

$\varphi_1, \varphi_3, \varphi_4, \varphi_6$ proviennent de faces, φ_2, φ_5 proviennent de points extrémaux, φ_7 provient d'une unité formelle.

$$|\mathcal{E}_x/\mathcal{U}_x^{(1)}| = 3, \quad E = E_0 = \{n; \varepsilon(n) \text{ entier de } K_n\}, \quad \overline{\lim}_{\substack{n \in E \\ n \rightarrow +\infty}} |\mathcal{E}_n/\mathcal{U}_n| = 7.$$

Remarque sur le théorème 1. Pour montrer le théorème 1, on pourrait penser utiliser le résultat de H. C. Williams [6] qui dit intuitivement la chose suivante:

“Si φ est un élément d'un ordre \mathcal{O} de petite norme alors φ est une meilleure approximation de G ”. Précisément:

Soit $\mathcal{O} = \mathbf{Z} + \mathbf{Z}\mu + \mathbf{Z}\nu$ un ordre de K (corps cubique pur) et D_0 son discriminant. Soit φ un élément non nul de \mathcal{O} . Si:

$$(40) \quad |N(\varphi)| < \sqrt[4]{D_0/27}$$

alors φ est une meilleure approximation de \mathcal{O} .

Malheureusement, l'approximation formelle obtenue de façon générale et le résultat de H. C. Williams ne sont pas suffisants pour démontrer le théorème 1. En effet, si φ est un point extrémal de \mathcal{O}_x on a:

$$\deg N(\varphi) \leq 3d - 2 \quad (\text{voir proposition 3}) \quad \text{et} \quad \deg(\text{Disc } \Delta) = 6d,$$

l'inégalité (40) n'est donc pas satisfaite asymptotiquement.

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A Dirichlet series for modular forms of degree n

by

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1. Introduction. In a recent paper Kohnen and Skoruppa [6] introduced a new type of Dirichlet series, which is associated with the Fourier-Jacobi expansion of a pair f, g of Siegel cusp forms of the same weight and degree 2. The proof is based on the Rankin-Selberg method. If $f = g$ is a simultaneous Hecke eigenform in the so-called Maaß Spezialschar, this Dirichlet series is proportional to the spinor zeta function attached to f by Andrianov.

Then Yamazaki [17], [18] derived the analytic continuation and functional equation for this type of Dirichlet series in the case of arbitrary degree n . The main point of his proof is the analytic continuation of an Eisenstein series of Klingen type, which is derived from Langlands' theory. Recently Raghavan and Sengupta [13] obtained analogous results for Hermitian modular forms of degree 2 with respect to the Gaussian number field.

In this paper we give an elementary proof of Yamazaki's result [17], which only makes use of well-known properties of Epstein zeta functions. A modification of the arguments and another application of the Rankin-Selberg method lead to the corresponding result for Hermitian modular forms of degree n with respect to the Gaussian number field. Finally, we deal with modular forms of quaternions of degree n (cf. [11], [7]). We derive analytic continuation and functional equation of the attached Dirichlet series. If f and g belong to the Maaß space (cf. [8]) and if f is a simultaneous eigenform under all Hecke operators, then the Dirichlet series possesses an Euler product expansion and is proportional to the Andrianov zeta function attached to f .

2. Preliminaries. A good deal of work can be done for all the three types of modular forms simultaneously just as in [7]. Therefore let F stand for the field \mathbf{R} of real numbers resp. \mathbf{C} of complex numbers resp. for the skew-field \mathbf{H} of Hamiltonian quaternions. Set $r = [F:\mathbf{R}]$ and denote the standard basis of F over \mathbf{R} by e_1, \dots, e_r , where

$$e_1 = 1, \quad e_2^2 = e_3^2 = e_4^2 = -1, \quad e_2e_3 = -e_3e_2 = e_4.$$

Let $a \mapsto \bar{a}$ be the canonical involution on F and define

$$\text{Re}(a) := \frac{1}{2}(a + \bar{a}), \quad \langle a, b \rangle := \text{Re}(\bar{a}b), \quad N(a) := a\bar{a}.$$

The letter I is reserved for the $n \times n$ identity matrix and J for the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Given matrices A, B of appropriate size set

$$A[B] := \bar{B}'AB,$$

where the prime stands for the transpose.

Let $\text{Sym}(n; F)$ denote the $n + \frac{1}{2}n(n-1)$ -dimensional real vector space of all over F Hermitian $n \times n$ matrices and $\text{Pos}(n; F) := \{Y \in \text{Sym}(n; F) \mid Y > 0\}$ the open subset of all positive definite matrices. Then the half-space of degree n is defined to be

$$H(n; F) := \text{Sym}(n; F) + i \text{Pos}(n; F) \subset \text{Sym}(n; F) \otimes_{\mathbf{R}} \mathbf{C}.$$

Clearly $H(n; \mathbf{R})$ is the Siegel half-space, $H(n; \mathbf{C})$ the Hermitian half-space, and $H(n; \mathbf{H})$ the half-space of quaternions. Moreover, $H := H(1; F)$ equals the upper half-plane in \mathbf{C} . The symplectic group (over F)

$$\text{Sp}(n; F) := \{M \in \text{Mat}(2n; F) \mid J[M] = J\}$$

acts on $H(n; F)$ in the usual way by

$$Z \mapsto M \langle Z \rangle := (AZ + B)(CZ + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n; F).$$

Given $Z = X + iY \in H(n; F)$ consider

$$P_Z := \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \left[\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \right] \in \text{Sp}(n; F) \cap \text{Pos}(2n; F).$$

Given $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n; F)$ and $Z \in H(n; F)$ with $M \langle Z \rangle = X_M + iY_M$, we get

$$(1) \quad P_{M \langle Z \rangle} = P_Z [\bar{M}'],$$

$$(2) \quad Y_M^{-1} = P_Z \left[\begin{pmatrix} \bar{C}' \\ \bar{D}' \end{pmatrix} \right]$$

from [7, II.1.9 and II.1.7].

We choose a particular order $\mathcal{O} = \mathcal{O}(F)$ for each case of F , namely

$$\mathcal{O}(\mathbf{R}) = \mathbf{Z}, \quad \mathcal{O}(\mathbf{C}) = \mathbf{Z}e_1 + \mathbf{Z}e_2, \quad \mathcal{O}(\mathbf{H}) = \mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. Then clearly $\mathcal{O}(\mathbf{C})$ equals the Gaussian integers

and $\mathcal{O}(\mathbf{H})$ the quaternions of Hurwitz. The dual lattice of \mathcal{O} with respect to \langle, \rangle is denoted by \mathcal{O}^* , i.e.,

$$\mathcal{O}^*(F) = \mathcal{O}(F) \quad \text{for } F = \mathbf{R}, \mathbf{C},$$

$$\mathcal{O}^*(\mathbf{H}) = \mathbf{Z}2e_1 + \mathbf{Z}(e_1 + e_2) + \mathbf{Z}(e_1 + e_3) + \mathbf{Z}(e_1 + e_4).$$

Moreover, let

$$\Gamma_n := \Gamma(n; F) := \text{Sp}(n; F) \cap \text{Mat}(2n; \mathcal{O}(F))$$

stand for the modular group of degree n .

3. Jacobi forms. If $n > 1$ and k is an integer, which is even for $F = \mathbf{H}$, the vector space $[\Gamma(n; F), k]$ of modular forms of degree n and weight k consists of all holomorphic functions $f: H(n; F) \rightarrow \mathbf{C}$ such that

$$f(M \langle Z \rangle) = (\det(CZ + D))^k f(Z)$$

for all $M \in \Gamma(n; F)$. If $F = \mathbf{H}$ we refer to [7, p. 78] for the definition of the determinant. Let τ denote the reduced trace form

$$\tau(A, B) := \frac{1}{2} \text{trace}(A\bar{B}' + B\bar{A}') \quad \text{for } n \times m \text{ matrices } A, B.$$

Then each $f \in [\Gamma(n; F), k]$ possesses a Fourier expansion of the form

$$(3) \quad f(Z) = \sum_{T \in \text{Sym}^r(n; \mathcal{O}), T \geq 0} \alpha_f(T) e^{2\pi i \tau(T, Z)}, \quad Z \in H(n; F),$$

where $\text{Sym}^r(n; \mathcal{O})$ denotes the dual lattice of $\text{Sym}(n; \mathcal{O})$ with respect to τ . The subspace $[\Gamma(n; F), k]_0$ of all cusp forms consists of those $f \in [\Gamma(n; F), k]$ with the Fourier expansion (3), where $\alpha_f(T) \neq 0$ implies $T \in \text{Pos}(n; F)$ (cf. [7] for details). Now consider a decomposition

$$Z = \begin{pmatrix} Z_1 & w \\ \bar{w}' & z \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & \frac{1}{2}t \\ \frac{1}{2}t' & m \end{pmatrix},$$

where $z \in H(1; F)$, $m \in \mathbf{N}_0$, $w = u + iv \in F_{\mathbf{C}}^{n-1}$, $\bar{w}' = \bar{u}' + i\bar{v}'$ and $F_{\mathbf{C}} = F \otimes_{\mathbf{R}} \mathbf{C}$. A rearrangement of (3) yields the Fourier-Jacobi expansion of f

$$(4) \quad f(Z) = \sum_{m=0}^{\infty} f_m(Z_1, w) e^{2\pi i m z},$$

where

$$f_m(Z_1, w) = \sum_{\substack{T_1 \in \text{Sym}^r(n-1; \mathcal{O}), t \in \mathcal{O}^{*n-1} \\ \text{such that } T \geq 0}} \alpha_f(T) e^{2\pi i (\tau(T_1, Z_1) + \tau(t, w))}.$$

Then $f_m(Z_1, w)$ is a Jacobi form of degree $n-1$, weight k and index m . It is characterized by the identities

$$f_m|_{k,m}[M] = f_m|_m[\lambda, \mu] = f \quad \text{for all } M \in \Gamma_{n-1} \text{ and } \lambda, \mu \in \mathcal{O}^{n-1}.$$

Given $\varphi: H(n-1; \mathbf{F}) \times \mathbf{F}_c^{n-1} \rightarrow \mathbf{C}$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n-1; \mathbf{F})$ and $\lambda, \mu \in \mathbf{F}^{n-1}$ we use the abbreviations

$$(5) \quad \begin{aligned} \varphi|_{k,m}[M](Z_1, w) &:= \varphi(M\langle Z_1 \rangle, \overline{(CZ_1 + D)^{-1}w}) \det(CZ_1 + D)^{-k} e^{-2nim((CZ_1 + D)^{-1}C)[w]}, \\ \varphi|_m[\lambda, \mu](Z_1, w) &= \varphi(Z_1, w + Z_1\lambda + \mu) e^{2nimZ_1[\lambda] + 4nimt(\lambda, w)}. \end{aligned}$$

The results on Jacobi forms are immediately verified along the same lines as in [4], [5], [16], [19] for arbitrary Siegel modular forms, resp. in [1] and [13] in the case of degree 2.

According to (5) we consider the group $\Gamma_n^* = \Gamma^*(n; \mathbf{F})$, which acts discontinuously on $H(n-1; \mathbf{F}) \times \mathbf{F}_c^{n-1}$ and is generated by

$$\begin{aligned} (Z_1, w) &\mapsto (M\langle Z_1 \rangle, \overline{(CZ_1 + D)^{-1}w}), \quad M \in \Gamma_{n-1}, \\ (Z_1, w) &\mapsto (Z_1, w + Z_1\lambda + \mu), \quad \lambda, \mu \in \mathcal{O}^{n-1}. \end{aligned}$$

A fundamental domain with respect to the action of $\Gamma^*(n; \mathbf{F})$ is for instance given by $\mathcal{F}_n^* := \mathcal{F}^*(n; \mathbf{F})$,

$$\mathcal{F}_n^* := \{(Z_1, w) \mid Z_1 \in \mathcal{F}_{n-1}, w = u + iv, u: \mathbf{F}^{n-1}/\mathcal{O}^{n-1}, v: \mathbf{F}^{n-1}/Y_1\mathcal{O}^{n-1}\},$$

where $\mathcal{F}_n = \mathcal{F}(n; \mathbf{F})$ is a fundamental domain of $H(n; \mathbf{F})$ with respect to the action of $\Gamma(n; \mathbf{F})$ (cf. [7]). Moreover, $dv = (\det Y_1)^{-2-r(n-1)} dX_1 dY_1 dudv$ is a volume element, which is invariant under the action of $\Gamma^*(n; \mathbf{F})$.

Given two cusp forms $f, g \in [\Gamma(n; \mathbf{F}), k]_0$, $n > 1$ and $m \in \mathbf{N}$, we define the Petersson scalar product of the attached Jacobi forms (cf. [4]) by

$$\langle f_m, g_m \rangle := \int_{\mathcal{F}_n^*} f_m(Z_1, w) \overline{g_m(Z_1, w)} (\det Y_1)^k e^{-4nmY_1^{-1}[v]} dv.$$

LEMMA 1. Given $f, g \in [\Gamma(n; \mathbf{F}), k]_0$ one has

$$\langle f_m, g_m \rangle = O(m^k),$$

where the O -constant only depends on f and g .

Proof. If $(Z_1, w) \in H(n-1; \mathbf{F}) \times \mathbf{F}_c^{n-1}$ is fixed, we get

$$f_m(Z_1, w) = \int_{ic}^{ic+1} f(Z) e^{-2nimz} dz,$$

where c is any real constant greater than $Y_1^{-1}[v]$. Since $(\det Y)^{k/2} |f(Z)|$ is bounded on $H(n; \mathbf{F})$ due to [7, III.2.4], we can choose $c = Y_1^{-1}[v] + 1/m$ and get

$$|f_m(Z_1, w)| = O\left(\left(\frac{1}{m} \det Y_1\right)^{-k/2} e^{2nmY_1^{-1}[v]}\right).$$

Thus the Petersson scalar product is majorized by

$$m^k \int_{\mathcal{F}_n^*} (\det Y_1)^{-2-r(n-1)} dX_1 dY_1 dudv = m^k (\text{vol } \mathcal{O})^{2n-2} \text{vol}(\mathcal{F}_{n-1}). \blacksquare$$

Given $f, g \in [\Gamma(n; \mathbf{F}), k]_0$, $n > 1$, we can therefore attach the Dirichlet series

$$D(f, g; s) := \sum_{m=1}^{\infty} \langle f_m, g_m \rangle m^{-s},$$

which converges absolutely for $\text{Re}(s) > k+1$ due to Lemma 1.

The analytic continuation of this Dirichlet series is derived by an integral representation. Therefore we have to introduce an Eisenstein series of Klingen type (cf. [3]).

Let $C_n := C(n; \mathbf{F})$ denote the subgroup of $\Gamma_n = \Gamma(n; \mathbf{F})$ consisting of all those matrices with $(0, \dots, 0, 1)$ as its last row. The subscript 1 denotes the upper left $(n-1) \times (n-1)$ corner of an $n \times n$ matrix. Then we define the Eisenstein series of Klingen type

$$(6) \quad E(Z; s) := \sum_{M: C_n \backslash \Gamma_n} \left(\frac{\det Y_M}{\det(Y_M)_1} \right)^s, \quad Z \in H(n; \mathbf{F}).$$

In analogy with [7, V.2.8], we conclude that the series converges absolutely for $\text{Re}(s) > 1 + r(n-1)$.

Given $f, g \in [\Gamma(n; \mathbf{F}), k]$, where at least one is a cusp form, the Petersson scalar product is defined by

$$\langle f, g \rangle := \int_{\mathcal{F}_n} f(Z) \overline{g(Z)} (\det Y)^k d\mu,$$

where $d\mu = (\det Y)^{-2-r(n-1)} dX dY$ is the invariant volume element. If $\Gamma(s)$ denotes the gamma function, we get

LEMMA 2. Given $f, g \in [\Gamma(n; \mathbf{F}), k]_0$, then

$$\langle fE(\cdot; s), g \rangle = 2 \cdot (4\pi)^{r(n-1)+1-k-s} \Gamma(s+k-1-r(n-1)) D(f, g; s+k-1-r(n-1))$$

holds for $\text{Re}(s) > 2 + r(n-1)$.

Proof. We apply the usual unfolding trick. Therefore we calculate

$$\begin{aligned} \langle fE(\cdot; s), g \rangle &= \int_{\mathcal{F}_n} f(Z) \overline{g(Z)} (\det Y)^k E(Z, s) d\mu \\ &= \sum_{M: C_n \backslash \Gamma_n} \int_{\mathcal{F}_n} f(M\langle Z \rangle) \overline{g(M\langle Z \rangle)} (\det Y_M)^{k+s} (\det(Y_M)_1)^{-s} d\mu \\ &= 2 \int_{\mathcal{O}_n} f(Z) \overline{g(Z)} (\det Y)^{k+s} (\det Y_1)^{-s} d\mu, \end{aligned}$$

where $\mathcal{C}_n := \mathcal{C}(n; F)$ is a fundamental domain of $H(n; F)$ with respect to the action of C_n . We can choose

$$\mathcal{C}_n = \{Z | (Z_1, w) \in \mathcal{F}_n^*, y > Y_1^{-1}[v], 0 \leq x \leq 1\},$$

since $M \in C_n$ has the form

$$M = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{U}' & 0 \\ 0 & U^{-1} \end{pmatrix} \left(N \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),$$

$$S \in \text{Sym}(n; \mathcal{O}), \quad S_1 = 0, \quad U = \begin{pmatrix} I & \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \in \mathcal{O}^{n-1}, \quad N \in \Gamma_{n-1}.$$

Hence the Petersson scalar product equals

$$2 \int_{\mathcal{F}_n^*} \left(\int_{y > Y_1^{-1}[v], 0 \leq x \leq 1} \sum_{l, m \geq 1} f_l(Z_1, w) \overline{g_m(Z_1, w)} e^{-2\pi(l+m)y} e^{2\pi i(l-m)x} \times (\det Y_1)^{k-2-r(n-1)} (y - Y_1^{-1}[v])^{s+k-2-r(n-1)} dx dy \right) dX_1 dY_1 dudv$$

in view of $\det Y = (\det Y_1)(y - Y_1^{-1}[v])$. Carrying out the integration over x and substituting $q = y - Y_1^{-1}[v]$, we get

$$\begin{aligned} \{fE(\cdot; s), g\} &= 2 \int_{\mathcal{F}_n^*} \sum_{m=1}^{\infty} f_m(Z_1, w) \overline{g_m(Z_1, w)} e^{-4\pi m Y_1^{-1}[v]} (\det Y_1)^k dv \\ &\quad \times \int_0^{\infty} q^{s+k-2-r(n-1)} e^{-4\pi m q} dq \\ &= 2 \sum_{m=1}^{\infty} \langle f_m, g_m \rangle \Gamma(s+k-1-r(n-1)) (4\pi m)^{r(n-1)+1-k-s}. \quad \blacksquare \end{aligned}$$

Analytic continuation and functional equation of $D(f, g; s)$ now follow from properties of $E(Z; s)$. The quotient $\det(Y_M)_1$ over $\det(Y_M)$ in (6) equals the (n, n) -entry of Y_M^{-1} . Using (2) we obtain

$$(7) \quad E(Z; s) = \sum_{\lambda} (P_Z[\lambda])^{-s},$$

where $\bar{\lambda}'$ runs through the last rows of all the matrices in $\Gamma(n; F)$.

4. Siegel modular forms. In this section let $F = \mathbf{R}$. Then it follows from [12, § 11] that λ in (7) runs through all coprime vectors in \mathbf{Z}^{2n} . If $\zeta(s)$ denotes the Riemann zeta function, we obtain an Epstein zeta function by

$$\zeta(2s)E(Z; s) = \sum_{0 \neq \lambda \in \mathbf{Z}^{2n}} (P_Z[\lambda])^{-s}.$$

In view of $P_Z^{-1} = P_Z[J]$ now well-known properties of Epstein zeta functions (cf. [15, p. 59]) yield that

$$E(Z; s) := \pi^{-s} \Gamma(s) \zeta(2s) E(Z; s)$$

is a holomorphic function of s except for two simple poles at $s = 0$ resp. $s = n$ with residue -1 resp. 1 and satisfies the functional equation $E(Z; n-s) = E(Z; s)$.

THEOREM 1 ([17], [18]). *Let $f, g \in [\Gamma(n; \mathbf{R}), k]_0$ be two Siegel cusp forms of weight k and degree $n > 1$. Then*

$$D(f, g; s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s+n-k) \zeta(2s+2n-2k) D(f, g; s)$$

is a holomorphic function of $s \in \mathbf{C}$ except for two possible simple poles at $s = k$ resp. $s = k-n$ with residue $\frac{1}{2} \pi^{n-k} \{f, g\}$ resp. $-\frac{1}{2} \pi^{n-k} \{f, g\}$ and satisfies the functional equation

$$D(f, g; 2k-n-s) = D(f, g; s).$$

Proof. The claim follows from the considerations above and Lemma 2 due to

$$\{fE(\cdot; s+n-k), g\} = 2\pi^{k-n} D(f, g; s). \quad \blacksquare$$

Clearly $D(f, g; s)$ possesses a meromorphic continuation to the whole complex s -plane. It follows from the well-known properties of $\zeta(s)$ and $\Gamma(s)$ that

$$D(f, g; s) = 0 \quad \text{for } s = k + \frac{1}{2} - n \text{ and } s = 0, -1, -2, \dots$$

as well as

$$D(f, g; k-n) = \frac{(4\pi)^{k-n}}{(k-n-1)!} \{f, g\} \quad \text{provided that } k > n.$$

5. Hermitian modular forms. In this section let $F = \mathbf{C}$, $\mathcal{O} = \mathcal{O}(\mathbf{C})$. In analogy with [12, § 11], we conclude that λ in (7) runs through all coprime vectors in \mathcal{O}^{2n} with $J[\lambda] = 0$. Hence we obtain

$$(8) \quad E^*(Z; s) := \zeta_{\mathcal{O}}(s) E(Z; s) = \sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s},$$

where

$$\zeta_{\mathcal{O}}(s) := \frac{1}{4} \sum_{0 \neq a \in \mathcal{O}} N(a)^{-s} = \zeta(s) L(s, \left(\frac{-4}{\cdot}\right)), \quad \text{Re}(s) > 1,$$

and $L(s, \left(\frac{-4}{\cdot}\right))$ is the Dirichlet L -series.

LEMMA 3. *Given $n > 1$ and $Z \in H(n; \mathbf{C})$; then*

$$E(Z; s) := \pi^{-2s} \Gamma(s) \Gamma(s+1-n) \zeta(2s+2-2n) E^*(Z; s)$$

becomes a holomorphic function of $s \in \mathbf{C}$ except for possible simple poles at $s = 0, n-1, n, 2n-1$ and satisfies the functional equation

$$E(Z; 2n-1-s) = E(Z; s).$$

Proof. We consider the theta series

$$\vartheta(Z, w) = v^n \sum_{\lambda \in \mathcal{O}^{2n}} e^{2\pi i(uJ + ivP_Z)[\lambda]}, \quad Z \in H(n; \mathbf{C}), w = u + iv \in H = H(1; \mathbf{R}).$$

As one easily checks for generators, $\vartheta(Z, w)$ is invariant under the action of $\Gamma(n; \mathbf{C}) \times \Gamma$, $\Gamma = \text{SL}(2; \mathbf{Z})$. Then

$$\Theta(Z, w) := \Delta \vartheta(Z, w) - n(n-1)\vartheta(Z, w),$$

where $\Delta = v^2(\partial^2/\partial u^2 + \partial^2/\partial v^2)$ is the hyperbolic Laplacian on H , is also invariant under the action of $\Gamma(n; \mathbf{C}) \times \Gamma$. If Z belongs to a compact subset of $H(n; \mathbf{C})$ one has

$$\Theta(Z, w) = O(e^{-v}) \quad \text{for all } w \in \mathcal{F}_1.$$

We denote the non-analytic Eisenstein series on the upper half-plane in \mathbf{C} by

$$\tilde{E}(w, s) := \sum_{M: \mathbf{C}_1/\Gamma} v_M^s, \quad w \in H(1; \mathbf{R}), \text{Re}(s) > 1.$$

Now the integral

$$\mathcal{I} := \int_{\mathcal{F}_1} \tilde{E}(w, s+1-n)\Theta(Z, w)d\mu(w)$$

converges absolutely for $\text{Re}(s) > n$. The usual unfolding trick (cf. [13]) yields

$$\mathcal{I} = 2s(s+1-2n)\pi^{-s}\Gamma(s)E^*(Z; s).$$

Now the claim follows by virtue of the results on the non-analytic Eisenstein series in [15, §3.5]. ■

A more detailed description of similar arguments will be given in Section 6.

A combination of Lemma 3 and Lemma 2 leads to

THEOREM 2. *Let $f, g \in [\Gamma(n; \mathbf{C}), k]_0$ be two Hermitian cusp forms of weight k and degree $n > 1$. Then*

$$\begin{aligned} \mathbf{D}(f, g; s) &:= (4\pi^3)^{-s}\Gamma(s)\Gamma(s-k-1+2n)\Gamma(s-k+n)\zeta(2s-2k+2n)\zeta(s-k-1+2n) \\ &\quad \times L(s-k-1+2n, (\overline{\cdot}^4))\mathbf{D}(f, g; s) \end{aligned}$$

becomes a holomorphic function of $s \in \mathbf{C}$ except for possible simple poles at $s = k, k+1-n, k-n, k+1-2n$ and satisfies the functional equation

$$\mathbf{D}(f, g; 2k+1-2n-s) = \mathbf{D}(f, g; s).$$

Again it is clear that $\mathbf{D}(f, g; s)$ possesses a meromorphic continuation to the whole s -plane and has trivial zeros at $s = 0, -1, -2, \dots$

The case $n = 2$ of this theorem coincides with Theorem 1 in [13]. It is evident from the proof that the cases of Hermitian modular forms of an arbitrary imaginary quadratic number field of class number 1 can be dealt with in the same way.

6. Modular forms of quaternions. In this section let $F = \mathbf{H}$, $\mathcal{O} = \mathcal{O}(\mathbf{H})$, and $\Gamma_n = \Gamma(n; \mathbf{H})$. In analogy with (8) we obtain

$$(9) \quad E^*(Z; s) := \zeta_{\mathcal{O}}(s)E(Z; s) = \sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s},$$

where

$$\zeta_{\mathcal{O}}(s) := \frac{1}{24} \sum_{0 \neq a \in \mathcal{O}} N(a)^{-s} = (1-2^{1-s})\zeta(s)\zeta(s-1), \quad \text{Re}(s) > 2.$$

In order to derive the analytic continuation of (9) we have to express the function as an integral of a modified theta series against an Eisenstein series on the four-dimensional hyperbolic space.

Let \mathcal{H} be the half-space model of the four-dimensional hyperbolic space (cf. [9]), i.e.,

$$\mathcal{H} = \{w = u + v = \sum_{j=1}^4 w_j e_j \in \mathbf{H} \mid w_1 = v > 0\}.$$

The modified symplectic group of degree 1

$$\text{MSp}(1; \mathbf{H}) := \{M \in \text{Mat}(2; \mathbf{H}) \mid Q[M] = Q\}, \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

acts on \mathcal{H} by fractional linear transformations, i.e.,

$$(10) \quad w \mapsto M\langle w \rangle := (aw + b)(cw + d)^{-1}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{MSp}(1; \mathbf{H}).$$

The attached hyperbolic Laplacian

$$\Delta = w_1^2 \sum_{j=1}^4 \frac{\partial^2}{\partial w_j^2} - 2w_1 \frac{\partial}{\partial w_1}$$

is invariant under all the transformations (10). Moreover, consider the modified modular group

$$\tilde{\Gamma} := \text{MSp}(1; \mathbf{H}) \cap \text{Mat}(2; \mathcal{O})$$

and set

$$\text{Im } \mathbf{H} = \{a \in \mathbf{H} \mid \text{Re}(a) = 0\} = \mathbf{Re}_2 + \mathbf{Re}_3 + \mathbf{Re}_4.$$

Let $Z \in H(n; \mathbf{H})$, $w = u + v \in \mathcal{H}$, $v = \text{Re}(w)$, and let Λ be a lattice in \mathbf{H}^{2n} . Then we define the *theta series*

$$\vartheta(Z, w; \Lambda) := v^{2n} \sum_{\lambda \in \Lambda} e^{\pi i \langle u, J[\lambda] \rangle - \pi v P_Z[\lambda]}.$$

We derive the theta transformation formula in

PROPOSITION 1. Let Λ be a lattice in \mathbf{H}^{2n} and let

$$\Lambda^\# := \{ \mu \in \mathbf{H}^{2n} \mid \text{Re}(\bar{\mu}' \lambda) \in \mathbf{Z} \text{ for all } \lambda \in \Lambda \}$$

be the dual lattice. Then one has for all $Z \in H(n; \mathbf{H})$ and $w \in \mathcal{H}$

$$\vartheta(-Z^{-1}, w^{-1}; \Lambda^\#) = (\text{vol } \Lambda) \vartheta(Z, w; \Lambda).$$

Proof. We express the theta series as a specialization of a symplectic theta series on the Siegel half-space $H(8n; \mathbf{R})$ of degree $8n$. Given a quaternion $a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$, we therefore define the real 4×4 matrices

$$\hat{a} := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & -a_4 & a_3 \\ -a_3 & a_4 & a_1 & -a_2 \\ -a_4 & -a_3 & a_2 & a_1 \end{bmatrix}, \quad \tilde{a} := \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_2 & a_1 & a_4 & -a_3 \\ -a_3 & -a_4 & a_1 & a_2 \\ -a_4 & a_3 & -a_2 & a_1 \end{bmatrix},$$

The mapping $\mathbf{H} \rightarrow \text{Mat}(4; \mathbf{R})$, $a \mapsto \hat{a}$, is a homomorphism, whereas the map $\mathbf{H} \rightarrow \text{Mat}(4; \mathbf{R})$, $a \mapsto \tilde{a}$, is an anti-homomorphism of the algebras. One has

$$\hat{a} \cdot \tilde{b} = \tilde{b} \cdot \hat{a} \quad \text{for all } a, b \in \mathbf{H}.$$

We extend the definition of $\hat{\cdot}$ to matrices as in [7, p. 15] and use \otimes for the Kronecker product of matrices. Then we obtain

$$(11) \quad (\tilde{a} \otimes J)[\hat{M}] = \tilde{a} \otimes J \quad \text{for all } a \in \mathbf{H}, M \in \text{Sp}(n; \mathbf{H}).$$

Setting

$$W := W(Z, w) := -\tilde{u} \otimes J + iv \hat{P}_Z \in H(8n; \mathbf{R}) \quad \text{for } Z \in H(n; \mathbf{H}), w \in \mathcal{H},$$

we get

$$\vartheta(Z, w; \Lambda) = v^{2n} \sum_{\lambda \in \Lambda_1} e^{\pi i W[\lambda]},$$

where Λ_1 is the lattice in \mathbf{R}^{8n} consisting of the first columns of the matrices $\hat{\lambda} \in \text{Mat}(8n, 4; \mathbf{R})$, $\lambda \in \Lambda$. One can use (11) in order to verify

$$-W^{-1} = W(-Z^{-1}, w^{-1}), \quad \det(i^{-1} W) = |w|^{8n}.$$

Then the assertion follows from the ordinary theta transformation formula as for instance in [7, IV.2.2]. ■

Moreover, we need to know the behavior of our theta series under modular substitutions. Therefore let $\Gamma_n = \Gamma(n; \mathbf{H})$ and set

$$\tilde{\Gamma}^\# := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma} \mid b \in \mathcal{O}^\# \right\}.$$

PROPOSITION 2. (a) $\vartheta(M \langle Z \rangle, N \langle w \rangle; \mathcal{O}^{2n}) = \vartheta(Z, w; \mathcal{O}^{2n})$ holds for all $M \in \Gamma_n$ and $N \in \tilde{\Gamma}^\#$.

(b) $\Theta(Z, w) := \sum_{N \in \tilde{\Gamma}^\# \setminus \tilde{\Gamma}} \vartheta(Z, N \langle w \rangle; \mathcal{O}^{2n}) = \vartheta(Z, w; \mathcal{O}^{2n}) + \vartheta(Z, w + e_2; \mathcal{O}^{2n}) + 2^{2n} \vartheta(Z, w; \mathcal{O}^{\#2n})$ is invariant under the transformations $Z \mapsto M \langle Z \rangle$, $M \in \Gamma_n$, as well as under $w \mapsto N \langle w \rangle$, $N \in \tilde{\Gamma}$.

Proof. (a) The invariance under $Z \mapsto M \langle Z \rangle$, $M \in \Gamma_n$, is clear due to (1). The invariance under the substitutions of $\tilde{\Gamma}^\#$ is demonstrated for generators. By the use of the Euclidean algorithm one checks that $\tilde{\Gamma}^\#$ is generated by

$$\varepsilon I, \varepsilon \in \mathcal{O}, N(\varepsilon) = 1, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathcal{O}^\# \cap \text{Im } \mathbf{H}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, c \in \mathcal{O} \cap \text{Im } \mathbf{H}.$$

(b) The claim follows from part (a) and Proposition 1, since $\tilde{\Gamma}$ is the disjoint union of the right cosets $\tilde{\Gamma}^\#, \tilde{\Gamma}^\# Q$ and $\tilde{\Gamma}^\# \begin{pmatrix} 1 & e_2 \\ 0 & 1 \end{pmatrix}$. ■

Given $n \in \mathbf{N}$ then

$$\Omega_n := \Delta - 2n(2n-3)\text{Id}$$

proves to be an $\text{MSp}(1; \mathbf{H})$ -invariant differential operator.

PROPOSITION 3. (a) Given $Z \in H(n; \mathbf{H})$, $w \in \mathcal{H}$ and a lattice Λ in \mathbf{H}^{2n} , one has

$$\begin{aligned} \Omega_n \vartheta(Z, w; \Lambda) &= v^{2n} \sum_{\lambda \in \Lambda} ((\pi v)^2 (P_Z[\lambda])^2 - |J[\lambda]|^2) - (4n-2)\pi v P_Z[\lambda] e^{\pi i \langle u, J[\lambda] \rangle - \pi v P_Z[\lambda]}. \end{aligned}$$

(b) Given a compact subset $\mathcal{C} \subset H(n; \mathbf{H})$, there exist positive constants C and δ depending only on \mathcal{C} such that

$$|\Omega_n \Theta(Z, w)| \leq C v^{-2n} e^{-\delta v}$$

for all $Z \in \mathcal{C}$ and $w \in \mathcal{H}$. Moreover, $\Omega_n \Theta(Z, w)$ is invariant under the transformations $Z \mapsto M \langle Z \rangle$, $M \in \Gamma_n$, as well as $w \mapsto N \langle w \rangle$, $N \in \tilde{\Gamma}$.

Proof. (a) The calculation is straightforward.

(b) Apply the same arguments as in [9, Proposition 3]. ■

Let $\tilde{\mathcal{F}}$ denote a fundamental domain of \mathcal{H} with respect to the action of $\tilde{\Gamma}$ (cf. [9]). Just as in [9] resp. [2] consider the Eisenstein series

$$\tilde{E}(w, s) := \sum_{N \in \tilde{\Gamma}_\infty \setminus \tilde{\Gamma}} (v_N)^s, \quad w \in \mathcal{H},$$

where $\tilde{\Gamma}_\infty$ denotes the subgroup of $\tilde{\Gamma}$ with $(0 \ 1)$ as its second row. Moreover, let $dv = w_1^{-4} dw_1 dw_2 dw_3 dw_4$ denote the invariant volume element. Then we get

LEMMA 4. Given $n > 1$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 4n - 3$, one has for all $Z \in H(n; \mathbf{H})$

$$(12) \quad \int_{\tilde{\mathcal{F}}_\infty} \tilde{E}(w, s + 3 - 2n) \Omega_n \Theta(Z, w) dv(w) = 4s(s + 3 - 4n)(1 + 2^{2n-1-s}) \pi^{-s} \Gamma(s) E^*(Z, s).$$

Proof. There exists a constant C such that $|\tilde{E}(w, s)| \leq C v^{\text{Re}(s)}$ for all $w \in \tilde{\mathcal{F}}$. Hence the integral exists due to Proposition 3 and $\text{vol}(\tilde{\mathcal{F}}) < \infty$. Let

$$\tilde{\mathcal{F}}_\infty := \{w = u + v \in \mathcal{H} \mid 0 \leq u_j \leq 1, j = 2, 3, 4\}$$

be a fundamental domain with respect to the action of $\tilde{\Gamma}_\infty$. Then the integral (12) has the value

$$\begin{aligned} & \int_{\tilde{\mathcal{F}}_\infty} \left(\sum_{N: \Gamma_\infty \backslash \Gamma} (v_N)^{s+3-2n} \Omega_n \Theta(Z, N \langle w \rangle) \right) dv(w) \\ &= 2 \int_{\tilde{\mathcal{F}}_\infty} v^{s+3-2n} \Omega_n \Theta(Z, w) dv(w) \\ &= 4 \sum_{\lambda \in \mathcal{O}^{2n}, J[\lambda]=0} \int_0^\infty v^{s-1} ((\pi v P_Z[\lambda])^2 - (4n-2)\pi v P_Z[\lambda]) e^{-\pi v P_Z[\lambda]} dv \\ & \quad + 2^{2n+1} \sum_{\lambda \in \mathcal{O}^{*2n}, J[\lambda]=0} \int_0^\infty v^{s-1} ((\pi v P_Z[\lambda])^2 - (4n-2)\pi v P_Z[\lambda]) e^{-\pi v P_Z[\lambda]} dv \\ &= 4(\Gamma(s+2) - (4n-2)\Gamma(s+1)) \pi^{-s} \Gamma(s) \\ & \quad \times \left(\sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s} + 2^{2n-1} \sum_{0 \neq \lambda \in \mathcal{O}^{*2n}, J[\lambda]=0} (P_Z[\lambda])^{-s} \right) \\ &= 4s(s+3-4n) \pi^{-s} \Gamma(s) (1 + 2^{2n-1-s}) \left(\sum_{0 \neq \lambda \in \mathcal{O}^{2n}, J[\lambda]=0} (P_Z[\lambda])^{-s} \right). \end{aligned}$$

Then (9) completes the proof. ■

Now set

$$\zeta(2s) = \pi^{-s} \Gamma(s) \zeta(2s), \quad \zeta_\theta(s) = (\sqrt{2} \pi)^{-s} \Gamma(s) \zeta_\theta(s),$$

which satisfy

$$\zeta(1-s) = \zeta(s), \quad \zeta_\theta(2-s) = \zeta_\theta(s).$$

Then the combination of Lemma 4 and [9, Theorem 3], resp. [2] implies

COROLLARY 1. Given $n > 1$ the Eisenstein series $E(Z, s)$ possesses a meromorphic continuation to the whole complex s -plane. More precisely,

$$E(Z, s) := 2^s (1 + 2^{2n-2-s}) (1 + 2^{2n-1-s}) \zeta_\theta(s+3-2n) \zeta_\theta(s) \zeta(2s+4-4n) E(Z, s)$$

becomes a holomorphic function of $s \in \mathbb{C}$ except for possible simple poles at $s = 0, 2n-3, 2n-2, 2n-1, 2n, 4n-3$ and satisfies the functional equation

$$E(Z, 4n-3-s) = E(Z, s).$$

Setting

$$\begin{aligned} \mathbf{D}(f, g; s) &:= (1 + 2^{k+1-2n-s}) (1 + 2^{k+2-2n-s}) \zeta_\theta(s+2n-k) \zeta_\theta(s+4n-3-k) \\ & \quad \times \zeta(2s+4n-2-2k) (2\pi)^{-s} \Gamma(s) \mathbf{D}(f, g; s), \end{aligned}$$

we obtain

$$\{f \mathbf{E}(\cdot, s+4n-3-k), g\} = 2^{4n-2-k} \mathbf{D}(f, g; s)$$

from Lemma 2. Thus Corollary 1 implies

THEOREM 3. Let $f, g \in [\Gamma(n; \mathbf{H}), k]_0$ be modular forms of quaternions of even weight k and degree $n > 1$, which are cusp forms. Then the Dirichlet series $\mathbf{D}(f, g; s)$ possesses a meromorphic continuation to the whole complex s -plane. More precisely, $\mathbf{D}(f, g; s)$ is a holomorphic function of $s \in \mathbb{C}$ except for possible simple poles at

$$s = k, k+3-2n, k+2-2n, k+1-2n, k-2n, k+3-4n$$

and satisfies the functional equation

$$\mathbf{D}(f, g; 2k+3-4n-s) = \mathbf{D}(f, g; s).$$

Again it is clear that $\mathbf{D}(f, g; s)$ has trivial zeros at $s = 0, -1, -2, \dots$. One can also cancel several factors in the definition of $\mathbf{D}(f, g; s)$, but the cancellation increases the possible number of poles. The function

$$\begin{aligned} \mathbf{D}^*(f, g; s) &:= (s+2n-1-k)(s+4n-4-k) (1 - 2^{k+1-2n-s}) (1 - 2^{k+4-4n-s}) \\ & \quad \times \zeta(s+2n-k) \zeta(s+4n-3-k) \zeta(s+4n-4-k) \\ & \quad \times \zeta(2s+4n-2-2k) (2\pi)^{-s} \Gamma(s) \mathbf{D}(f, g; s) \end{aligned}$$

is also invariant under $s \mapsto 2k+3-4n-s$.

7. The Maaß space. In this section we consider modular forms in the Maaß space (cf. [8]). Therefore let $n = 2, \mathbf{F} = \mathbf{H}$. We always assume Z, T to be

$$Z = \begin{pmatrix} z & w \\ \bar{w} & z^* \end{pmatrix} \in H(2; \mathbf{H}), \quad T = \begin{pmatrix} l & \frac{1}{2}t \\ \frac{1}{2}\bar{t} & m \end{pmatrix} \in \text{Sym}^\tau(2; \mathcal{O}).$$

The Maaß space $\mathcal{M}(k; \mathbf{H})$ consists of all $f \in [\Gamma(2; \mathbf{H}), k]$ with the Fourier expansion (3) such that a function $\alpha_f^*: N_0 \rightarrow \mathbb{C}$ exists satisfying

$$\alpha_f(T) = \sum_{d \in N, d^{-1}T \in \text{Sym}^\tau(2; \mathcal{O})} d^{k-1} \alpha_f^*(2d^{-2} \det T)$$

for all $0 \neq T \in \text{Sym}^\tau(2; \mathcal{O}), T \geq 0$. The subspace $\mathcal{M}(k; \mathbf{H})_0$ of cusp forms is characterized by the condition $\alpha_f^*(0) = 0$.

Given $m \in N$ set

$$\mathcal{T}(m) = \{M \in \text{Mat}(2; \mathbf{Z}) \mid \det M = m\}, \quad \mathcal{T}_0(m) = \{M \in \mathcal{T}(m) \mid \gcd(M) = 1\}.$$

Due to [8, Lemma 2], we have

$$(13) \quad f_m(z, w) = f_{1|k}[\mathcal{T}(m)](z, w) := m^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \backslash \mathcal{T}(m)} d^{-k} f_1\left(\frac{az+b}{d}, \frac{mw}{d}\right),$$

where $\Gamma = \text{SL}(2; \mathbf{Z})$. We can determine the adjoint $[\mathcal{T}(m)]^*$ of this operator with respect to the Petersson scalar product. Given $f, g \in \mathcal{M}(k; \mathbf{H})_0, m \geq 1$, we obtain

$$\langle f_m, g_{1|k}[\mathcal{T}(m)] \rangle = \langle f_{m|k}[\mathcal{T}(m)]^*, g_1 \rangle,$$

where

$$(14) \quad f_{m|k}[\mathcal{T}(m)]^*(z, w) = m^{k-9} \sum_{\lambda, \mu: \mathcal{O}/m\mathcal{O}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \backslash \mathcal{T}(m)} d^{-k} e^{2\pi i(zN(\lambda) + 2\langle \lambda, w \rangle)} f_m\left(\frac{az+b}{d}, \frac{w + \lambda z + \mu}{d}\right)$$

is a Jacobi form of index 1. The proof follows the same lines as for the Proposition in [6] and can therefore be omitted.

In analogy with [1, §4], we introduce a Hecke operator \mathcal{T}_m^0 on the space of Jacobi forms of index 1 via

$$f_{1|k} \mathcal{T}_m^0(z, w) := m^{k-10} \sum_{\lambda, \mu: \mathcal{O}/m\mathcal{O}} \sum_{M: \Gamma \backslash \mathcal{T}_0(m^2)} f_{1|k,1}[m^{-1}M]_{|1}[\lambda, \mu],$$

where we use the notation (5).

PROPOSITION 4. Given $f \in \mathcal{M}(k; \mathbf{H})$ and $m \geq 1$ one has

$$(15) \quad f_{1|k}[\mathcal{T}(m)]_{|k}[\mathcal{T}(m)]^* = \sum_{l|m} \sigma_1\left(\frac{m}{l}\right) \left(\frac{m}{l}\right)^{k-2} f_{1|k} \mathcal{T}_l^0.$$

Proof. Using (13) and (14) the left hand side of (15) is equal to

$$m^{k-10} \sum_M \sum_N \sum_{\lambda, \mu} f_{1|k,1}[m^{-1}MN]_{|1}[\lambda, \mu],$$

where M and N run through sets of representatives of $\Gamma \backslash \mathcal{T}(m)$. Now we apply Theorem 3.24 in [14] and see that each right coset $\Gamma L, L \in \mathcal{T}(m^2)$, appears exactly $\sigma_1(m/l)$ times as ΓMN , where l is the gcd of the entries of L . Then a simple calculation completes the proof. ■

Given $f \in \mathcal{M}(k; \mathbf{H})$ with Fourier–Jacobi coefficients f_m in (4) we consider the mapping

$$\Omega: f_1 \mapsto F(z) := \sum_{l=0}^{\infty} \alpha_f^*(l) e^{2\pi i l z}.$$

Ω maps the space of Jacobi forms of index 1 onto a certain subspace \mathfrak{M}_{k-2} of elliptic modular forms for $\Gamma_0 := \Gamma_0(4)$ of weight $k-2$ (cf. [10]). Using the same definition for Hecke operators as in [10] without any normalizing factors, we obtain

PROPOSITION 5. Let $f \in \mathcal{M}(k; \mathbf{H})$.

(a) Given relatively prime $m, n \in N$ one has

$$f_{1|k} \mathcal{T}_m^0 |_{k} \mathcal{T}_n^0 = f_{1|k} \mathcal{T}_{mn}^0.$$

(b) If p is an odd prime, then

$$f_{1|k} \mathcal{T}_p^0 = \Omega^{-1} \left(p^{2l(k-3)} F \Big|_{k-2} \Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & p^{2l} \end{pmatrix} \Gamma_0 \right)$$

and

$$f_{1|k} \mathcal{T}_2^0 = \Omega^{-1} \left(2^{2l(k-3)} F \Big|_{k-2} \left(\Gamma_0 \begin{pmatrix} 1 & 0 \\ 0 & 2^{2l} \end{pmatrix} \Gamma_0 + \Gamma_0 \begin{pmatrix} 2^{2l} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0 \right) \right)$$

for all $l \geq 1$.

Proof. (a) This part is easily verified (cf. [1, §4]).

(b) Proceed in the same way as Raghavan and Sengupta [13] in the proof of Lemma 2. The arising Gauss sums are evaluated to be

$$G(\alpha, h, p^l) := \sum_{g: \mathcal{O}/p^l\mathcal{O}} e^{2\pi i \alpha p^{-l} N(g+h/2)} = \begin{cases} (-1)^{\alpha N(h)/2} p^{2l} & \text{if } p \text{ is odd,} \\ -2^{2l+1} & \text{if } p = 2, h \in 2\mathcal{O}, \\ 0 & \text{if } p = 2, h \notin 2\mathcal{O}, \end{cases}$$

where $\alpha \in \mathbf{Z}, p \nmid \alpha, l \geq 1, h \in \mathcal{O}^*$. ■

Due to [10] the Maaß space $\mathcal{M}(k; \mathbf{H})$ is invariant under all Hecke operators. More precisely, it can be concluded from Theorem 2 in [10] that f_1 is a simultaneous eigenform under all operators $\mathcal{T}_m^0, m \geq 1$, provided that f is a simultaneous eigenform under all Hecke operators. Thus Proposition 5 leads to

COROLLARY 2. Let $f \in \mathcal{M}(k; \mathbf{H})$ be a simultaneous eigenform under all Hecke operators with $\alpha_f^*(1) = 1$. Then

$$f_{1|k} \mathcal{T}_p^0 = (\alpha_f^*(p^{2l}) - p^{k-4} \alpha_f^*(p^{2l-2})) f_1, \quad f_{1|k} \mathcal{T}_2^0 = \alpha_f^*(2^{2l}) f_1$$

for $l \geq 1$ and all odd primes p .

Now we apply all the results above in order to evaluate the Dirichlet series $D(f, g; s)$.

THEOREM 4. Let $f, g \in \mathcal{M}(k; \mathbf{H})_0$, where f is a simultaneous eigenform under all Hecke operators with $\alpha_f^*(1) = 1$. Then the Dirichlet series $D(f, g; s)$ possesses an Euler product expansion of the form

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{1 - 2^{k-2-s}}{1 - 2^{k-4-s}} \times \prod_p \frac{(1 - p^{k-4-s})(1 + p^{k-3-s})}{(1 - p^{k-2-s})(1 - p^{k-1-s})(1 + \alpha_f^*(p)p^{-s/2} + p^{k-3-s})(1 - \alpha_f^*(p)p^{-s/2} + p^{k-3-s})}$$

Proof. Propositions 4, 5 and Corollary 2 imply

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{\zeta(s+2-k)\zeta(s+1-k)}{\zeta(s+4-k)(1-2^{k-4-s})} \prod_p \sum_{l=1}^{\infty} \alpha_f^*(p^{2l})p^{-ls}$$

Then Corollary 1 in [10] completes the proof. ■

Remark. Under the assumptions of Theorem 4 the Dirichlet series $D(f, g; s)$ is proportional to an Andrianov zeta function attached to f , namely

$$D(f, g; s) = \langle f_1, g_1 \rangle \frac{\zeta(s+2-k)}{\zeta(s+4-k)(1-2^{k-4-s})} \sum_{l=1}^{\infty} \alpha_f(lT_0)l^{-s}$$

where

$$2T_0 = \begin{pmatrix} 2 & e_1 + e_2 \\ e_1 - e_2 & 2 \end{pmatrix}$$

If $\alpha_f^*(2) \neq 0$, it follows from [10, Theorem 7] that

$$D(f, g; s) = \frac{\langle f_1, g_1 \rangle}{\alpha_f^*(2)} \frac{1 + 2^{k-3-s} \zeta(s+2-k)}{1 - 2^{k-4-s} \zeta(s+4-k)} A(f, s)$$

where $A(f, s)$ denotes the Andrianov zeta function attached to f in [10].

If additionally $\langle f_1, g_1 \rangle \neq 0$, then $D(f, g; s)$ possesses a pole at $s = s_0$ if and only if $A^*(f, s_0) \neq 0$, whenever $s_0 = k, k-1, k-2, k-3, k-4, k-5$.

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