

A Dirichlet series for Hermitian modular forms of degree 2

by

S. RAGHAVAN and J. SENGUPTA (Bombay)

Introduction. In a recent paper [5], Kohnen and Skoruppa, using the Rankin–Selberg method, investigated a novel Dirichlet series $D_{F,G}(s)$ associated with a pair F, G of Siegel cusp forms of integral weight for the modular group $\mathrm{Sp}_2(\mathbf{Z})$ of degree 2; prompted by the form of a functional equation it satisfies, they established its “proportionality” with the “spinor” zeta function $Z_F(s)$ attached to F by Andrianov, whenever F is a Hecke eigenform and G is in the Maass space.

An analogous zeta function $Z_F(s)$ (but with “Euler factors of degree 6”) for eigen cusp forms F of weight k with respect to the Hermitian modular group Γ_2 of degree 2 over $\mathcal{O} := \mathbf{Z}[\sqrt{-1}]$ has been studied by Gritsenko [2], [3]. Using the results of Kojima [6] on the “Saito–Kurokawa descent” $F \mapsto f$ for F in the relevant Maass space, Gritsenko [2] proved (for k divisible by 4) the identity

$$Z_F(s) = \zeta(s-k+1) L\left(s-k+2, \left(\frac{-4}{\cdot}\right)\right) \zeta(s-k+3) R_f(s)$$

where R_f is the (Rankin) symmetric square zeta function associated to f and $L(\cdot, \left(\frac{-4}{\cdot}\right))$ is a Dirichlet L -series for the character $\left(\frac{-4}{\cdot}\right)$.

This article is concerned with an analogue of $D_{F,G}$ for Hermitian cusp forms of weight k (divisible by 4) for Γ_2 and its possible relationship with Z_F . Even when F and G are both in the Maass space, we find that $D_{F,G}$ and Z_F are no longer “proportional”, unlike in [5]. Although both $D_{F,G}(s)$ and $Z_F(s)$ admit functional equations under $s \mapsto 2k-3-s$, one faces at the same time a disturbing difference in the respective Γ -factors involved therein. Nevertheless, we can prove that

$$D_{F,G}(s) = \text{constant} \times \frac{\zeta(s-k+2)}{L(s-k+2, \left(\frac{-4}{\cdot}\right))} Z_F(s).$$

For a fundamental operator identity needed in the arguments, §4 provides a proof (quite algebraic and) different from the rather sketchy proof in [5] for the corresponding identity there; of course, both these identities have the same

structure. We also need to work out anew, for the ‘‘Saito–Kurokawa descent’’, detailed arguments in regard to the aspect of its Hecke-equivariance which is somewhat slurred over in [6] and barely mentioned in [2].

1. Notation and terminology. For any complex matrix $P = (p_{ij})$, let $\bar{P} = (\bar{p}_{ij})$ where the bar denotes complex conjugation in \mathbb{C} ; let tP denote the transpose of P . For a square matrix P over \mathbb{C} , let $\sigma(P)$, $\det P$ and $\|P\|$ stand respectively for the trace of P , determinant of P and absolute value of $\det P$. For any $(2, 2)$ matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, we define $P^* = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix}$. By $\text{Diag}(a_1, \dots, a_n)$ we mean the (n, n) diagonal matrix with a_1, \dots, a_n as its entries in that order. For any (commutative) ring R , let $\mathcal{M}_n(R)$ denote the ring of all (n, n) matrices with entries from R . We denote the (n, n) identity matrix in $\mathcal{M}_n(\mathbb{C})$ by E_n ; by 0 , we mean a matrix of the appropriate size with all entries equal to 0 . Whenever the product tGHG of complex matrices G, H is defined, we write $H[G]$ for the same. For $t \in \mathbb{N}$, let $\mathcal{M}_2(\mathbb{Z})_t := \{M \in \mathcal{M}_2(\mathbb{Z}) \mid \det M = t\}$ and $\Gamma_1 := \mathcal{M}_2(\mathbb{Z})_1 = \text{SL}(2, \mathbb{Z})$, the elliptic modular group. For any matrix M with entries from \mathbb{Z} , let $\text{gcd}(M)$ denote the greatest common divisor of the entries of M . If $n \in \mathbb{N}$ and $n = m^2$ for m in \mathbb{Z} , we write $n = \square$.

The ring $\mathbb{Z}[\sqrt{-1}]$ of algebraic integers in $\mathbb{K} := \mathbb{Q}(\sqrt{-1})$ is denoted by \mathcal{O} ; by $\sqrt{-1}$, we always mean that square root with argument $\pi/2$. For α in \mathbb{K} its ‘‘trace’’ $\alpha + \bar{\alpha}$ is denoted by $\text{tr}(\alpha)$. Moreover, for $z = (z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ and $\alpha \in \mathbb{K}$, we shall simply write $\text{tr}(\alpha z)$ for $\alpha z_1 + \bar{\alpha} z_2$, without risk of confusion. By a positive-definite matrix H , we mean a complex matrix $H = {}^t\bar{H}$ with all its eigenvalues positive and we then write $H > 0$. For any $\gamma \in \mathbb{C}$, $e(\gamma)$ stands for $\exp(2\pi\sqrt{-1}\gamma)$.

Let $\mathcal{H} = \mathcal{H}_1$ denote the upper half-plane $\{\tau = u + \sqrt{-1}v \in \mathbb{C} \mid u, v \in \mathbb{R}, v > 0\}$. For $k \in \mathbb{N}$ and any Dirichlet character ε (on \mathbb{Z}) modulo N , the space of holomorphic cusp forms of weight k and character ε for the congruence subgroup $\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid c \in N\mathbb{Z}\}$ is denoted by $S_k(\Gamma_0(N); \varepsilon)$.

The generalized upper half-plane $\{Z \in \mathcal{M}_2(\mathbb{C}) \mid -\sqrt{-1}(Z - {}^t\bar{Z}) > 0\}$ is denoted by \mathcal{H}_2 . The Hermitian modular group Γ_2 of degree 2 over \mathcal{O} consists of all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathcal{M}_4(\mathcal{O})$ with $(2, 2)$ matrices A, B, C, D for which $A {}^t\bar{B} = B {}^t\bar{A}$, $C {}^t\bar{D} = D {}^t\bar{C}$ and $A {}^t\bar{D} - B {}^t\bar{C} = E_2$. For any such M in Γ_2 and Z in \mathcal{H}_2 , let $M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}$. The analytic homeomorphisms $Z \mapsto M\langle Z \rangle$ for M in Γ_2 give a representation of Γ_2 as a discontinuous group of automorphisms of \mathcal{H}_2 and we denote by \mathcal{F}_2 a standard fundamental domain for Γ_2 in \mathcal{H}_2 . Writing $X := \frac{1}{2}(Z + {}^t\bar{Z})$ and $Y = \text{Im}(Z) := (1/2\sqrt{-1})(Z - {}^t\bar{Z}) > 0$ for Z in \mathcal{H}_2 , we have on \mathcal{H}_2 an invariant volume element $(\det Y)^{-4} dX dY$. A complex-valued function F holomorphic on \mathcal{H}_2 is called a modular form of integral weight k for Γ_2 if $F(M\langle Z \rangle) \det(CZ + D)^{-k} = F(Z)$ for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in Γ_2 . Such an F has a Fourier expansion

$$(1) \quad F(Z) = \sum_T a(T) e(\sigma(TZ))$$

indexed by $(2, 2)$ Hermitian matrices $T = (t_{ij})$ with t_{ii} and $2t_{ij}$ in \mathcal{O} and having (only) non-negative eigenvalues. We call F a cusp form of weight k for Γ_2 if in (1), $a(T)$ vanishes for all T which are not positive-definite. The space of (holomorphic) cusp forms of weight k for Γ_2 is denoted by $S_k(\Gamma_2)$; in the following, we assume, as in [6], that k is divisible by 4.

We denote the subgroup $\{M \in \Gamma_2 \mid M \text{ has } (0 \ 0 \ 0 \ 1) \text{ as its last row}\}$ by \mathcal{C} .

Writing any Z in \mathcal{H}_2 as $Z = \begin{pmatrix} z & z' \\ z_2 & z_2' \end{pmatrix}$, the Jacobi modular group $\Gamma_1^J = \text{SL}(2, \mathbb{Z}) \rtimes \mathcal{O}^2$ acts [1] in accordance with its imbedding in Γ_2 , taking

$$(\tau, z_1, z_2) \in \mathcal{H} \times \mathbb{C} \times \mathbb{C} \quad \text{to} \quad \left(\frac{\alpha\tau + b}{c\tau + d}, \frac{z_1 + \lambda\tau + \mu}{c\tau + d}, \frac{z_2 + \bar{\lambda}\tau + \bar{\mu}}{c\tau + d} \right)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SL}(2, \mathbb{Z})$ and $(\lambda, \mu) \in \mathcal{O}^2$. For $(k, m) \in \mathbb{N}^2$, let $J_{k,m}^0$ denote the space of Jacobi cusp forms φ of weight k and index m for Γ_1^J , viz. holomorphic $\varphi: \mathcal{H} \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ such that for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\text{SL}(2, \mathbb{Z})$ and $(\lambda, \mu) \in \mathcal{O}^2$,

$$(2) \quad (\varphi|_{k,m} M)(\tau, z_1, z_2) := \varphi\left(\frac{\alpha\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, \frac{z_2}{c\tau + d}\right) e\left(-\frac{m\bar{c}z_1z_2}{c\tau + d}\right) = \varphi(\tau, z_1, z_2),$$

$$(3) \quad (\varphi|[\lambda, \mu])(\tau, z_1, z_2) := \varphi(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}) e(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2) = \varphi(\tau, z_1, z_2)$$

and the function $\varphi(\tau, z_1, z_2) e(m\tau)$ has on \mathcal{H}_2 a Fourier expansion of the form (1) indexed by $T > 0$. We note that any F in $S_k(\Gamma_2)$ has a Fourier–Jacobi expansion

$$F(Z) = \sum_{m \geq 1} \varphi_m(\tau, z_1, z_2) e(m\tau)$$

with $\varphi_m(\tau, z_1, z_2)$ in $J_{k,m}^0$ which we shall refer to as the m th Fourier–Jacobi coefficient of F .

For $(\tau, z_1, z_2) \in \mathcal{H} \times \mathbb{C} \times \mathbb{C}$, writing $\tau = u + \sqrt{-1}v$, $z_j = x_j + \sqrt{-1}y_j$ ($j = 1, 2$) with $u, v (> 0), x_1, y_1, x_2, y_2$ in \mathbb{R} , we have on $\mathcal{H} \times \mathbb{C} \times \mathbb{C}$ an invariant volume element $d\mu := v^{-4} du dv dx_1 dy_1 dx_2 dy_2$. For φ, ψ in $J_{k,m}^0$, we have the Petersson inner product

$$(4) \quad \langle \varphi, \psi \rangle := \int_{\mathcal{F}_2^J} \varphi(\tau, z_1, z_2) \overline{\psi(\tau, z_1, z_2)} v^k e^{-\pi m|z_1 - z_2|^2/v} d\mu$$

where \mathcal{F}_2^J is a fundamental domain for Γ_1^J .

Besides the zeta functions $\zeta(s) := \sum_{n \in \mathbb{N}} n^{-s}$ and $\zeta_{\mathbb{K}}(s) := \frac{1}{4} \sum_{0 \neq \lambda \in \mathcal{O}} (\lambda\bar{\lambda})^{-s}$ and the Dirichlet L -series $\sum_{n \in \mathbb{N}} \chi(n) n^{-s}$ for a Dirichlet character χ , we recall also (from [3]), the ‘‘spinor’’ zeta function $Z_F(s)$ defined, for any Hecke eigenform F in $S_k(\Gamma_2)$, by

$$Z_F(s) := \prod_p (1 + p^{k-2-s})^{-2} Q_{F,p}^{(2)}(p^{-s})^{-1} \prod_p Q_{F,p}^{(2)}(p^{-s})^{-1}$$

with the accent on the first infinite product indicating that p (in N) runs over all primes for which $p\mathcal{O}$ is prime in \mathcal{O} , while the second product is extended over all the remaining primes from N . Moreover, in the foregoing, the polynomial $Q_{F,p}^{(2)}(t)$ with t indeterminate is defined by $Q_p^{(2)}(t)F = Q_{F,p}^{(2)}(t)F$ for the following explicitly given operator-valued polynomials $Q_p^{(2)}(t)$ involving the standard operators

$$T_p := \Gamma_2 \text{Diag}(1, 1, p, p) \Gamma_2, \quad T_{1,p} := \Gamma_2 \text{Diag}(1, p, p^2, p) \Gamma_2,$$

$$T_\pi := \Gamma_2 \text{Diag}(1, \pi, p, \pi) \Gamma_2 \quad \text{for } p = \pi\bar{\pi} \text{ in } \mathcal{O},$$

$$A_\lambda := \Gamma_2(\lambda E_4) \Gamma_2 \quad \text{for } \lambda \in \mathcal{O}$$

in the Hecke ring for Γ_2 :

$$(i) \quad Q_p^{(2)}(t) := 1 - T_p t + (pT_{1,p} + p(p^3 + p^2 - p + 1)A_p)t^2 - p^4 A_p T_p t^3 + p^8 A_p^2 t^4$$

for $p\mathcal{O}$ prime in \mathcal{O} , i.e. $(\frac{-4}{p}) = -1$,

$$(ii) \quad Q_p^{(2)}(t) := 1 - T_p t + p(T_\pi T_\pi - p^4 A_p)t^2 - p^3(T_\pi^2 A_\pi + T_\pi^2 A_\pi - 2pA_p T_p)t^3$$

+ $p^4 A_p(pT_\pi T_\pi - p^4 A_p)t^4 - p^8 A_p^2 T_p t^5 + p^{12} A_p^3 t^6$,

for $p = \pi\bar{\pi}$ in \mathcal{O} , i.e. $(\frac{-4}{p}) = 1$, and

$$(iii) \quad Q_2^{(2)}(t) := 1 - (T_2 - 3A_{1+i})t + 2(T_{1+i}^2 - 8A_{1+i}(T_2 + A_{1+i}))t^2$$

- $(4A_{1+i})^2(T_2 - 3A_{1+i})t^3 + (4A_{1+i})^4 t^4$ with $i := \sqrt{-1}$.

We also know from [3] that if $\xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s)$, then

$$Z_F^*(s) := \pi^{-3s} \Gamma(s) \Gamma(s-k+3) \Gamma^2\left(\frac{s-k+3}{2}\right) \xi(s-k+2) Z_F(s) = Z_F^*(2k-3-s).$$

2. An Eisenstein series. Towards obtaining the meromorphic continuation and functional equation of $D_{F,g}(s)$, we first address ourselves precisely to these two questions but in the context of an Eisenstein series $E_s(Z)$ on \mathcal{H}_2 , of the Klingen-Siegel type, relative to the subgroup \mathcal{G} of Γ_2 . The latter is defined (cf. [4]) for Z in \mathcal{H}_2 and s in \mathbb{C} with $\text{Re}(s) > 3$, by

$$(5) \quad E_s(Z) := \sum_{M \in \mathcal{G}/\Gamma_2} (\det \text{Im}(M\langle Z \rangle)) / (\text{Im}(M\langle Z \rangle))_1^s$$

where $(\text{Im}(M\langle Z \rangle))_1$ is the first diagonal element of $\text{Im}(M\langle Z \rangle)$. Let us rewrite the general term of the series in (5) in a more convenient form, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ_2 with $(b \ a \ d \ c)$ as its last row. For any $(2, 2)$ matrix P , we have $PP^* = P^*P = (\det P)E_2$ and in particular, for $P = Y = \text{Im}(Z)$ and $P = CZ + D$; also we note that $Z^* = X^* + \sqrt{-1}Y^*$ and further $b\bar{d} + a\bar{c} - d\bar{b} - c\bar{a} = 0$. It is now easy to check that

$$\text{Im}(M\langle Z \rangle) = Y[(CZ + D)^{-1}] = \|CZ + D\|^{-2} Y[(CZ + D)^*],$$

$$(\text{Im}(M\langle Z \rangle))_1 = \|CZ + D\|^{-2} Y \left[Z^* \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} c \\ -d \end{pmatrix} \right],$$

$$(6) \quad \det Y(\text{Im}(M\langle Z \rangle))_1 / \det \text{Im}(M\langle Z \rangle)$$

$$= Y \left[X^* \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} c \\ -d \end{pmatrix} \right] + Y \left[Y^* \begin{pmatrix} a \\ -b \end{pmatrix} \right]$$

$$= (\det Y) \left(Y^{*-1} \left[X^* \begin{pmatrix} a \\ -b \end{pmatrix} + \begin{pmatrix} c \\ -d \end{pmatrix} \right] + Y^* \begin{pmatrix} a \\ -b \end{pmatrix} \right).$$

Under $M \rightarrow (0 \ 0 \ 0 \ 1)M$, we can identify \mathcal{G}/Γ_2 with

$$(7) \quad \{(a, b, c, d) \in \mathcal{O}^4 \mid a\bar{c} + b\bar{d} - c\bar{a} - d\bar{b} = 0, a\mathcal{O} + b\mathcal{O} + c\mathcal{O} + d\mathcal{O} = \mathcal{O}\}$$

which is invariant with respect to $(b, d) \mapsto (-b, -d)$. If we take $g_1 := (a \ b)$, $g_2 := (c \ d)$ and $g := (g_1 \ g_2)$ the defining conditions in (7) go over into

$$(8) \quad {}^i\bar{g}_2 g_1 - {}^i\bar{g}_1 g_2 = 0, \quad g \text{ is "primitive" over } \mathcal{O}$$

where by a "primitive" column over \mathcal{O} , we mean a column whose entries together generate \mathcal{O} . Let S, H be $(4, 4)$ Hermitian matrices defined by

$$(9) \quad S = \begin{pmatrix} 0 & \sqrt{-1}E_2 \\ -\sqrt{-1}E_2 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} Y^* & 0 \\ 0 & Y^{*-1} \end{pmatrix} \begin{pmatrix} E_2 & 0 \\ X^* & E_2 \end{pmatrix}.$$

Then $H > 0$ and it is a "majorant" of the indefinite (Hermitian) matrix S of "signature" $(2, 2)$, i.e. $HS^{-1}H = S$. After replacing (b, d) by $(-b, -d)$ in (6), we see that

$$(10) \quad (\text{Im}(M\langle Z \rangle))_1 / \det \text{Im}(M\langle Z \rangle) = H[g]$$

with, of course g "primitive" and $S[g] = 0$, by (8). Any non-zero column over \mathcal{O} becomes "primitive" under multiplication by λ^{-1} for a suitable $\lambda \neq 0$ from \mathcal{O} ; thus, from (5), (10) and multiplication by $\zeta_{\mathbf{K}}(s)$ to eliminate the inconvenient "primitivity" condition in (8), we obtain

$$(11) \quad \zeta_{\mathbf{K}}(s) E_s(Z) = \sum'_g (H[g])^{-s}$$

where the accent indicates that g runs over all non-zero $(4, 1)$ columns over \mathcal{O} with $S[g] = 0$. This formula provides a link with a theta series associated with S and H and the possibility of using towards our own objectives, its (transformation) properties, by exhibiting (11) as a "transform" of the theta series; tackling the condition $S[g] = 0$ has to be done at the same time, by adopting an idea from [3].

For S, H in (9) and $\tau = u + \sqrt{-1}v \in \mathcal{H}$ with $u, v \in \mathbb{R}$, let us consider the theta series

$$\vartheta(\tau) = \vartheta(\tau, Z) := \sum'_g e(\frac{1}{2}(uS[g] + \sqrt{-1}vH[g]))$$

where g runs over all $(4, 1)$ columns over \mathcal{O} ; the absolute convergence of the series is ensured by " $H > 0$ ". There exists an invertible $(4, 4)$ complex matrix V such that $H = E_4[V]$ and $S = D[V]$ with D diagonal; also $D^2 = E_4$, in view of the relation $(HS^{-1})^2 = E_4$. Since S has "signature" $(2, 2)$, we may, after multiplying V on the left by an appropriate permutation matrix, assume already that $D = \text{Diag}(1, 1, -1, -1)$. If now $K = K(\tau) := vH - \sqrt{-1}uS$, then it is easily verified that

$$K^{-1} = \frac{v}{v^2 + u^2} H^{-1} + \frac{\sqrt{-1}u}{v^2 + u^2} S^{-1} = (K(-1/\tau)) [S^{-1}], \quad \det K = (u^2 + v^2)^2.$$

We are thus led to the theta transformation formula

$$\begin{aligned} (12) \quad \vartheta(-1/\tau, Z) &= \sum_g \exp(-\pi K(-1/\tau)[g]) = \sum_g \exp(-\pi K^{-1}[S^{-1}g]) \\ &= \sum_g \exp(-\pi K^{-1}[g]) \quad \text{since } S \in \text{GL}(4, \mathcal{O}) \\ &= (\det K) \sum_g \exp(-\pi K[g]) = |\tau|^4 \vartheta(\tau, Z). \end{aligned}$$

This transformation formula can also be proved by going over to the theta series associated with the $(8, 8)$ matrix $vP - \sqrt{-1}uQ$ and the lattice Z^8 (in lieu of \mathcal{O}^4 as above) where

$$2P := {}^t A \begin{pmatrix} 0 & \bar{H} \\ H & 0 \end{pmatrix} A, \quad 2Q := {}^t A \begin{pmatrix} 0 & \bar{S} \\ S & 0 \end{pmatrix} A, \quad A := \begin{pmatrix} E_4 & E_4 \\ \sqrt{-1}E_4 & -\sqrt{-1}E_4 \end{pmatrix}$$

and applying the well-known theta transformation formula from ([6], §2). Now, for any $(4, 1)$ column g over \mathcal{O} , $S[g]$ is always in $2Z$ and $s^0 \vartheta(\tau+1) = \vartheta(\tau)$. This together with the transformation formula (12) implies the invariance of $v^2 \vartheta(\tau, Z)$ for every substitution $\tau \mapsto (a\tau + b)(c\tau + d)^{-1}$ from Γ_1 . Consequently, the same invariance holds good also for

$$\Theta(\tau, Z) := L(v^2 \vartheta(\tau, Z)) + 2v^2 \vartheta(\tau, Z)$$

where L is the invariant differential operator $-v^2(\partial^2/\partial u^2 + \partial^2/\partial v^2)$. Writing α, β for $H[g], S[g]$ respectively and taking $f(u, v) := \exp(-\pi\alpha v + \pi\sqrt{-1}\beta u)$, it is immediate that

$$(13) \quad L(v^2 f(u, v)) + 2v^2 f(u, v) = (4\pi\alpha v^3 - \pi^2(\alpha^2 - \beta^2)v^4) f(u, v).$$

For s in C with large enough $\text{Re}(s)$, integration over a standard fundamental domain for the subgroup $\Gamma_{1,\infty} := \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_1 \}$ in \mathcal{H} yields

$$\begin{aligned} (14) \quad \int_0^1 \int_0^1 v^{s-1} \Theta(\tau, Z) v^{-2} du dv &= \int_0^1 v^{s-3} dv \int_0^1 \sum_g (4\pi H[g] v^3 - \pi^2 (H[g]^2 - S[g]^2) v^4) \\ &\quad \times \exp(-\pi v H[g] + 2\pi \sqrt{-1} u S[g]) du, \quad \text{by (13)} \\ &= \sum_{\substack{0 \neq g \\ S[g]=0}} \int_0^1 \exp(-\pi v H[g]) \\ &\quad \times (4\pi H[g] v^{s+1} - \pi^2 H[g]^2 v^{s+2}) \frac{dv}{v} \\ &= \pi^{-s} s(3-s) \Gamma(s) \zeta_{\mathbf{K}}(s) E_s(Z), \quad \text{in view of (11)}. \end{aligned}$$

On "folding back" the domain of integration for the left-hand side to recover the (usual) fundamental domain \mathcal{F}_1 for Γ_1 in \mathcal{H} , we see that the left-hand side is nothing but

$$\int_{\mathcal{F}_1} \Theta(\tau, Z) E_{\Gamma_1}(\tau, 2s-2) v^{-2} du dv$$

where

$$E_{\Gamma_1}(\tau, \rho) := v^{\rho/2} \sum_{(c,d) \in \Gamma_{1,\infty} \backslash \Gamma_1} |c\tau + d|^{-\rho}$$

is the usual Eisenstein series for Γ_1 . It is well known that $\pi^{-\rho/2} \Gamma(\rho/2) \zeta(\rho) E_{\Gamma_1}(\tau, \rho)$ is holomorphic in the entire ρ -plane except for simple poles at $\rho = 0, 2$ and is further invariant under $\rho \mapsto 2 - \rho$. For $\tau \in \mathcal{F}_1$ and s in any compact set, we have

$$|E_{\Gamma_1}(\tau, 2s-2)| \leq c_1 v^s, \quad |\Theta(\tau, Z)| \leq c_2 \exp(-c_3 v)$$

for suitable constants v, c_1, c_2, c_3 independent of τ . These facts give rise (as usual) to the required meromorphic continuation as well as a functional equation under $s \mapsto 3-s$ (or correspondingly $2s-2 \mapsto 4-2s$). Multiplying the expression in (14) by $\pi^{-(s-1)} \Gamma(s-1) \zeta(2s-2)$, we see that

$$\pi^{-(2s-1)} \Gamma(s) \Gamma(s-1) s(3-s) \zeta(2s-2) \zeta_{\mathbf{K}}(s) E_s(Z)$$

is holomorphic in the entire s -plane except for possible simple poles at $s = 1, 2$ and is further invariant under $s \mapsto 3-s$. We have hence the following

LEMMA 1. The function

$$E_s^*(Z) := \pi^{-2s} \Gamma(s) \Gamma(s-1) \zeta(2s-2) \zeta_{\mathbf{K}}(s) E_s(Z)$$

admits analytic continuation in s to all of C and is holomorphic except for possible simple poles at $s = 0, 1, 2, 3$. Moreover, $E_s^*(Z) = E_{3-s}^*(Z)$.

Remark. For Z in \mathcal{F}_2 (and hence, due to Γ_2 -invariance, also for all Z in \mathcal{H}_2), $E_s(Z)$ behaves at most like a power of $\det(\text{Im}(Z))$ as the latter goes to infinity.

3. The Dirichlet series $D_{F,G}(s)$. For any two holomorphic cusp forms F, G of integral weight k for the Hermitian modular group Γ_2 over \mathcal{O} , we associate, following [5], a Dirichlet series $D_{F,G}(s)$ of the Rankin–Selberg type. Namely, if

$$F(Z) = \sum_{m \in \mathbb{N}} \varphi_m(\tau, z_1, z_2) e(m\tau') \quad \text{and} \quad G(Z) = \sum_{n \in \mathbb{N}} \psi_n(\tau, z_1, z_2) e(n\tau')$$

we first consider, for s in \mathbb{C} with $\text{Re}(s) > k+1$, the Dirichlet series $\sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s}$ whose absolute convergence is immediate from " $\langle \varphi_m, \psi_m \rangle = O(m^k)$ ". To get this estimate, we start from the formula $\varphi_m(\tau, z_1, z_2) = \int_{\gamma}^{+1} F(Z) e(-m\tau') d\tau'$ with $\gamma = \sqrt{-1}c$ and any $c > 0$ and note that for all Z in \mathcal{H}_2 , $(\det Y)^{k/2} F(Z) = (vv' - |z_1 - \bar{z}_2|^2/4)^{k/2} F(Z)$ is bounded where $v := \text{Im}(\tau)$, $v' := \text{Im}(\tau')$. We then choose $c = (|z_1 - \bar{z}_2|^2/4v) + 1/m$ and are led, from the foregoing, to the estimate

$$\varphi_m(\tau, z_1, z_2) = O((v/m)^{-k/2} \exp(\pi m |z_1 - \bar{z}_2|^2/2v))$$

and eventually to conclude that

$$\varphi_m(\tau, z_1, z_2) \psi_m(\tau, z_1, z_2) v^k \exp(-\pi m |z_1 - \bar{z}_2|^2/v) = O(m^k) \quad \text{on } \mathcal{F}^J,$$

yielding the estimate asserted for $\langle \varphi_m, \psi_m \rangle$ at once.

For s in \mathbb{C} with $\text{Re}(s) > k+1$, we can now define

$$D_{F,G}(s) = \zeta_{\mathbb{K}}(s-k+3) \zeta(2s-2k+4) \sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-s}$$

and proceed to state

THEOREM 1. *The Dirichlet series $D_{F,G}(s)$ associated to F, G in $S_k(\Gamma_2)$ can be continued meromorphically to the entire s -plane. The function*

$$D_{F,G}^*(s) := (4\pi^3)^{-s} \Gamma(s) \Gamma(s-k+2) \Gamma(s-k+3) D_{F,G}(s)$$

is holomorphic in s except for possible simple poles at $s = k-3, k-2, k-1, k$ and satisfies the functional equation $D_{F,G}^*(s) = D_{F,G}^*(2k-3-s)$.

Proof. Clearly $D_{F,G}(s)$ is holomorphic for $\text{Re}(s) > k+1$. To continue it meromorphically to the left, we first note that for F, G in $S_k(\Gamma_2)$ and $\text{Re}(s) > k+1$,

$$\langle FE_s, G \rangle := \int_{\mathcal{F}_2} (\det Y)^k F(Z) E_s(Z) \overline{G(Z)} (\det Y)^{-4} dX dY$$

is well-defined and by the (definition of $E_s(Z)$ and the) usual "unfolding argument", it is equal to

$$\int_{\mathcal{G} \setminus \mathcal{H}_2} F(Z) \overline{G(Z)} v^{-s} (\det Y)^{k-4+s} dX dY.$$

Now, a fundamental domain for \mathcal{G} in \mathcal{H}_2 is given by the set

$$\left\{ Z = \begin{pmatrix} \tau & z_1 \\ z_2 & \tau' \end{pmatrix} \in \mathcal{H}_2 \mid \begin{array}{l} (\tau, z_1, z_2) \in \mathcal{F}^J, \\ \tau' = u' + \sqrt{-1}v', v' > |z_1 - \bar{z}_2|^2/4v, 0 \leq u' \leq 1 \end{array} \right\}.$$

Hence $\langle FE_s, G \rangle$ equals

$$\begin{aligned} (15) \quad & \int_{\mathcal{F}^J} \int_{v' > |z_1 - \bar{z}_2|^2/4v} du dv dv' dx_1 dy_1 dx_2 dy_2 \int_0^1 \sum_{m, n \in \mathbb{N}} \varphi_m(\tau, z_1, z_2) \overline{\psi_n(\tau, z_1, z_2)} \\ & \times \exp(-2\pi(m+n)v') e((m-n)u') v^{k-4} \left(v' - \frac{|z_1 - \bar{z}_2|^2}{4v} \right)^{k-4+s} du' \\ & = \int_{\mathcal{F}^J} \sum_{m \in \mathbb{N}} \varphi_m(\tau, z_1, z_2) \overline{\psi_m(\tau, z_1, z_2)} \exp(-\pi m |z_1 - \bar{z}_2|^2/v) v^k d\mu \\ & \quad \times \int_0^\infty \exp(-4\pi nt) t^{k-4+s} dt \quad (t := v' - |z_1 - \bar{z}_2|^2/4v) \\ & = (4\pi)^{-(s+k-3)} \Gamma(s+k-3) \sum_{m \in \mathbb{N}} \langle \varphi_m, \psi_m \rangle m^{-(s+k-3)}, \end{aligned}$$

as the interchange of the summation over m and the integration over \mathcal{F}^J is easily justified. From the "rapid decay at infinity" of F and G and the "polynomial growth" of E_s in \mathcal{F}_2 (see the Remark following Lemma 1), the left-hand side of (15) represents a meromorphic function of s all over \mathbb{C} and provides the meromorphic continuation required for $D_{F,G}$, as well. The asserted properties of $D_{F,G}^*(s) = \pi^{6-2k} \langle FE_{s-k+3}^*, G \rangle$ by (15) including its functional equation under $s \mapsto 2k-3-s$ are now immediate, proving Theorem 1.

4. A proposition. This section is devoted to the proof of a basic identity (needed in §6) involving certain operators \mathcal{V}_l and \mathcal{F}_l in the spaces $J_{k,1}^0$. For $l \in \mathbb{N}$, the operators $\mathcal{V}_l: J_{k,1}^0 \rightarrow J_{k,1}^0$ and $\mathcal{F}_l: J_{k,1}^0 \rightarrow J_{k,1}^0$ are defined as follows: namely, for any φ in $J_{k,1}^0$ (cf. [1], [6]),

$$\begin{aligned} (16) \quad & (\varphi|_{k,1} \mathcal{V}_l)(\tau, z_1, z_2) \\ & := l^{k-1} \sum_{s=(c,d) \in \Gamma_1 \setminus \mathcal{H}_1(\mathbb{Z})} e\left(-\frac{lcz_1 z_2}{c\tau+d}\right) \varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{lz_1}{c\tau+d}, \frac{lz_2}{c\tau+d}\right) (c\tau+d)^{-k}, \\ & (\varphi|_{k,1} \mathcal{F}_l)(\tau, z_1, z_2) := l^{-4} \sum_{\lambda, \mu \in \mathcal{O}/l\mathcal{O}} e(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2) \cdot l^{k-2} \\ & \quad \times \sum_{\substack{R \in \Gamma_1 \setminus \mathcal{H}_1(\mathbb{Z}), \\ \text{gcd}(R) = \square}} (\varphi|_{k,1} R)(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}). \end{aligned}$$

For the representatives S, R here we can choose ones in the triangular form $\begin{pmatrix} * & \\ & * \end{pmatrix}$. The definition of the operator $\mathcal{T}_l^0: J_{k,1}^0 \rightarrow J_{k,1}^0$ is the same as that of \mathcal{T}_l , except that the condition $\gcd(R) = \square$ is replaced by $\gcd(R) = 1$. We have between \mathcal{T}_l and \mathcal{T}_l^0 the identity $\mathcal{T}_l = \sum_{d^2|l} d^{2k-4} \mathcal{T}_{l/d^2}^0$ just as in [1].

For the adjoint operator $\mathcal{V}_l^*: J_{k,1}^0 \rightarrow J_{k,1}^0$ corresponding to \mathcal{V}_l , we have the formula

$$(17) \quad (\psi|\mathcal{V}_l^*)(\tau, z_1, z_2) = l^{k-5} \sum_{\lambda, \mu \in \mathcal{O}/l\mathcal{O}} e(l(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2)) \times \sum_{S = \begin{pmatrix} * & \\ & * \end{pmatrix} \in \Gamma_1 \backslash \mathcal{M}_2(\mathcal{Z})_l} \psi\left(\frac{a\tau + b}{d}, \frac{z_1 + \lambda\tau + \mu}{d}, \frac{z_2 + \bar{\lambda}\tau + \bar{\mu}}{d}\right) d^{-k}.$$

We omit its proof, since it is on the same lines as in [5].

The following proposition deals with an identity connecting $\mathcal{V}_l, \mathcal{V}_l^*$ with the Hecke operators \mathcal{T}_r on $J_{k,1}^0$ and it corresponds to assertion (ii) of a proposition due to Kohlen and Skoruppa ([5], p. 549), where, however, a good part of the details in the proof has been left to the reader. Our proof is different and algebraic in nature.

PROPOSITION. For $l \in N$,

$$\mathcal{V}_l^* \mathcal{V}_l = \sum_{1 \leq t|l} \psi(l/t) (l/t)^{k-2} \mathcal{T}_t \quad \text{on } J_{k,1}^0,$$

where, for any $r \in N$,

$$\psi(r) := r \prod_{p|r} (1 + 1/p),$$

the product being extended over all primes p dividing r .

Proof. It is clear from the definition (16) of \mathcal{V}_l (taking the representatives S in upper triangular form as is more convenient) that for $l = l_1 l_2$ and $(l_1, l_2) = 1, \mathcal{V}_l = \mathcal{V}_{l_1} \mathcal{V}_{l_2} = \mathcal{V}_{l_2} \mathcal{V}_{l_1}$ and $\mathcal{V}_l^* = \mathcal{V}_{l_2}^* \mathcal{V}_{l_1}^* = \mathcal{V}_{l_1}^* \mathcal{V}_{l_2}^*$. Moreover, the mapping $t \mapsto t^{k-2} \psi(t)$ from N is multiplicative. It is not hard to verify directly from the definitions that, for $(l_1, l_2) = 1, \mathcal{T}_{l_1}$ commutes both with \mathcal{V}_{l_2} and with \mathcal{T}_{l_2} (cf. [1], p. 51). Thus it suffices to prove the proposition for $l = p^n$ for any given prime number p ; we assume $l = p^n$ in the sequel. From (16) and (17), we have

$$(18) \quad ((\mathcal{V}_l^* \mathcal{V}_l)(\varphi))(\tau, z_1, z_2) = l^{2k-6} \sum_{\substack{\lambda, \mu \in \mathcal{O}/l\mathcal{O} \\ S = \begin{pmatrix} * & \\ & * \end{pmatrix} \in \Gamma_1 \backslash \mathcal{M}_2(\mathcal{Z}), i=1,2}} e(l(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2)) \times \varphi\left(\frac{a_1 a_2 \tau + a_1 b_2 + a_2 b_1}{d_1 d_2}, \frac{l(z_1 + \lambda\tau + \mu)}{d_1 d_2}, \frac{l(z_2 + \bar{\lambda}\tau + \bar{\mu})}{d_1 d_2}\right) (d_1 d_2)^{-k}$$

$$= l^{k-6} \sum_{\lambda, \mu} e(l(\lambda\bar{\lambda}\tau + \bar{\lambda}z_1 + \lambda z_2)) \sum_{S_1, S_2} (\varphi|_{k,1} S_1 S_2)(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu}) = \frac{1}{l^4} \sum_{\lambda, \mu \in \mathcal{O}/l\mathcal{O}} \left((l^{k-2} \sum_{S_1, S_2} \varphi|_{k,1} S_1 S_2 | [\lambda, \mu])(\tau, z_1, z_2) \right)$$

where λ, μ run independently over a complete system of residues of \mathcal{O} modulo $l\mathcal{O}$ and S_1, S_2 run again independently over

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid 1 \leq d|l, 0 \leq b < d, ad = l \right\} \simeq \Gamma_1 \backslash \mathcal{M}_2(\mathcal{Z})_l.$$

For given $r \in \mathcal{Z}$ with $0 \leq r \leq n$, let

$$\mathcal{R}(r, n-r) = \mathcal{R}_p(r, n-r) := \left\{ \begin{pmatrix} p^r & a \\ 0 & p^{n-r} \end{pmatrix} \mid 0 \leq a < p^{n-r}, a \in \mathcal{Z} \right\}$$

and $\mathcal{R}^*(r, n-r) = \mathcal{R}_p^*(r, n-r)$ the subset for which $\gcd(a, p^r, p^{n-r}) = 1$. For $r = 0$ or n , we clearly have $\mathcal{R}(r, n-r) = \mathcal{R}^*(r, n-r)$. To deal further with (18), we need to know the structure of

$$\mathcal{S}_{r,s} := \mathcal{R}(r, n-r) \cdot \mathcal{R}(s, n-s) := \{AB \mid A \in \mathcal{R}(r, n-r), B \in \mathcal{R}(s, n-s)\}.$$

We now show that indeed modulo factors from Γ_1 on the left

$$(19) \quad \mathcal{R}(r, n-r) \cdot \mathcal{R}(s, n-s) = \begin{cases} \perp \prod_{0 \leq u \leq s} p^{r+u} \mathcal{R}^*(s-u, 2n-2r-s-u) & \text{for } r+s \leq n, \\ \perp \prod_{0 \leq v \leq n-r} p^{n-s+v} \mathcal{R}^*(r+2s-n-v, n-r-v) & \text{for } r+s \geq n \end{cases}$$

where \perp signifies that the sets involved occur with the indicated multiplicity $*$ and $t\mathcal{R}^*(a, b) := \{tA \mid A \in \mathcal{R}^*(a, b)\}$. The left-hand side is just $\{M = \begin{pmatrix} p^{r+s} & ap^{n-s-r} + bp^r \\ 0 & p^{2n-2r-s} \end{pmatrix} \mid 0 \leq a < p^{n-r}, 0 \leq b < p^{n-s}\}$. First, when $r+s = n$, we see that the general element M becomes $p^r \begin{pmatrix} p^s & a + bp^{r+s-n} \\ 0 & p^{n-r} \end{pmatrix}$; (modulo factors from Γ_1 on the left) this covers $p^r \cdot \mathcal{R}(s, s)$ exactly once, for each fixed b , as a runs over $\mathcal{Z}/p^{n-r}\mathcal{Z}$. Hence, for $r+s = n$,

$$\mathcal{S}_{r,s} = \perp p^r \cdot \mathcal{R}(s, s) = \perp \prod_{0 \leq u \leq s} p^{r+u} \mathcal{R}^*(p^{s-u}, p^{s-u})$$

Proving (19) in this case. When $r+s < n$ (respectively $r+s > n$), we have

$$M = p^r \begin{pmatrix} p^s & ap^{n-s-r} + b \\ 0 & p^{2n-2r-s} \end{pmatrix}, \quad \text{resp. } M = p^{n-s} \begin{pmatrix} p^{2s+r-n} & a + bp^{r+s-n} \\ 0 & p^{n-r} \end{pmatrix}.$$

Writing $b = b_1 + p^{n-s-r} b_2$ with $b_1 \in \mathcal{Z}/p^{n-s-r}\mathcal{Z}$ and $b_2 \in \mathcal{Z}/p^r\mathcal{Z}$, $ap^{n-s-r} + b$

$= (b_1 + ap^{n-s-r}) + b_2 p^{n-s-r} = c + b_2 p^{n-s-r}$ covers, for each fixed b_2 , the elements of $Z/p^{2n-2r-s}Z$ once, as a and b_1 run respectively over $Z/p^{n-r}Z$ and $Z/p^{n-s-r}Z$. Thus, for $r+s < n$,

$$\mathcal{S}_{r,s} = \perp p^r \mathcal{R}(s, 2n-r-s).$$

If $r+s > n$ on the other hand, $a + bp^{r+s-n}$ covers $Z/p^{n-r}Z$ once along with a , for each fixed b from $Z/p^{n-s}Z$ and hence

$$\mathcal{S}_{r,s} = \perp p^{n-s} \mathcal{R}(2s+r-n, n-r).$$

The proof of (19) is complete, since $\mathcal{R}(e, f) = \bigsqcup_{0 \leq g \leq e} p^g \mathcal{R}^*(e-g, f-g)$ for $e \leq f$.

Remark. The second relation in (19) can formally be obtained from the first by using $(r, s, u) \mapsto (n-s, n-r, v)$ followed by $\mathcal{R}^*(a, b) \mapsto \mathcal{R}^*(b, a)$. Thus $\mathcal{S}_{r',s'}$ with $r'+s' > n$ are determined (already) by $\mathcal{S}_{r,s}$ with $r+s < n$.

For $l^{k-2} \sum_{S_1, S_2} \varphi|_{k,1} S_1 S_2$ in (18), we get by (19) the expression

$$(20) \quad p^{n(k-2)} \left\{ \sum_{\substack{0 \leq r,s \\ r+s \leq n}} p^r \sum_{\substack{0 \leq u \leq s \\ M_1}} \varphi|p^{r+u} M_1 + \sum_{\substack{0 \leq r',s' \leq n \\ r'+s' > n}} p^{n-s'} \sum_{\substack{0 \leq u' \leq s' \\ M_2}} \varphi|p^{n-s'+u'} M_2 \right\}$$

where M_1 runs over $\mathcal{R}^*(s-u, 2n-2r-s-u)$ and M_2 over $\mathcal{R}(2r'+2s-n-r'-u', n-r'-u')$. Under $(r', s') \mapsto (n-s, n-r)$, the condition " $r'+s' > n$ " goes over precisely into " $r+s < n$ ". The part of the sum over r, s, u and M_1 in (20) corresponding to the condition $r+s = n$ clearly reduces under $u \mapsto v$ ($:= s-u$) to

$$(21) \quad \sum_{0 \leq r \leq n} p^r \sum_{0 \leq v \leq n-r} \sum_{M \in \mathcal{R}^*(v,v)} \varphi|p^{n-v} M = \sum_{0 \leq v \leq n} (1+p+\dots+p^{n-v}) \varphi|_{k,1} \mathcal{R}^*(v, v),$$

using the abbreviation $\varphi|p^\lambda \mathcal{R}^*(\cdot, \cdot)$ for $\sum_{M \in \mathcal{R}^*(\cdot, \cdot)} \varphi|_{k,1} p^\lambda M$ and noting that $\varphi|_{k,1} p^\lambda M = \varphi|_{k,1} M$. The rest of the sum over r, s, u, M_1 as well as over r', s', u', M_2 in (20) yields in all

$$(22) \quad \sum_{\substack{0 \leq r,s \\ r+s < n}} p^r \sum_{0 \leq u \leq s} (\varphi|_{k,1} \mathcal{R}^*(s-u, 2n-2r-s-u) + \varphi|_{k,1} \mathcal{R}^*(2n-2r-s-u, s-u)) = \sum_{t=0}^{n-1} \sum_{\substack{0 \leq r,v \\ r+v \leq t}} p^r (\varphi| \mathcal{R}^*(v, 2(n-t)+v) + \varphi| \mathcal{R}^*(2(n-t)+v, v))$$

(with $t := r+s, v := s-u$)

$$= \sum_{w=1}^n \sum_{\substack{0 \leq r,v \\ w+r+v \leq n}} p^r (\varphi| \mathcal{R}^*(v, v+2w) + \varphi| \mathcal{R}^*(v+2w, v)) \quad (\text{with } w := n-t) = \sum_{\substack{1 \leq w, 0 \leq v \\ v+w \leq n}} \left(\sum_{0 \leq r \leq n-v-w} p^r \right) (\varphi| \mathcal{R}^*(v, v+2w) + \varphi| \mathcal{R}^*(v+2w, v)).$$

Hence, for the innermost sum over S_1, S_2 in (18), we have, in view of (20)–(22), derived the expression

$$(23) \quad p^{n(k-2)} \sum_{\substack{0 \leq v,w \\ v+w \leq n}} (1+p+\dots+p^{n-v-w}) (\varphi|_{k,1} \mathcal{R}^*(v, v+2w) + \varphi|_{k,1} \mathcal{R}^*(v+2w, v)) \frac{1}{1+\delta}$$

with $\delta = \delta_{w,0} := 1$ for $w = 0$ and 0, otherwise. Taking $a = n-v-w, b = v$ and $c = v+2w$, (23) becomes

$$(24) \quad p^{n(k-2)} \sum_{\substack{0 \leq a,b,c \\ 2a+b+c=2n}} (1+p+\dots+p^a) \varphi|_{k,1} \mathcal{R}^*(b, c) = p^{n(k-2)} \sum_{\substack{0 \leq a,b,c \\ 2a+b+c=2n}} \{ \psi(p^a) + \psi(p^{a-2}) + \psi(p^{a-4}) + \dots + \psi(p^{1-(-1)^a/2}) \} \varphi|_{k,1} \mathcal{R}^*(b, c)$$

$$= p^{n(k-2)} \sum_{0 \leq d \leq n} \psi(p^d) \sum_{\substack{0 \leq b,c,r \in \mathbb{Z} \\ b+c+4r=2(n-d)}} \varphi|_{k,1} \mathcal{R}^*(b, c),$$

collecting together $\varphi| \mathcal{R}^*(\cdot, \cdot)$ corresponding to the same d

$$= \sum_{0 \leq d \leq n} \psi(p^{n-d}) p^{(n-d)(k-2)} p^{d(k-2)} \sum_{\substack{0 \leq b,c,r \in \mathbb{Z} \\ b+c+4r=2d}} \varphi| \mathcal{R}^*(b, c).$$

We know, from the definition of \mathcal{T}_{p^d} , that, for $\varphi \in J_{k,1}^0$,

$$\mathcal{T}_{p^d}(\varphi) = p^{d(k-6)} \sum_{\lambda, \mu \in \mathcal{O}/p^d \mathcal{O}} \sum_{\substack{0 \leq b,c,r \\ b+c+4r=2d}} \varphi| \mathcal{R}^*(b, c) |[\lambda, \mu].$$

Consequently, from (18), (23) and (24), we obtain finally, for $l = p^n$,

$$(\mathcal{V}^* \mathcal{V}_l)(\varphi) = (1/l^4) \sum_{\lambda, \mu \in \mathcal{O}/l \mathcal{O}} \left(\sum_d \psi(p^{n-d}) p^{(n-d)(k-2)} p^{d(k-2)} \sum_{b,c,r} \varphi| \mathcal{R}^*(b, c) \right) |[\lambda, \mu] = \sum_{0 \leq d \leq n} \psi(p^{n-d}) p^{(n-d)(k-2)} p^{d(k-2)} \cdot (1/p^{4n}) \sum_{\lambda, \mu \in \mathcal{O}/p^n \mathcal{O}} \left(\sum_{b,c,r} \varphi| \mathcal{R}^*(b, c) \right) |[\lambda, \mu]$$

$$\begin{aligned}
 &= \sum_d \psi(p^{n-d}) p^{(n-d)(k-2)} p^{d(k-2)} \cdot (1/p^{4d}) \cdot \sum_{\lambda', \mu' \in \mathcal{O}/p^d \mathcal{O}} \left(\sum_{b, c, r} \varphi | \mathcal{R}^*(b, c) | [\lambda', \mu'] \right) \\
 &= \sum_{0 \leq d \leq n} \psi(p^{n-d}) (p^{n-d})^{k-2} \mathcal{T}_{p^d}(\varphi)
 \end{aligned}$$

and the proposition is proved.

5. The Maass space and Saito-Kurokawa descent. The subspace

$$M_k(\Gamma_2) := \{F \in S_k(\Gamma_2) \text{ for whose Fourier coefficients } a(T) \text{ in (1), there}$$

$$\text{exists } \tilde{a} \text{ defined on } \mathbf{Z} \text{ with } a\left(\begin{pmatrix} n & \alpha/2 \\ \bar{\alpha}/2 & m \end{pmatrix}\right) = \sum_{1 \leq d | (m, n, \alpha)} d^{k-1} \tilde{a}((4mn - |\alpha|^2)/d^2)\}$$

is called the *Maass space* for Γ_2 and weight k . It is stable under the action of all the Hecke operators and if φ is the first Fourier-Jacobi coefficient of F , the map $F \mapsto \varphi$ from $M_k(\Gamma_2)$ to $J_{k,1}^0$ is a Hecke-equivariant isomorphism. If F is a Hecke eigenform in $M_k(\Gamma_2)$, so is φ and let us denote by $\lambda_\varphi(l)$ the eigenvalue of φ corresponding to the Hecke operator \mathcal{T}_l . Defining the theta series

$$\theta_h(\tau, z_1, z_2) := \sum_{m \in \mathcal{O}} e(\tau |m+h|^2 + \text{tr}((m+h)z))$$

for $(\tau, z_1, z_2) \in \mathcal{H} \times \mathcal{C} \times \mathcal{C}$ and any h in $\frac{1}{2}\mathcal{O}/\mathcal{O}$, we know that any φ in $J_{k,1}^0$ can be written as a finite linear combination

$$\varphi(\tau, z_1, z_2) = \sum_{h \in \frac{1}{2}\mathcal{O}/\mathcal{O}} \varphi_h(\tau) \theta_h(\tau, z_1, z_2)$$

and the “ θ_0 -component” φ_0 of φ actually belongs to $S_{k-1}(\Gamma_0(4); \chi)$ with $\chi(n) := (\frac{-4}{n})$.

The map $\mathcal{D}: \varphi(\tau, z_1, z_2) \mapsto \varphi_0(\tau)$ from $J_{k,1}^0$ to $S_{k-1}(\Gamma_0(4); \chi)$ is injective. Defining $\tilde{\varphi}_0$ by $(\tilde{\varphi}_0|_{k-1} W_4)(\tau) = \varphi_0(\tau)$ under the Fricke involution $z \mapsto -1/4z$ corresponding to $W_4 := \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$, one obtains an isomorphism $\tilde{D}: J_{k,1}^0 \xrightarrow{\sim} S_{k-1}^+(\Gamma_0(4); \chi)$ where $S_{k-1}^+(\Gamma_0(4); \chi) := \{g \in S_{k-1}(\Gamma_0(4); \chi) \mid g(\tau) = \sum_{n \geq 1} b(n) e(n\tau), b(n) = 0 \text{ for } \chi(n) = 1\}$. This subspace has a basis given by $\{f_i - f_i^\rho \mid i = 1, 2, \dots, a\}$ where each f_i is a normalized Hecke eigenform in $S_{k-1}(\Gamma_0(4); \chi)$ in the sense of Atkin-Lehner, $f_i^\rho(\tau) := \tilde{f}_i(-\bar{\tau})$ and $a := \dim_{\mathcal{C}} S_{k-1}^+(\Gamma_0(4); \chi)$. The isomorphism $I: F \mapsto \tilde{\varphi}_0$ is precisely the “Saito-Kurokawa descent” from $M_k(\Gamma_2)$ to $S_{k-1}^+(\Gamma_0(4); \chi)$ [6].

We will show that the map \mathcal{D} is Hecke-equivariant with respect to a homomorphism, say δ , between the associated Hecke algebras and thereby compute the eigenvalues $\lambda_\varphi(l)$ of φ in $J_{k,1}^0$ under \mathcal{T}_l , where, in view of our proposition, it clearly suffices to take $l = p^n$ for prime $p \in N$ and $n \geq 1$. (This direct computation of $\lambda_\varphi(p^n)$ spares us from having to find explicit generators and relations in the local Hecke algebra for $J_{k,1}^0$.) Again, in view of the relation between \mathcal{T}_l and \mathcal{T}_l^0 , it is enough to find the eigenvalues $\lambda_\varphi^0(l)$ of φ under \mathcal{T}_l^0 ,

for $l = p^n$; along with these simplifying steps (and symbols) we shall also take, in the sequel, φ to be of the form $\mathcal{D}^{-1}(f - f^\rho)$ for $f = f_i$ with some i ($1 \leq i \leq a$).

With obvious notation, we have the following

LEMMA 2. With $l = p^n$, φ and f as above and $f(\tau) = \sum_{t \geq 1} a(t) e(t\tau)$,

$$\lambda_\varphi^0(l) = \begin{cases} a(p^{2n}) - p^{k-3} \chi(p) a(p^{2n-2}) & \text{for any odd prime } p, \\ a(2)^{2n} + \overline{a(2)^{2n}} & \text{for } p = 2. \end{cases}$$

Proof. By definition, $(\mathcal{T}_{p^n}^0 \varphi)(\tau, z_1, z_2)$ equals

$$\begin{aligned}
 &p^{n(k-6)} \sum_{0 \leq s \leq 2n} p^{k(s-n)} \sum_{u_{2n-s}; \lambda; \mu} \varphi\left(\frac{p^s \tau + u_{2n-s}}{p^{2n-s}}, \frac{z_1 + \lambda \tau + \mu}{p^{n-s}}, \frac{z_2 + \bar{\lambda} \tau + \bar{\mu}}{p^{n-s}}\right) \\
 &\hspace{15em} \times e(\lambda \bar{\lambda} \tau + \text{tr}(\bar{\lambda} z)) \\
 &= p^{n(k-6)} \sum_{0 \leq s \leq 2n} p^{k(s-n)} \sum_{u_{2n-s}; \lambda; \mu} \sum_{h \in \mathcal{O}} \sum_{m \in \mathcal{O}} \varphi_h\left(\frac{p^s \tau + u_{2n-s}}{p^{2n-s}}\right) \\
 &\hspace{15em} \times e((p^s \tau + u_{2n-s}) p^{s-2n} |m+h|^2 + (z_1 + \lambda \tau + \mu)(m+h) p^{s-n} \\
 &\hspace{15em} + (z_2 + \bar{\lambda} \tau + \bar{\mu})(\bar{m} + \bar{h}) p^{s-n} + \lambda \bar{\lambda} \tau + \text{tr}(\bar{\lambda} z))
 \end{aligned}$$

where, as in the rest of this proof, in the summations carried out, u , runs over $\mathbf{Z}/p^r \mathbf{Z}$ subject to the additional proviso that $p \nmid \chi u$, for $0 < r < 2n$, h runs over $\frac{1}{2}\mathcal{O}/\mathcal{O}$ and λ, μ run independently over $\mathcal{O}/p^n \mathcal{O}$.

We need to compute the θ_0 -component of $\mathcal{T}_{p^n}^0 \varphi$. First, we restrict ourselves to that part of the sum above defining it where $n \leq s \leq 2n$ and take up the remaining part later. For $0 \leq s \leq 2n$, the coefficient of $\varphi_h\left(\frac{p^s \tau + u_{2n-s}}{p^{2n-s}}\right)$ therein (for h, s, u_{2n-s} fixed) is, upto the factor $p^{n(k-6) + k(s-n)}$, precisely

$$\begin{aligned}
 (25) \quad &\sum_{\lambda, \mu \in \mathcal{O}} e(\tau (|p^{s-n}(m+h)|^2 + \lambda \bar{\lambda} + \text{tr}(p^{s-n}(m+h)\lambda))) \\
 &+ \text{tr}((\bar{\lambda} + p^{s-n}(m+h)z)) e(u_{2n-s} |m+h|^2 p^{s-2n} + \text{tr}((m+h) p^{s-n} \mu)).
 \end{aligned}$$

If we write $m = m_1 + p^{2n-s} m_2$ with m_2 running over \mathcal{O} and m_1 over $\mathcal{O}/p^{2n-s} \mathcal{O}$ then

$$e(u_{2n-s} |m+h|^2 p^{s-2n}) = e(u_{2n-s} |p^{s-n}(m_1+h)|^2 p^{-s})$$

is independent of m_2 and therefore the sum over m, λ, μ in (25) reduces, for $n \leq s \leq 2n$, to

$$\begin{aligned}
 &p^{2n} \sum_{\substack{m_2 \in \mathcal{O}; \lambda \\ m_1 \in \mathcal{O}/p^{2n-s} \mathcal{O}}} e(\tau (|p^n m_2 + \bar{\lambda} + p^{s-n}(m_1+h)|^2 + \text{tr}((p^n m_2 + \bar{\lambda} + p^{s-n}(m_1+h)z))) \\
 &\hspace{15em} \times e(u_{2n-s} |m_1+h|^2 p^{s-2n}).
 \end{aligned}$$

When m_2 runs over \mathcal{O} and λ over $\mathcal{O}/p^n\mathcal{O}$, $x := p^n m_2 + \bar{\lambda}$ (and likewise, for every fixed $m_1 \in \mathcal{O}/p^{2n-s}\mathcal{O}$, also $x + p^{s-n}m_1$) runs through \mathcal{O} exactly once. Thus the last-mentioned sum (for $n \leq s \leq 2n$) becomes

$$p^{2n} \sum_{x \in \mathcal{O}} e(\tau |x + p^{s-n}h|^2 + \text{tr}((x + p^{s-n}h)z)) \sum_{m_1 \in \mathcal{O}/p^{2n-s}\mathcal{O}} e(u_{2n-s} |m_1 + h|^2 p^{s-2n}).$$

Here, let us note that the series over x is nothing but $\theta_h(\tau, z_1, z_2)$ for odd p and $n \leq s \leq 2n$ or $p = 2$ with $s = n$ and $\theta_0(\tau, z_1, z_2)$ for $p = 2$ and $s > n$. The sum over m_1 is a generalized Gauss sum which can be evaluated. In fact, it can be shown that, for $p \nmid u \in \mathbf{Z}$, $r \geq 0$ and $h \in \frac{1}{2}\mathcal{O}/\mathcal{O}$,

$$G(u, h; p^r) := \sum_{m \in \mathcal{O}/p^r\mathcal{O}} e(u |m + h|^2 p^{-r}) = \begin{cases} (p\chi(p))^r & \text{for odd primes } p \text{ and } r > 0, \\ 2^{r+1} \chi(u) \sqrt{-1} & \text{if } h = 0, r \geq 2, p = 2 \text{ or} \\ & \text{if } h = \frac{1}{2}(1 + \sqrt{-1}), r = 1, p = 2, \\ 0 & \text{otherwise (for } p = 2 \text{ and } r > 0). \end{cases}$$

We see therefore that the total contribution to the θ_0 -component of $\mathcal{F}_{p^n}^0 \varphi$ from terms for which $n \leq s \leq 2n$ is

$$\sum_{n \leq s \leq 2n} p^{n(k-6) + k(s-n) + 2n} (p\chi(p))^{2n-s} \sum_{u_{2n-s}} \varphi_0((p^s \tau + u_{2n-s}) p^{s-2n})$$

for all odd primes p ; the relevant contribution for $p = 2$ from terms with $n \leq s \leq 2n$ can be seen to be

$$2^{n(k-6) + 2n} \left\{ \sum_{u_n} \varphi_0(\tau + u_n 2^{-n}) G(u_n, 0; 2^n) + \sum_{n < s \leq 2n} 2^{k(s-n)} \sum_{h; u_{2n-s}} \varphi_h((2^s \tau + u_{2n-s}) 2^{s-2n}) G(u_{2n-s}, h; 2^{2n-s}) \right\}.$$

The contribution arising from terms with $0 \leq s < n$ can be treated similarly. We give the detailed arguments for the case $p = 2$, since the case of odd p is simpler. Carrying out in (25) the summation over μ (while keeping the other summation-indices fixed) and noting that

$$\sum_{\mu} e(\text{tr}(\mu(m+h) 2^{s-n})) = \begin{cases} 2^{2n} & \text{if } (m+h) 2^{s-n} \in \frac{1}{2}\mathcal{O}, \\ 0 & \text{otherwise,} \end{cases}$$

we can have non-zero contribution only from terms with $h = 0$ and $m \in 2^{n-s-1}\mathcal{O}$ (as the first alternative above on the right-hand side would entail, under " $n > s$ "). Consequently, writing $m = 2^{n-s-1}l$ with $l \in \mathcal{O}$, (25) now takes the form

$$2^{2n} \sum_{l \in \mathcal{O}; \lambda} e(\tau |(l/2) + \bar{\lambda}|^2 + \text{tr}(((l/2) + \bar{\lambda})z) + u_{2n-s} |l/2|^2 2^{-s}).$$

Again, we put $l/2 = x + h_1$ with $x \in \mathcal{O}$ and $h_1 \in \frac{1}{2}\mathcal{O}/\mathcal{O}$ and further $x = x_1 + 2^s x_2$ with $x_1 \in \mathcal{O}/2^s\mathcal{O}$ and $x_2 \in \mathcal{O}$. We also note that

$$e(u_{2n-s} |x_1 + 2^s x_2 + h_1|^2 2^{-s}) = e(u_{2n-s} |x_1 + h_1|^2 2^{-s})$$

is independent of $x_2 \in \mathcal{O}$ and moreover, as λ and x_2 run respectively over $\mathcal{O}/2^n\mathcal{O}$ and \mathcal{O} , $\bar{\lambda} + 2^s x_2$ runs through \mathcal{O} precisely $2^{2(n-s)}$ times. Thus (25) finally reduces to

$$2^{2n+2n-2s} \sum_{h_1 \in \frac{1}{2}\mathcal{O}/\mathcal{O}} G(u_{2n-s}, h_1; 2^s) \theta_{h_1}(\tau, z_1, z_2)$$

and using the above-mentioned values of the Gauss sums, we see that the contribution to the θ_0 -component of $\mathcal{F}_{2^n}^0 \varphi$ from the terms with $0 \leq s < n$ is 0 for $s = 1$,

$$2^{-2n} \sum_{u_{2n}} \varphi_0((\tau + u_{2n})/2^{2n}) \quad \text{for } s = 0$$

and

$$2^{n(2k-4)+1} \sqrt{-1} \sum_{u_{2n-s}} \chi(u_{2n-s}) \varphi_0((2^s \tau + u_{2n-s}) 2^{s-2n}) 2^{(s-2n)(k-1)}$$

for $1 < s < n$. Now, using the relation $\sum_h \varphi_h(\tau) = -2\sqrt{-1} \varphi_0(-1/\tau) \tau^{1-k}$ (see [6], p. 221), we conclude that the θ_0 -component of $\mathcal{F}_{2^n}^0 \varphi$ is

$$(26) \quad 2^{n(2k-4)} \times \left\{ \sum_{u_{2n}} \varphi_0\left(\frac{\tau + u_{2n}}{2^{2n}}\right) 2^{-2n(k-1)} + 2\sqrt{-1} \left(\sum_{1 < s < 2n-1, u_{2n-s}} \chi(u_{2n-s}) \varphi_0\left(\frac{2^s \tau + u_{2n-s}}{2^{2n-s}}\right) \times 2^{-(2n-s)(k-1)} + \varphi_{(1+\sqrt{-1})/2}\left(\frac{2^{2n-1}\tau+1}{2}\right) 2^{-(k-1)} - \varphi_0\left(\frac{-1}{2^{2n}\tau}\right) (2^{2n}\tau)^{-(k-1)} \right) \right\}.$$

The θ_0 -component of $\mathcal{F}_{p^n}^0 \varphi$ for odd primes p is given by

$$\sum_{0 \leq s \leq 2n} p^{n(k-4) + k(s-n) + 2n-s} \chi(p^{2n-s}) \varphi_0((p^s \tau + u_{2n-s}) p^{s-2n}) = p^{n(2k-4)} \sum_{0 \leq s \leq 2n} \sum_{u_{2n-s}} \varphi_0\left(\frac{p^s \tau + u_{2n-s}}{p^{2n-s}}\right) \chi(p^{2n-s}) p^{-(2n-s)(k-1)}$$

which is nothing but $\varphi_0 |T(1, p^{2n})$ where, for $a, b \in \mathbf{N}$, we denote the Hecke operator corresponding to the double coset $\Gamma_0(4) \text{Diag}(a, b) \Gamma_0(4)$ by $T(a, b)$. Now $T(1, p^{2n})$ commutes with W_4 and hence $\delta(\mathcal{F}_{p^n}^0) = T(1, p^{2n})$. This gives us $\lambda_{\varphi}^0(p^n) = a(p^{2n}) - p^{k-3} \chi(p) a(p^{2n-2})$, at once for any odd prime p .

Using also the relation $\varphi_{(1+\sqrt{-1})/2}(\tau) = \varphi_0(\tau/(2\tau+1))/(2\tau+1)^{k-1}$ (see [6], p. 222), we obtain from (26),

$$(27) \quad (\mathcal{F}_{2^n}^0 \varphi)_0 = \tilde{\varphi}_0 |T(2^{2n}, 1) + 2\sqrt{-1} \\ \times \left\{ \sum_{2 \leq s \leq 2n-2; u_{2n-s}} \chi(u_{2n-s}) \tilde{\varphi}_0 |W_4 \begin{pmatrix} 2^s & u_{2n-s} \\ 0 & 2^{2n-s} \end{pmatrix} W_4^{-1} \right. \\ \left. + \tilde{\varphi}_0 |W_4 \begin{pmatrix} 2^{2n-1} & 1 \\ 2^{2n} & 4 \end{pmatrix} W_4^{-1} - \tilde{\varphi}_0 |W_4 \begin{pmatrix} 0 & -1 \\ 2^{2n} & 0 \end{pmatrix} W_4^{-1} \right\}.$$

From

$$2\sqrt{-1} \tilde{\varphi}_0 |W_4 = - \sum_{c \in \mathbb{Z}/4\mathbb{Z}} \tilde{\varphi}_0 | \begin{pmatrix} 1 & c \\ 0 & 4 \end{pmatrix}$$

(see [6], p. 226), it is seen that

$$-2\sqrt{-1} \tilde{\varphi}_0 |W_4 \begin{pmatrix} 0 & -1 \\ 2^{2n} & 0 \end{pmatrix} W_4^{-1} = \sum_{c \in \mathbb{Z}/4\mathbb{Z}} \tilde{\varphi}_0 | \begin{pmatrix} 1 & 2^{2n-2}c \\ 0 & 2^{2n} \end{pmatrix};$$

likewise,

$$-2\sqrt{-1} \tilde{\varphi}_0 |W_4 \begin{pmatrix} 2^{2n-1} & 1 \\ 2^{2n} & 4 \end{pmatrix} W_4^{-1} = \sum_{c \in \mathbb{Z}/4\mathbb{Z}} \tilde{\varphi}_0 | \begin{pmatrix} 1 & 2^{2n-2}c - 2^{2n-3} \\ 0 & 2^{2n} \end{pmatrix},$$

if we note that

$$\begin{pmatrix} -4c-1 & c^2 \\ -16 & 4c-1 \end{pmatrix} \in \Gamma_0(4) \quad \text{and} \quad \chi(4c-1) = -1.$$

Now for $2 \leq s \leq 2n-2$, let

$$t := s-2 \quad \text{and} \quad P_1 := \begin{pmatrix} -u_{2n-s} - 2^{2n-2-t}c & 2^t \\ -2^{2n-t} & 0 \end{pmatrix}$$

with given odd u_{2n-s} modulo 2^{2n-s} and c modulo 4. There exists (an odd) u'_{2n-s} unique modulo 2^{2n-2-t} such that $w := u'_{2n-s}(u_{2n-s} + 2^{2n-2-t}c) \equiv -1 \pmod{2^{2n-2-t}}$ and again $w \equiv -(1 + 2^{2n-2-t}u_{2n-s}c_1) \pmod{2^{2n-t}}$ for a uniquely determined c_1 modulo 4. If we set $r := 2^t(2^{2n-2-t}c_1 + u'_{2n-s})$, then $r/2^t \in \mathbb{Z}$ and $2^t + (u_{2n-s} + 2^{2n-2-t}c)r \in 2^{2n}\mathbb{Z}$. Moreover,

$$P_2 := P_1 \begin{pmatrix} 1 & -r/2^{2n} \\ 0 & 2^{-2n} \end{pmatrix} = \begin{pmatrix} * & * \\ -2^{2n-t} & r/2^t \end{pmatrix} \in \Gamma_0(4).$$

The pair (u_{2n-s}, c) determines (u'_{2n-s}, c_1) uniquely and this gives rise to a bijection of $(\mathbb{Z}/2^{2n-s}\mathbb{Z})^\times \times \mathbb{Z}/4\mathbb{Z}$, where the first factor is the group of odd residue classes modulo 2^{2n-s} . Since $\chi(u'_{2n-s}) = -\chi(u_{2n-s})$ and $P_2 \in \Gamma_0(4)$, we have

$$\tilde{\varphi}_0 |P_2 = \chi(r/2^t) \tilde{\varphi}_0 = -\chi(u_{2n-s}) \tilde{\varphi}_0.$$

Thus, for $2 \leq s \leq 2n-2$, we obtain

$$(28) \quad 2\sqrt{-1} \chi(u_{2n-s}) \tilde{\varphi}_0 |W_4 \begin{pmatrix} 2^s & u_{2n-s} \\ 0 & 2^{2n-s} \end{pmatrix} W_4^{-1} = -\chi(u_{2n-s}) \sum_{c \in \mathbb{Z}/4\mathbb{Z}} \tilde{\varphi}_0 |P_1 \\ \text{(with } P_1 \text{ as above)} \\ = \sum_{c_1 \in \mathbb{Z}/4\mathbb{Z}} \tilde{\varphi}_0 | \begin{pmatrix} 1 & 2^t(2^{2n-2-t}c_1 + u'_{2n-s}) \\ 0 & 2^{2n} \end{pmatrix}.$$

Together with the residue classes $2^{2n-2}c, 2^{2n-2}c - 2^{2n-3}$, the residues r above cover all the residue classes modulo 2^{2n} . Hence we get by (27) and (28),

$$(\mathcal{F}_{2^n}^0 \varphi)_0 = \tilde{\varphi}_0 | \Gamma_0(4) \begin{pmatrix} 2^{2n} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4) + \tilde{\varphi}_0 | \Gamma_0(4) \begin{pmatrix} 1 & 0 \\ 0 & 2^{2n} \end{pmatrix} \Gamma_0(4) \\ = (a(2)^{2n} + \overline{a(2)^{2n}}) \tilde{\varphi}_0,$$

i.e.

$$\delta(\mathcal{F}_{2^n}^0) = (a(2)^{2n} + \overline{a(2)^{2n}}) \text{Id}$$

where Id denotes the identity operator. As a result, we have

$$\lambda_\varphi^0(2^n) = a(2)^{2n} + \overline{a(2)^{2n}}$$

and this concludes the proof of Lemma 2.

Remarks. For $n = 1$, the results in Lemma 2 appear as assertions (without detailed proof) in Propositions 4.2 and 4.3 of [2], wherein one should read 2^{k-3} instead of 2^{2k-6} in the expression for $\psi_0(\tau)$ in Proposition 4.3 on page 19 and $3 \cdot 2^{k-2}$ instead of $3 \cdot 2^{k-4}$ in the last line of page 20; the second term in the formula for $\text{Desc}(T_{1,p})$ on page 20 should read $p^{2k-8}(p^3 + p^4 + p - 1)$.

We are thankful to Professor A. Krieg for having rushed to us, at our request, a copy of [2] from Münster.

6. A relation between $D_{F,G}$ and Z_F . For the Fourier-Jacobi coefficients φ_m, ψ_m respectively of a Hecke eigenform F and any cusp form G in $M_k(\Gamma_2)$, we have $\langle \varphi_m, \psi_m \rangle = \langle \mathcal{V}_m \varphi_1, \mathcal{V}_m \psi_1 \rangle = \langle \mathcal{V}_m^* \mathcal{V}_m \varphi_1, \psi_1 \rangle$ and so, by the proposition,

$$D_{F,G}(s) = \zeta_{\mathbb{K}}(s-k+3) \zeta(2s-2k+4) \langle \varphi_1, \psi_1 \rangle \sum_{t \in \mathbb{N}} \sum_{d|t} \psi(d) d^{k-2} \lambda_{\varphi_1}(t/d) t^{-s} \\ = \langle \varphi_1, \psi_1 \rangle \zeta_{\mathbb{K}}(s-k+3) \zeta(2s-2k+4) \sum_{d \in \mathbb{N}} \psi(d) d^{k-2-s} \sum_{t \in \mathbb{N}} \lambda_{\varphi_1}(t) t^{-s} \\ = \langle \varphi_1, \psi_1 \rangle \zeta_{\mathbb{K}}(s-k+3) \zeta(s-k+1) \zeta(s-k+2) \sum_{t \in \mathbb{N}} \lambda_{\varphi_1}(t) t^{-s},$$

on noting that

$$\sum_{d \in \mathbb{N}} \psi(d) d^{-\varrho} = \zeta(\varrho-1) \zeta(\varrho) / \zeta(2\varrho).$$

Now

$$\sum_{t \in \mathbb{N}} \lambda_{\varphi_1}(t) t^{-s} = \prod_p \left(\sum_{n \geq 0} \lambda_{\varphi_1}(p^n) p^{-ns} \right)$$

where the product is taken over all primes $p \in \mathbb{N}$. We know from Lemma 2 that for odd primes p ,

$$\begin{aligned} & \lambda_{\varphi_1}(p^n) \\ &= \sum_{0 \leq r \leq [n/2]} p^{2r(k-2)} a(p^{2(n-2r)}) - p^{-1} \chi(p) \sum_{0 \leq r \leq [(n-1)/2]} p^{(2r+1)(k-2)} a(p^{2(n-2r-1)}) \end{aligned}$$

where $[x] :=$ largest integer not exceeding $x \in \mathbb{R}$. Therefore

$$\sum_{n \geq 0} \lambda_{\varphi_1}(p^n) p^{-ns} = \left(\sum_{j \geq 0} p^{2j(k-2-s)} - p^{-1} \chi(p) \sum_{j \geq 0} p^{(2j+1)(k-2-s)} \right) \left(\sum_{u \geq 0} a(p^{2u}) p^{-us} \right)$$

which is precisely the p th Euler factor of $R_f(s)/L(s-k+3, \chi)$, where

$$\begin{aligned} & R_f(s) \\ & := \left\{ (1-a(2)^2 2^{-s}) (1-\overline{a(2)^2} 2^{-s}) \prod_{p \neq 2} (1-\alpha_p^2 p^{-s}) (1-\chi(p) \alpha_p \bar{\alpha}_p p^{-s}) (1-\bar{\alpha}_p^2 p^{-s}) \right\}^{-1} \end{aligned}$$

is the symmetric square zeta function associated to f and $\alpha_p + \chi(p) \bar{\alpha}_p = a(p)$, $\alpha_p \bar{\alpha}_p = p^{k-2}$. By Lemma 2 again

$$\begin{aligned} \sum_{n \geq 0} \lambda_{\varphi_1}(2^n) 2^{-ns} &= \sum_{n \geq 0} \left(\sum_{\substack{n_1, n_2 \geq 0 \\ n_1 + n_2 = n}} a(2)^{2n_1} \overline{a(2)^{2n_2}} \right) 2^{-ns} \\ &= \left\{ (1-a(2)^2 2^{-s}) (1-\overline{a(2)^2} 2^{-s}) \right\}^{-1}. \end{aligned}$$

Hence

$$\sum_{t \in \mathbb{N}} \lambda_{\varphi_1}(t) t^{-s} = R_f(s)/L(s-k+3, \chi)$$

which implies that

$$D_{F,G}(s) = \langle \varphi_1, \psi_1 \rangle \zeta(s-k+3) \zeta(s-k+2) \zeta(s-k+1) R_f(s).$$

Since Gritsenko [2] has shown that $Z_F(s) = \zeta(s-k+1) L(s-k+2, \chi) \times \zeta(s-k+3) R_f(s)$, we have finally the following

THEOREM 2. *Let $F, G \in M_k(\Gamma_2)$, F being a Hecke eigenform. Then*

$$D_{F,G}(s) = \langle \varphi_1, \psi_1 \rangle \zeta(s-k+2) L\left(s-k+2, \left(\frac{-4}{\cdot}\right)^{-1}\right) Z_F(s).$$

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SCHOOL OF MATHEMATICS
 TATA INSTITUTE OF FUNDAMENTAL RESEARCH
 Homi Bhabha Road
 Bombay 400 005
 India

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