below, using the following terminology for the two-line

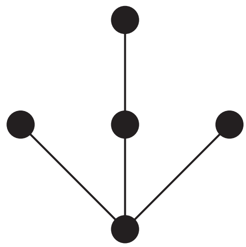

Fig. 2. A tree in $\mathcal{T}_5$.

representation of σ^{-1}: we regard this two-line representation as a table with 2 rows and $2p$ columns, and note that the entry in the first row of column c is equal to c, for each $c = 1, \ldots, 2p$. We place vertical bars in the table to separate the $\omega^{(j)}$'s in the second row, as displayed here:

$1 \quad \ldots \quad \sigma(m^{(1)})$	$\sigma(m^{(1)})+1 \quad \ldots \quad \sigma(m^{(2)})$	$\ldots$	$\sigma(m^{(k-1)})+1 \quad \ldots \quad 2p$
$\omega^{(1)}$	$\omega^{(2)}$	$\ldots$	$\omega^{(k)}$

Our task is to uniquely reconstruct this table from $(\sigma(\mu), \sigma(\pi))$. Begin with entry c in the first row of column c, for each $c = 1, \ldots 2p$, and the second row empty. Note that since we are given $\sigma(\pi)$, then we know the $\sigma(m^{(j)})$'s, so we can place the vertical bars in the table, and we now translate the information above into an iterative process that uniquely places each i in the second row of the table, for $i = 1, \ldots, 2p$.

- First, place 1 in the second row of the table, in the left-most position of $\omega^{(k)}$ (so 1 is located in column $\sigma(m^{(k-1)}) + 1$ of the second row in the table);
- Repeat for $i = 1, \ldots, 2p - 1$: Suppose that i has been placed in column c of the second row in the table, that c is paired with d in $\sigma(\mu)$, and that $d \in \sigma(\pi^{(j)})$. Then place $i + 1$ in the second row of the table, in the left-most unoccupied position of $\omega^{(j)}$.

This process is simply a translation, into the terminology of the table, of the information deduced from the relationship between μ, π, γ above, and thus uniquely determines σ^{-1}. Applying this permutation to $(\sigma(\mu), \sigma(\pi))$, we uniquely recover (μ, π), and the result follows. □

The description of $\psi_{p,k}^{-1}$ given in the above proof can be checked by applying it to the pair $(\sigma(\mu_1), t)$ that was created in Example 2.2. Indeed, the pair (μ, π) is uniquely recovered if we do so. We now give a second example, in which we apply $\psi_{p,k}^{-1}$ to an arbitrary element of $\mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$.

Example 3.2. In the case $p = 7$ and $k = 5$, consider $\{\{2, 13\}, \{3, 6\}, \{9, 12\}\} \in \mathcal{R}_{7,3}$ and the tree in $\mathcal{T}_5$ given in Fig. 2.

After reverse-labelling the tree we find that (in notation similar to that used above) $\mu(m^{(1)}) \in \pi^{(3)}$ and that $\mu(m^{(2)}) < \mu(m^{(3)}) < \mu(m^{(4)})$ are all elements of $\pi^{(5)}$. As a consequence, the order of appearance of the elements of β in the sequence ω must be as follows:

$$\cdots m^{(1)} \cdots m^{(2)} \cdots \mu(m^{(1)}) \cdots m^{(3)} \cdots m^{(4)} \cdots \mu(m^{(2)}) \cdots \mu(m^{(3)}) \cdots \mu(m^{(4)}) \cdots$$

The set of positions where these elements appear (in the sequence ω) is $\alpha = \{1, 4, 5, 7, 8, 10, 11, 14\}$. This implies that we have

$$\sigma(m^{(1)}) = 1, \quad \sigma(m^{(2)}) = 4, \quad \sigma(m^{(3)}) = 7, \quad \sigma(m^{(4)}) = 8$$

and hence that $\sigma(\pi) = \{\sigma(\pi^{(1)}), \ldots, \sigma(\pi^{(5)})\}$ is described as follows: $\sigma(\pi^{(1)}) = \{1\}$, $\sigma(\pi^{(2)}) = \{2, 3, 4\}$, $\sigma(\pi^{(3)}) = \{5, 6, 7\}$, $\sigma(\pi^{(4)}) = \{8\}$, $\sigma(\pi^{(5)}) = \{9, 10, 11, 12, 13, 14\}$. Moreover, the given partial pairing $\{ \{2, 13\}, \{3, 6\}, \{9, 12\} \}$ is completed to the pairing $\sigma(\mu) = \{ \{1, 5\}, \{2, 13\}, \{3, 6\}, \{4, 10\}, \{7, 11\}, \{8, 14\}, \{9, 12\} \}$. In order to finish the process of finding μ and π such that $\psi_{7,5}(\mu, \pi)$ equals the given data, we need to place elements $i + 1 = 2, \ldots, 14$ in the second row of the following table.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
								1					

This is straightforward, as described above. The reader willing to practice implementing the algorithm should have no difficulty checking that after two iterations, for instance, the table becomes:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	3							1	2				

The next entry to be placed in this partial table is 4, which appears in the second row under 11; this is because 3 appears below 2, which is paired in $\sigma(\mu)$ with $13 \in \sigma(\pi^{(5)})$, and the left-most unoccupied position in $\pi^{(5)}$ appears under 11. Completing the table, we obtain the two-line representation of σ^{-1} below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
6	3	8	12	5	7	9	14	1	2	4	10	11	13

Now we have obtained $(\mu, \pi) \in \mathcal{B}_{p,k}$, where $\mu = \{\{1, 10\}, \{2, 12\}, \{3, 11\}, \{4, 9\}, \{5, 6\}, \{7, 8\}, \{13, 14\}\}$, and $\pi^{(1)} = \{6\}$, $\pi^{(2)} = \{3, 8, 12\}$, $\pi^{(3)} = \{5, 7, 9\}$, $\pi^{(4)} = \{14\}$ and $\pi^{(5)} = \{1, 2, 4, 10, 11, 13\}$. (Note that $\pi_i = \pi^{(i)}$ for $i = 1, 4, 5$, but that $\pi_2 = \pi^{(3)}$ and $\pi_3 = \pi^{(2)}$.)

In the next result, we prove that, as in the example above, $\psi^{-1}_{p,k}$ can be applied to *any* element of $\mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$ to yield an element of $\mathcal{B}_{p,k}$, and thus deduce that $\psi_{p,k}$ is also a surjection.

Proposition 3.3. *For* $p \geqslant k - 1 \geqslant 0$, $\psi_{p,k} : \mathcal{B}_{p,k} \to \mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$ *is a surjection.*

Proof. Consider the description of $\psi^{-1}_{p,k}$ given in the proof of Proposition 3.1. Clearly, this can be applied to an arbitrary element of $\mathcal{R}_{p,p-k+1} \times \mathcal{T}_k$ up to the stage where the permutation σ remains to be determined. Now, we examine the process of determining σ:

first, element 1 is placed in $\omega^{(k)}$; then, for $i = 1, \dots, 2p-1$, element $i+1$ is placed in $\omega^{(j)}$ (equivalently, in $\pi^{(j)}$), where j satisfies the condition that $\mu(\sigma(i)) \in \sigma(\pi^{(j)})$. Thus, we will never try to place more than $|\pi^{(j)}|$ of the elements $i+1 = 2, \dots, 2p$ in $\omega^{(j)}$ for any $j = 1, \dots, k$. Moreover, because of the initial placement of element 1 in $\omega^{(k)}$, the only way in which the process can terminate prematurely (and unsuccessfully), is if we try to place $|\pi^{(k)}|$ elements from $i+1 = 2, \dots, 2p$ in $\omega^{(k)}$.

Now, vertex u is a descendant of vertex v in the reverse-labelled tree implies that the $|\pi^{(v)}|$th element in $\omega^{(v)}$ cannot be placed before the $|\pi^{(u)}|$th element of $\omega^{(u)}$ is placed. Also, there is a path from the root vertex k to every vertex in the reverse-labelled tree, so the $|\pi^{(k)}|$th element in $\omega^{(k)}$ cannot be placed until after $|\pi^{(j)}|$ elements have been placed in $|\omega^{(j)}|$ for every $j = 1, \dots, k-1$. Thus, from the fact that $|\pi^{(1)}| + \cdots + |\pi^{(k)}| = 2p$, we deduce that, of the elements $i+1 = 2, \dots, 2p$, exactly $|\pi^{(j)}|$ are placed in $|\omega^{(j)}|$, for $j = 1, \dots, k-1$, and $|\pi^{(k)}| - 1$ are placed in $|\omega^{(k)}|$. The result follows. □

From Propositions 3.1 and 3.3, we immediately deduce our main result, recorded as Theorem 1.4.

4. Additional remarks

4.1. Shift-symmetric partitions and the case $k = 2$

We begin with a result that explains the usage of *shift-symmetric* for partitions π for which μ exists with $(\mu, \pi) \in \mathcal{B}_{p,k}$.

Proposition 4.1. *Let π be a partition of $[2p]$ into k blocks, denoted by π_i, $i = 1, \dots, k$, with any indexing convention. Then there exists a pairing $\mu \in \mathcal{P}_p$ for which $(\mu, \pi) \in \mathcal{B}_{p,k}$ if and only if π satisfies the following conditions*:

(i) $|\pi_i \cap \gamma^{-1}(\pi_j)| = |\pi_j \cap \gamma^{-1}(\pi_i)|, \quad 1 \leqslant i < j \leqslant k,$
(ii) $|\pi_i \cap \gamma^{-1}(\pi_i)|$ *is even*, $\quad 1 \leqslant i \leqslant k.$

Proof. First, note that condition (3) for $(\mu, \pi) \in \mathcal{B}_{p,k}$ can be restated as saying that every block of π is invariant under the permutation $\mu\gamma^{-1}$, or, equivalently,

$$\mu(\pi_i) = \gamma^{-1}(\pi_i), \quad 1 \leqslant i \leqslant k, \tag{7}$$

since $\mu = \mu^{-1}$. But condition (7) is equivalent to:

$$\mu(\pi_i \cap \gamma^{-1}(\pi_j)) = \pi_j \cap \gamma^{-1}(\pi_i), \quad 1 \leqslant i, j \leqslant k, \tag{8}$$

where the equivalence is proved as follows. For (7)⇒(8): $\mu(\pi_i \cap \gamma^{-1}(\pi_j)) = \mu(\pi_i) \cap \mu\gamma^{-1}(\pi_j) = \gamma^{-1}(\pi_i) \cap \pi_j$. For (8)⇒(7): For every block π_i of π we have $\mu(\pi_i) = \bigcup_{j=1}^{k} \mu(\pi_i \cap \gamma^{-1}(\pi_j)) = \bigcup_{j=1}^{k} (\pi_j \cap \gamma^{-1}(\pi_i)) = \gamma^{-1}(\pi_i)$.

Thus, the necessity of (i) and (ii) follows immediately from (8), for $i < j$ because μ is a permutation, and for $i = j$ because μ is a pairing. For the sufficiency, suppose we are given π satisfying (i) and (ii). Then it is easy to construct a μ satisfying (8); simply pair the elements of $\pi_i \cap \gamma^{-1}(\pi_i)$ arbitrarily for each $i = 1, \ldots, k$, and pair the elements of $\pi_i \cap \gamma^{-1}(\pi_j)$ arbitrarily with the elements of $\pi_j \cap \gamma^{-1}(\pi_i)$, for each $1 \leqslant i < j \leqslant k$. $\square$

The proof of Proposition 4.1 also allows us to count the number of pairings μ that are compatible with a shift-symmetric π. To state this, suppose that π is a shift-symmetric partition of $[2p]$, with

$$|\pi_i \cap \gamma^{-1}(\pi_i)| = 2c_i, \quad 1 \leqslant i \leqslant k, \qquad |\pi_i \cap \gamma^{-1}(\pi_j)| = q_{i,j}, \quad 1 \leqslant i < j \leqslant k.$$

Then an immediate counting argument gives us the explicit formula

$$|\{\mu \in \mathcal{P}_p \mid (\mu, \pi) \in \mathcal{B}_{p,k}\}| = \left(\prod_{i=1}^{k}(2c_i - 1)!!\right) \cdot \left(\prod_{1 \leqslant i < j \leqslant k} q_{i,j}!\right). \tag{9}$$

Now we consider the special case $k = 2$. The following result gives an especially simple necessary and sufficient condition for a partition to be shift-symmetric in this case.

Proposition 4.2. *The partition $\pi = \{\pi_1, \pi_2\}$ with two blocks is shift-symmetric if and only if $|\pi_1 \cap \gamma^{-1}(\pi_1)|$ is even.*

Proof. The condition that $|\pi_1 \cap \gamma^{-1}(\pi_1)|$ is even is clearly necessary, from Proposition 4.1 with $k = 2$. For sufficiency, suppose that $|\pi_1 \cap \gamma^{-1}(\pi_1)| = 2c_1$, that π is a partition of $[2p]$, and that $|\pi_1| = n$. Then we immediately determine that

$$|\pi_1 \cap \gamma^{-1}(\pi_2)| = |\pi_1| - |\pi_1 \cap \gamma^{-1}(\pi_1)| = n - 2c_1$$

and similarly that $|\pi_2 \cap \gamma^{-1}(\pi_1)| = n - 2c_1$, $|\pi_2 \cap \gamma^{-1}(\pi_2)| = 2(p - n + c_1)$. Thus we conclude that π is shift-symmetric, since π satisfies conditions (i) and (ii) of Proposition 4.1. $\square$

We can give an elementary expression for $b_{p,k}$ in the case $k = 2$ by means of (9) and Proposition 4.2, as follows. First, for an arbitrary partition $\pi = \{\pi_1, \pi_2\}$ with two blocks, with the indexing convention that $1 \in \pi_2$, we can write π_1 uniquely in the form

$$\pi_1 = \{i_1 + 1, \ldots, i_1 + j_1, i_2 + j_1 + 1, \ldots, i_2 + j_2, \ldots, \\ i_m + j_{m-1} + 1, \ldots, i_m + j_m\},$$

where $1 \leqslant i_1 < \cdots < i_m \leqslant 2p - n$, and $1 \leqslant j_1 < \cdots < j_m = n$. Note that

$$|\pi_1| = j_1 + (j_2 - j_1) + \ldots + (j_m - j_{m-1}) = j_m = n$$

and that the elements of π_1 that are *not* also in $\gamma^{-1}(\pi_1)$ are precisely the elements $i_1 + j_1, i_2 + j_2, \ldots, i_m + j_m$. Thus we have $|\pi_1 \cap \gamma^{-1}(\pi_1)| = n - m$, so both m and n are arbitrary positive integers, and for π to be shift-symmetric, we need only require that $n - m$

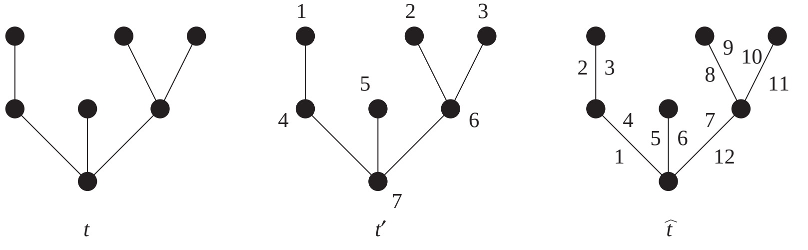

Fig. 3. Three trees $t, t', \widehat{t}$.

is even. Then (9) immediately gives

$$b_{p,2} = \sum_{n\geqslant 1} \sum_{\substack{m\geqslant 1\\ 2|n-m}} \binom{n-1}{m-1}\binom{2p-n}{m}(n-m-1)!!\,(2p-n-m-1)!!\,m!$$
$$= \sum_{n\geqslant 1} (2p-n) \sum_{\substack{m\geqslant 1\\ 2|n-m}} \binom{n-1}{m-1}\binom{2p-n-1}{m-1}$$
$$\times (n-m-1)!!\,(2p-n-m-1)!!\,(m-1)!$$

But the inner summation over m above is always equal to $(2p-3)!!$, since it equals the number of pairings on $[2p-2]$ for each fixed n – a pairing on $[2p-2]$ always pairs a subset of size $m-1$ from $[n-1]$ with a subset of size $m-1$ from $[2p-2]\setminus[n-1]$, for some unique $m\geqslant 1$, and then forms a pairing on the remaining $n-m$ elements of $[n-1]$, and a pairing on the remaining $2p-n-m$ elements of $[2p-2]\setminus[n-1]$. (Thus $n-m$ is even, and then $2p-n-m = 2(p-n)+n-m$ is also even.) This implies that

$$b_{p,2} = (2p-3)!! \sum_{n=1}^{2p-1} (2p-n) = (2p-3)!!\binom{2p}{2} = p(2p-1)!!$$

But this is exactly the expression for $b_{p,2}$ that is given by (5) in the case $k=2$, so we have been able to prove (5) for $k=2$ by elementary counting, independently of the bijection developed in Sections 2 and 3. We are unable to give an elementary explanation for $k\geqslant 3$.

4.2. A tree bijection and the case $p = k-1$

We conclude by considering the special case $p=k-1$ of our bijection. In this case we have $p-k+1=0$, so $\mathcal{R}_{p,p-k+1}$ contains the empty pairing only. Thus $\psi_{p,k}$ maps $\mathcal{B}_{p,k}$ to the set $\mathcal{T}_k$ of trees. First, we give an example of this tree bijection $\psi_{p,k}^{-1}$ in this case.

Example 4.3. In the case $p=6$ and $k=7$, consider $t\in\mathcal{T}_7$, given in Fig. 3, and apply $\psi_{6,7}^{-1}$ to (ε, t), where ε is the empty pairing.

From the reverse-labelled tree t', given in Fig. 3, proceeding as in Example 3.2, we obtain $\sigma(\mu) = \{\{1,4\},\{2,7\},\{3,8\},\{5,10\},\{6,11\},\{9,12\}\}$, and $\sigma(\pi^{(1)}) = \{1\}$, $\sigma(\pi^{(2)}) = \{2\}$,

$\sigma(\pi^{(3)}) = \{3\}$, $\sigma(\pi^{(4)}) = \{4, 5\}$, $\sigma(\pi^{(5)}) = \{6\}$, $\sigma(\pi^{(6)}) = \{7, 8, 9\}$, $\sigma(\pi^{(7)}) = \{10, 11, 12\}$, and the two-line representation of σ^{-1} follows, as given in the completed table below.

1	2	3	4	5	6	7	8	9	10	11	12
3	9	11	2	4	6	8	10	12	1	5	7

Thus we obtain $\psi_{6,7}^{-1}(\varepsilon, t) = (\mu, \pi) \in \mathcal{B}_{6,7}$, where

$$\mu = \{\{1, 4\}, \{2, 3\}, \{5, 6\}, \{7, 12\}, \{8, 9\}, \{10, 11\}\},$$
$$\pi = \{\{1, 5, 7\}, \{2, 4\}, \{3\}, \{6\}, \{8, 10, 12\}, \{9\}, \{11\}\}.$$

In general, the mapping $\psi_{k-1,k}^{-1}$ has a very simple direct description in terms of the tree t. Some notation is needed in order to give this description: for a tree $t \in \mathcal{T}_k$, traverse the outside of the tree in a clockwise direction, beginning on the left side of the edge between the root vertex and its leftmost descendant, and ending on the right-hand side of the edge between the root vertex and its rightmost descendant. In this traversal, an alternating sequence of vertices and edges will be encountered, with each edge appearing twice in the sequence, once for each side. Assign the numbers $1, \ldots, 2k-2$ to the sides of the edges, in the order that they are encountered in the traversal. For example, for the tree t considered in Example 4.3 above, the numbers assigned in the traversal of t are placed on $\widehat{t}$, as given in Fig. 3.

In terms of the numbers assigned to the sides of edges, the mapping has the following direct description: each pair in μ is the pair of numbers assigned to the two sides of an edge of t; each block of π consists of the numbers assigned to the counterclockwise side of the edges incident with a vertex in t. For example, it is straightforward to check that this description accounts for the action of the mapping in Example 4.3. The proof that this works in general is straightforward, and is omitted.

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