A Directional Total Variation

Ilker Bayram and Mustafa E. Kamasak
Istanbul Technical University

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Aims of the Work

The aims of this work are:

- To define a directional TV – modified TV that is sensitive to directions .
- To demonstrate the utility of the directional TV.
  - To develop an algorithm for the implementation of directional TV for image denoising problem.
  - To demonstrate the performance of the directional TV against regular (isotropic) TV on images with a dominant direction.

Total Variation (TV)

- TV is a commonly used prior for images
- TV of a discrete-space image f is defined as,

$$\text{TV}(f) = \sum_{i,j} \|\Delta f(i,j)\|_2$$

or

$$\text{TV}(f) = \sum_{i,j} \sup_{t \in B_2} \langle [\Delta f(i,j), t \rangle$$

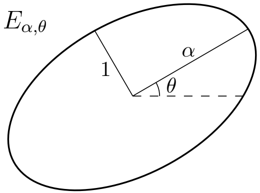
where Δ is defined as,

$$\Delta f(i,j) = \begin{bmatrix} f(i,j) - f(i-1,j) \\ f(i,j) - f(i,j-1) \end{bmatrix}$$

and B_2 is the unit ball of the ℓ_2 norm.

Directional TV

- TV is isotropic as it is invariant under a rotation in the image.
- It is possible to obtain directional TV by replacing B_2 with some other set.
- Replace B_2 with an ellipse, $E_{\alpha,\theta}$, that is characterized with the ratio of major axis to the minor axis (α) and the angle of orientation (θ).

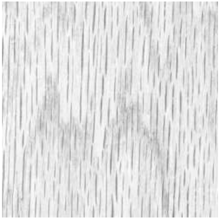


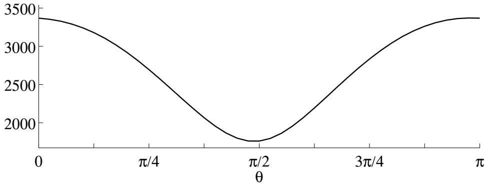
- The resulting norm is more sensitive to variations along θ.

$$\mathsf{TV}_{\alpha,\theta}(f) = \sum_{i,j} \sup_{t \in E_{\alpha,\theta}} \langle \Delta f(i,j), t \rangle$$

Directional TV

- Consider an image with a dominant direction. The $TV_{\alpha,\theta}$ with $\alpha = 3$ is:





- It is possible to use directional TV as a prior for images with a dominant direction.

Image Denoising with Directional TV

- Regular TV is used in image denoising as

$$f^* = \underset{f}{\operatorname{argmin}} \frac{1}{2} \|y - f\|_2^2 + \lambda \, TV(f)$$

- Replace $TV(f)$ with $TV_{\alpha,\theta}(f)$

$$f^* = \underset{f}{\operatorname{argmin}} \frac{1}{2} \|y - f\|_2^2 + \lambda \, TV_{\alpha,\theta}(f)$$

- Directional TV forces solutions to have less variations in the chosen direction θ.
- Amount of forcing can be adjusted with α.

Implementation of Directional TV for Image Denoising

Cost function

$$f^* = \underset{f}{\operatorname{argmin}} \frac{1}{2} \|y - f\|_2^2 + \lambda \, TV_{\alpha,\theta}(f)$$

If

$$\mathrm{TV}_{\alpha,\theta}(f) = \sup_{v(i,j) \in E_{\alpha,\theta}} \langle \Delta \, f, v \rangle$$

Proposition

For

$$v^* = \underset{v(i,j) \in E_{\alpha,\theta}}{\operatorname{argmin}} \left\| f - \lambda \, \Delta^T \, v \right\|_2^2,$$

set $P_f = \lambda \Delta^T v^*$. Then, $f - P_f$ minimizes the cost function.

Implementation of Directional TV for Image Denoising

- If we define the rotation and scaling matrices R_θ, Λ_α as,

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \Lambda_\alpha = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}.$$

- Using these, $E_{\alpha,\theta}$ and B_2 are related as $E_{\alpha,\theta} = R_\theta \, \Lambda_\alpha \, B_2$.
- Note that $\mathbf{R}_\theta^T = \mathbf{R}_{-\theta}$ and $\boldsymbol{\Lambda}_\alpha^T = \boldsymbol{\Lambda}_\alpha$.
- $\mathrm{TV}_{\alpha,\theta}$ becomes

$$\begin{aligned} \mathrm{TV}_{\alpha,\theta} &= \sup_{v(i,j) \in R_\theta \, \Lambda_\alpha \, B_2} \langle \Delta \, f, v \rangle \\ &= \sup_{v(i,j) \in B_2} \langle \Delta \, f, \mathbf{R}_\theta \, \boldsymbol{\Lambda}_\alpha \, v \rangle \\ &= \sup_{v(i,j) \in B_2} \langle \boldsymbol{\Lambda}_\alpha \, \mathbf{R}_{-\theta} \Delta \, f, v \rangle. \end{aligned}$$

Implementation of Directional TV for Image Denoising

Corollary

For

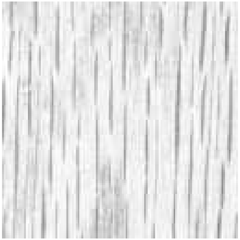
$$v^* = \operatorname*{argmin}_{v(i,j) \in B_2} \left\| f - \lambda \, \Delta^T \, \mathbf{R}_\theta \, \mathbf{\Lambda}_\alpha \, v \right\|_2^2,$$

set $P_f = \lambda \, \Delta^T \, \mathbf{R}_\theta \, \mathbf{\Lambda}_\alpha \, v^*$. *Then,* $f - P_f$ *minimizes the cost function.*

- Further details of the algorithm are given in the paper.
- Matlab code for the implementation can be found at `http://web.itu.edu.tr/ibayram/DTV/`.

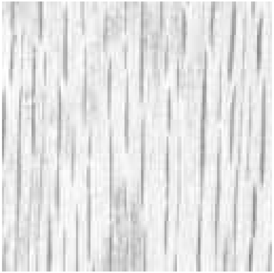
Image Denoising Experimental Setup

- Images used for denoising experiments

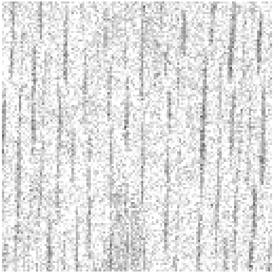


- Image pixel values are normalized to [0,1].
- Iid Gaussian noise with $\sigma = 0.1$ is added to the images.
- Images are denoised using regular TV and directional TV.
- TV parameter λ is chosen to minimize the RMSE of the denoised image.
- Parameters of directional TV (α,θ) are chosen manually.
- RMSE is used to compare the denoising results.

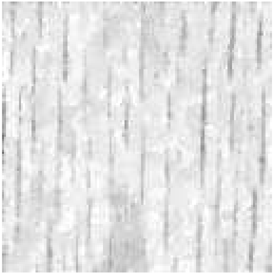
Results – Regular vs. Directional TV



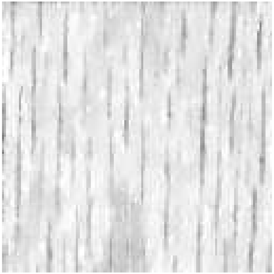
(a)Original



(b) Noisy – RMSE=0.1009



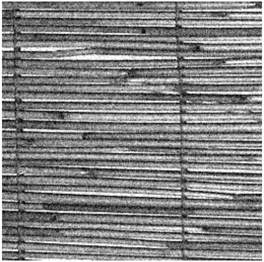
(c) TV – RMSE=0.0489



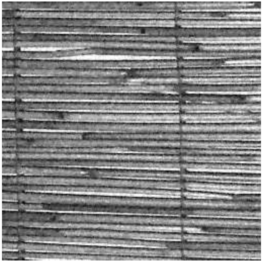
(d) $TV_{5,\pi/2}$ – RMSE=0.0429

Results – Regular vs. Directional TV

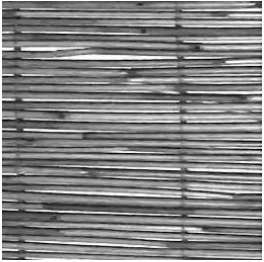
(a)Original



(b) Noisy – RMSE=0.1005

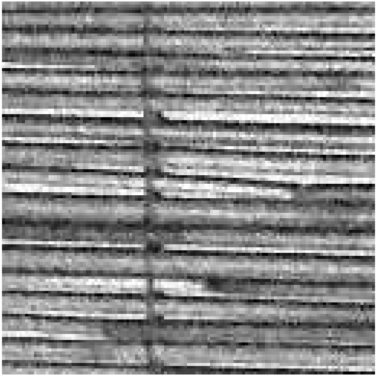


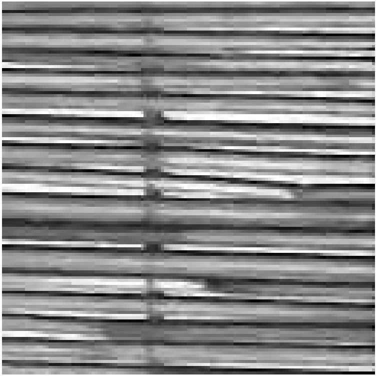
(c) TV – RMSE=0.0354



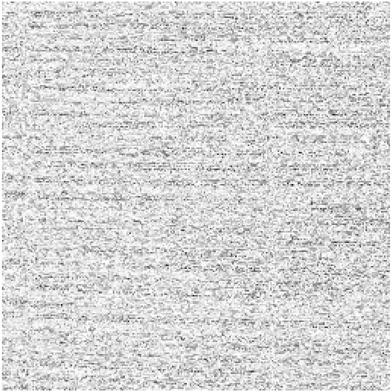
(d) $TV_{5,0}$ – RMSE=0.0279

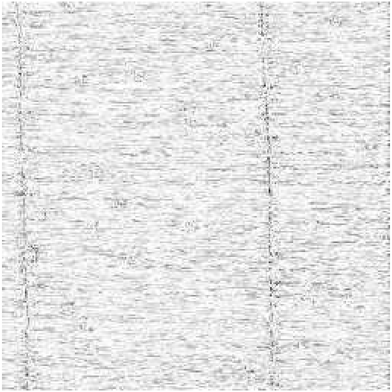
Results – Regular vs. Directional TV


(a) TV


(b) $TV_{5,0}$

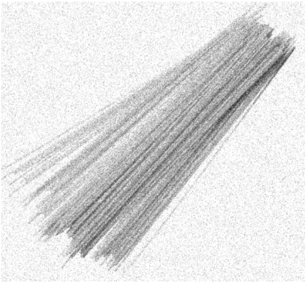
Results – Regular vs. Directional TV


(a) Difference TV

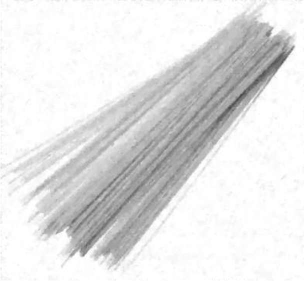

(b) Difference $TV_{5,0}$

Results – Regular vs. Directional TV

(a) Original



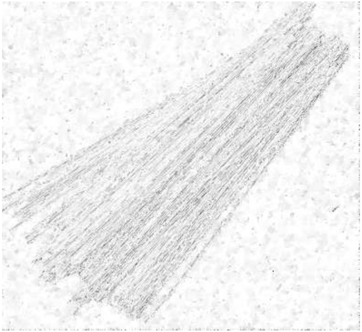
(b) Noisy – (RMSE=0.1002)



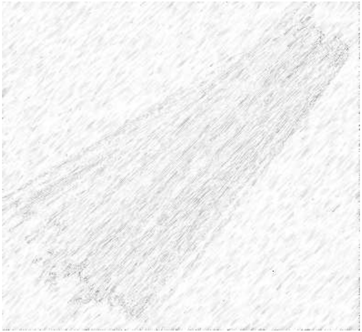
(c) TV – (RMSE=0.0431)

(d) $TV_{5,\pi/4}$ – (RMSE=0.0269)

Results – Regular vs. Directional TV



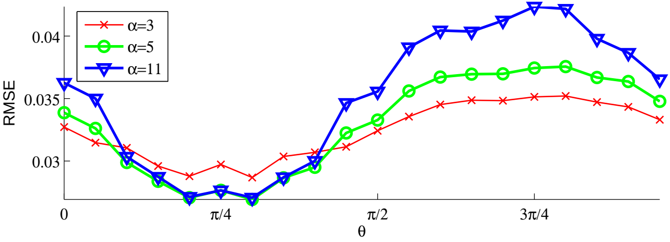
(a) Difference (TV)



(b) Difference ($TV_{5,\pi/4}$)

Effect of Parameter Selection for Directional TV

- Different parameters of directional TV (α,θ) are used to denoise the image.
- RMSE values of the denoised images are



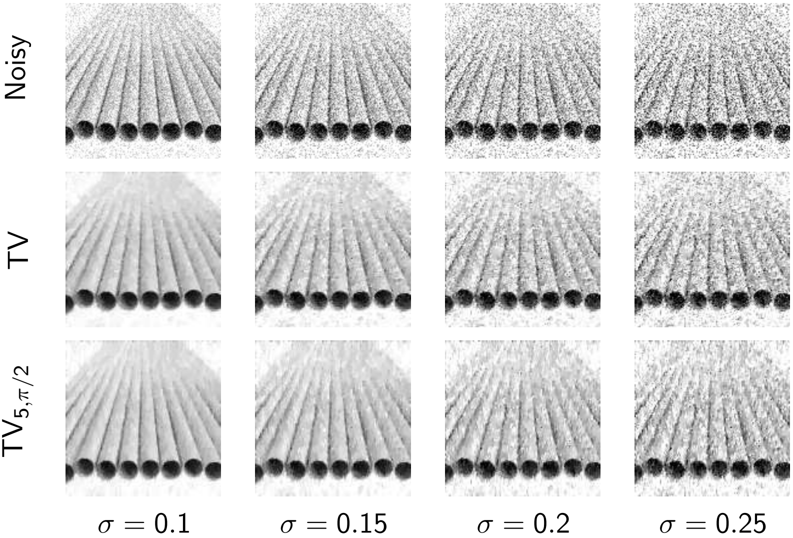
- If α is chosen small, it is not effective.
- If α is chosen to be large number, RMSE is high if θ is not correctly chosen.

Performance of TV vs Directional TV at Different Noise Levels

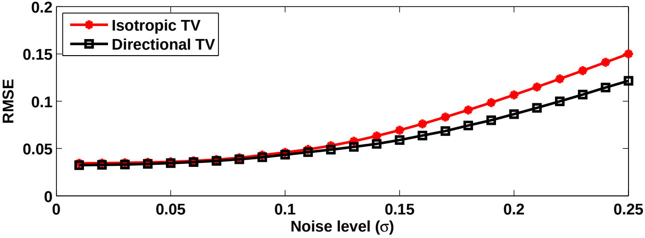
- Denoising performance of regular TV vs directional TV is investigated at different noise levels.
- Noise with $\sigma = \{0.01, 0.02, \cdots, 0.25\}$ are added to the pipe image.

- Noisy images are denoised with regular TV and directional TV.

Performance of TV vs Directional TV at Different Noise Levels



Performance of TV vs Directional TV at Different Noise Levels



- RMSE values for denoised pipes images obtained with directional TV with different α and θ parameters.
- As the noise level increases, denoising with directional TV prior outperforms the regular TV in terms of RMSE.
- The significance of the prior term increases with the level of noise.

Conclusions

- A directional TV and its implementation is described.
- Image denoising with directional TV prior outperforms regular TV in terms of RMSE on images (with a dominant direction).
- Parameters of directional TV (α,θ) has to be chosen correctly. Otherwise, the denoising fails.
- The denoising performance between directional TV and regular TV increases with level of noise.

Future Work

- Parameters of directional TV (that characterizes the ellipse) is same for all pixels.
- Limitation: Directional TV works on images with a dominant direction.
- Parameters (α,θ) can be changed for each pixel (requires edge detection).
- With such a modification, directional TV can be used for other images.
- Applications of directional TV can be extended to other applications such as sparse sample reconstruction, deconvolution etc.

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