

## A Discovery System for Trigonometric Functions

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### Abstract

This paper describes a discovery system for trigonometric functions (DST), which has abilities to acquire new knowledge in the form of theorems and formulas in a plane geometry domain. The system is composed of two subsystems: a plane geometry analysis system and a mathematical formula transformation system. The former changes the length and angles of a figure and extracts geometric relations, and the latter transforms the relations to acquire useful formulas. With little basic knowledge such as the definition of the congruence of triangles and the definition of fundamental trigonometric functions, our system has rediscovered many trigonometric formulas and geometric theorems, including the Pythagorean theorem.

### Introduction

Machine discovery elucidates human's intelligent activities, and it automates creative tasks of finding new knowledge. This paper proposes a method for discovery in a plane geometry domain, and describes a discovery system for trigonometric functions (DST) we developed. DST's initial knowledge is the definitions of similarity and congruence, and the concept of a triangle, such that a triangle consists of three lines. Definitions of fundamental trigonometric functions, such as sine, cosine, and tangent, are also given. Our DST finds relations of a figure which is obtained from a triangle by drawing additional lines and by changing its shape. The obtained relations among its angles and sides are transformed into geometric theorems. With a simple method, our system can rediscover many theorems.

Most of the previous discovery systems operate in physics and chemistry domains. Only few attempts have so far been made at machine discovery in a geometric domain. The domain has many research topics, such as the representation of geometric relations, the integration of figures and formulas, and the evaluation of acquired formulas.

AM (Lenat 1983) and BACON (Langley, Bradshaw, & Simon 1983) are well-known discovery systems. Nec-

essary data and heuristics are given to the system in advance. AM has many heuristics to guide its discovery. BACON discovers relations among the variables only within the given data.

In discovery systems, experiments play an important role to acquire useful data. COAST (Rajamoney 1990) and DEED (Rajamoney 1993) can design experiments to acquire data that discriminate between competing theories. COAST uses heuristics to modify the given situation so that it can discriminate between different theories. DEED uses the difference between the explanations of the competing theories as a clue for modifying the given situation. These knowledge-intensive approaches to experiment design is useful only when the knowledge for making explanations is sufficient.

KEKADA (Kulkarni & Simon 1988) proposes experiments based on the heuristics of surprising phenomena that constrain the search for new knowledge. Surprise arises when there are differences between an experimental result and its expectation. Defining surprising phenomena, however, is difficult when there is little amount of knowledge.

DST uses neither explanation nor surprise; it manipulates its environment, observes the effects, and relates the effects with their causes. Such a method is generally applicable to domains with little knowledge.

## A Discovery System for Trigonometric Functions

### Structure of DST

To discover geometric theorems, two kinds of activities are required. One is to extract relations of geometric elements, such as angles and length of sides, from a figure. The other activity is to transform the extracted relations.

In our discovery system, DST, these activities are performed by two subsystems: a plane geometry analysis system (PGA) and a mathematical formula transformation system (MFT), as shown in Figure 1. PGA extracts relations from a figure, changes its length of sides and angles, and draws an additional line. MFT transforms the extracted relations to discover new geometric theorems.

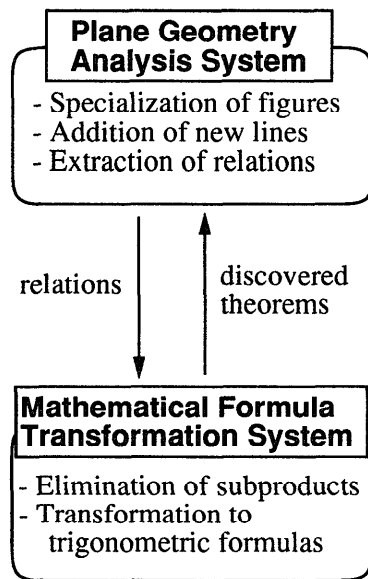


Figure 1: Two subsystems of DST

DST's initial knowledge is only for analyzing figures:

- Definitions of sine, cosine, and tangent as ratios of two sides in a right triangle
- Axioms of plane figures
  - If two triangles are congruent, their corresponding sides and angles are equal.
  - If two triangles are similar, their corresponding angles and the ratios of length of their corresponding sides are equal.

The former definitions are necessary to express the relations of sides and angles. The latter characteristics between two triangles, such as the similarity and congruence, are needed since a triangle is the fundamental element.

The amount of initial knowledge varies according to the purpose of discovery systems. Knowledge-intensive discovery systems use considerable amount of basic domain knowledge to acquire advanced knowledge. Discovery with too much initial knowledge is, however, nothing more than the transformation of the given knowledge. While our DST aims at initial exploration and can discover theorems with very few initial knowledge.

### Plane Geometry Analysis System

The role of PGA is to generate figures from an arbitrary triangle and to extract relations among sides and angles for MFT. PGA has the following mechanisms:

- Specialization of figures
- Addition of new lines

- Extraction of relations

**Specialization of figures** Performing specialization is one way to make experiments in a geometric domain. Specialization of a triangle can be done in the following way:

- changing a triangle to a right triangle
- making two angles equal
- making two sides equal

In some cases, giving one specialization causes other relations among sides and angles. For example, making two angles of a triangle equal can be a result of making two sides equal. The former specialization implies the latter, which can be considered as a new theorem. Among specialization, changing a triangle to a right triangle is often useful since trigonometric functions are defined with a right triangle.

**Addition of new lines** Addition of new lines to a figure often clarifies relations among its basic elements. An additional line divides a figure into two smaller figures, and the relations extracted from each smaller figure are combined to find relations in the figure. Additional lines drawn by PGA are as follows:

- a bisector of an angle
- a bisector of a side
- a perpendicular lines from a vertex

Additional lines generate new elements, such as sides and angles, which are called *subproducts*. Although subproducts are useful for extracting new relations, they cannot be used to express new theorems and should be eliminated later.

**Extraction of relations** A figure consists of basic elements, such as points, lines, angles and triangles. A change in one element often causes side effects to other parts of the figure.

In PGA, the same kind of basic elements are grouped together. The elements in the same group are combined together to form a different kind of element, and relations among them are found. After finding the relations, PGA sends them to MFT for further process. As shown in Figure 2, an additional line CH is drawn on a triangle ABC, and relations among the length of lines are extracted. Two lines which share the same end are combined to make an angle, and relations of adjacent angles are extracted. Similarly, three lines are combined to make a triangle, and relations of congruent or similar triangles are also extracted. Lines, angles, and triangles are mutually related in this way so that specialization to an element is propagated to the rest of the elements in the figure.

### Mathematical Formula Transformation System

The role of MFT is to discover theorems from the formulas which express the extracted relations. The dis-

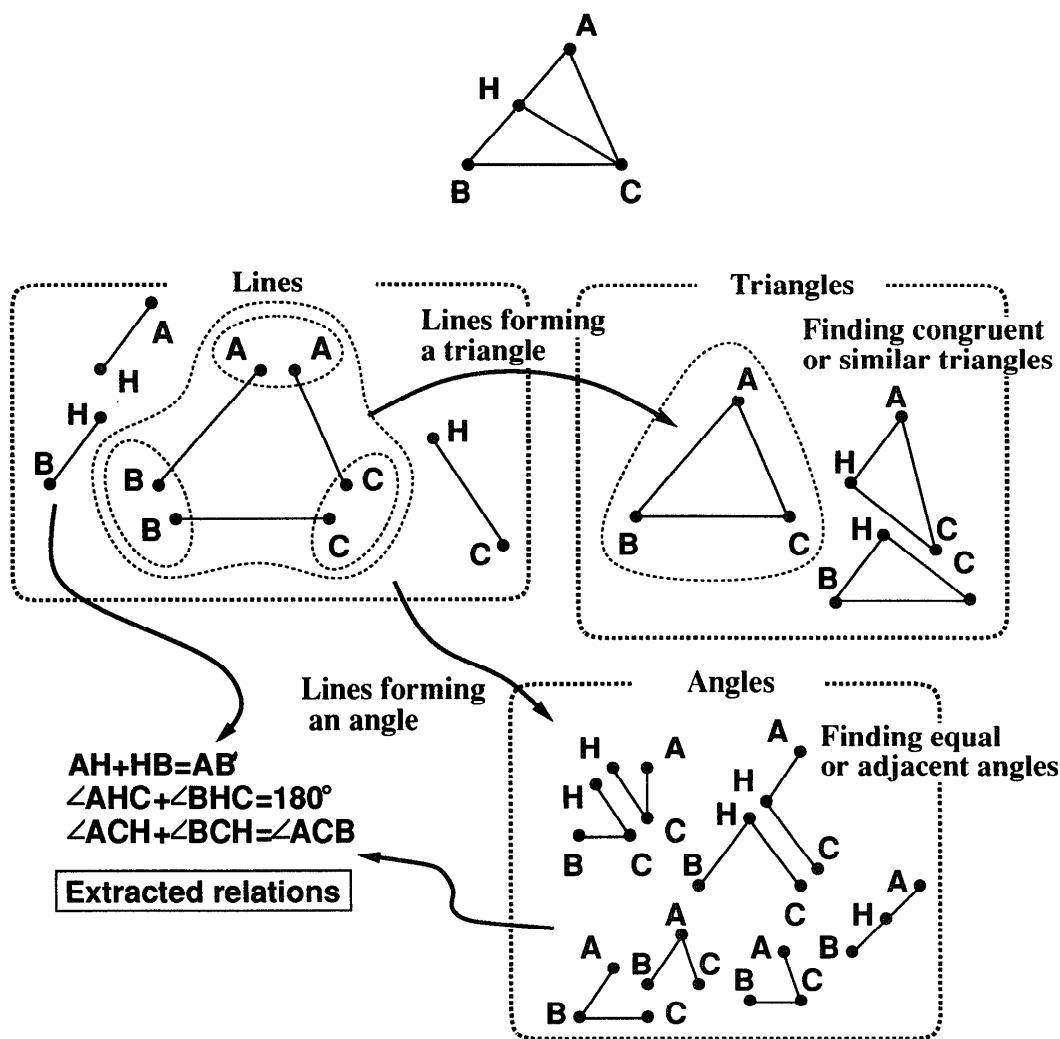


Figure 2: Extraction of relations

covery of MFT is performed by transposition, substitution, and fundamental arithmetic operations.

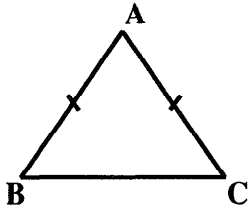
**1. Generation of a theorem** To generate a theorem, MFT eliminates subproducts in the obtained relations by repeating substitution. If the result of substitution contains no subproduct variable, it can be considered that a theorem is generated. The conditional part of the theorem is obtained from the formulas of the figure before drawing additional lines, and the consequent part is from the acquired formulas.

Figure 3 shows a process of discovering a theorem, "An isosceles triangle has equal base angles." The relation obtained from a triangle ABC before drawing an additional line is the given condition  $AB=AC$ . MFT draws a perpendicular line AH from vertex A to side

BC, and generates two congruent triangles,  $\triangle AHB$  and  $\triangle AHC$ . Among newly extracted relations,  $\angle ABC = \angle ACB$  contains no subproducts. Since the line does not add any new constraints to the original triangle, the relation is considered as a characteristic of the isosceles triangle. Therefore, the theorem is obtained of which conditional part is  $AB=AC$  and consequent part is  $\angle ABC = \angle ACB$ . As seen in the above process, it is found that the theorem is obtained from the relations which are extracted before and after drawing the line.

As a result of subproduct elimination, formulas with many terms or with complex terms are often generated. MFT does not use these formulas for subsequent transformation since they cause explosive increase of formulas. To put it more concretely, the results of the sub-

**Before drawing  
an additional line**



**After drawing it**

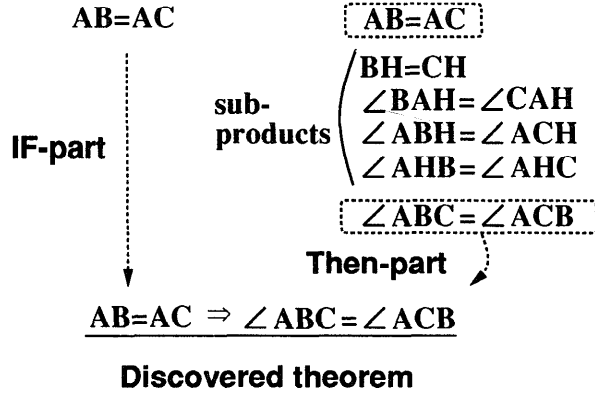
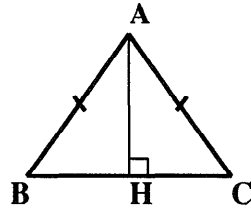


Figure 3: Discovery of a geometric theorem

stitution that increases the number of subproducts are discarded. Formulas whose dimension is higher than three are also discarded.

The number of formula transformation is also constrained in LEX (Mitchell et al. 1981) which treats symbolic integration. Heuristics about the application of its operators are the criteria for formula transformation. In MFT, the dimension of formulas and the number of variables are used as the criteria, since the formula transformation depends solely on the selection of formulas.

**2. Trigonometric representation** In order to acquire trigonometric theorems, discovered theorems in the above process have to be transformed. That is, variables expressing sides of a right triangle are eliminated from the theorem by using the definition of trigonometric functions. By introducing basic trigonometric functions such as sine, cosine, and tangent, a formula  $\sin^2\theta + \cos^2\theta = 1$  is acquired from the Pythagorean theorem which is discovered previously.

## Results

The items listed below are some of the theorems that DST has rediscovered from a triangle.

- $\tan \theta = \sin \theta / \cos \theta$

- $\tan \theta = \sin \theta / \sin(90^\circ - \theta)$
- $\tan \theta = 1 / \tan(90^\circ - \theta)$
- $\sin^2\theta + \cos^2\theta = 1$
- An isosceles triangle  
→ Its base angles are equal.
- A triangle of equal base angles  
→ It is an isosceles triangle.
- A right triangle  
→ The Pythagorean theorem holds.

The first four theorems are well-known fundamental laws of trigonometric functions, and the rest are obtained by interpreting the relation of acquired formulas. In the manner described above, MFT generates a number of formulas. For example, the Pythagorean theorem is found as EXPR4024 formed by EXPR3804 and EXPR4023, as shown in Figure 4. In general, Pythagorean theorem can be proved based on the comparison of area of additional squares. It should be noted that DST finds the theorem only by mathematical transformation without using the concept of area. The figure is generated by changing an angle to a right angle and by drawing a perpendicular line from a vertex to its opposite side. PGA finds that the triangles  $\triangle ABC$ ,  $\triangle PBA$ , and  $\triangle PAC$  are similar triangles since they have corresponding angles. Extracted relations are that the ratios of adjacent sides of the corresponding angles are equal. Other geometric relations, such as  $BP + CP = BC$ , are also extracted. Finally, MFT discovers the theorem  $AB^2 + AC^2 = BC^2$ , which includes no subproduct such as AP, BP, and CP.

In Figure 4, EXPR3636 and EXPR3810 involve subproduct variables AP, BP, and CP. MFT eliminates BP from these two expressions and produces EXPR4022. Then MFT finds EXPR3806 which involves the same subproduct variables in EXPR4022 to eliminate AP. In the same manner, EXPR3804 is used to eliminate CP. Finally MFT discovers EXPR4024 which involves no subproduct, and this formula is the well-known Pythagorean theorem.

PGA generates figures by performing all possible specialization and addition of lines to the given figure. All the figures constitute a tree structure whose node shows a generated figure and whose arc shows specialization or addition of lines. PGA selects one of the node for its analysis in the breadth-first order. Geometrically equal figures possibly appear in different node in the tree. These figures are, however, considered as different figures in PGA, since they are identified as a sequence of specialization and addition of lines. Generally the discovered theorem would be the same if the analyzed figures are equal, even if they appear in different nodes. However, theorems actually discovered from the figures are not always the same. For example, the fifth and the sixth theorems listed above are rediscovered separately in DST, though the figures used for the discovery are geometrically equal.

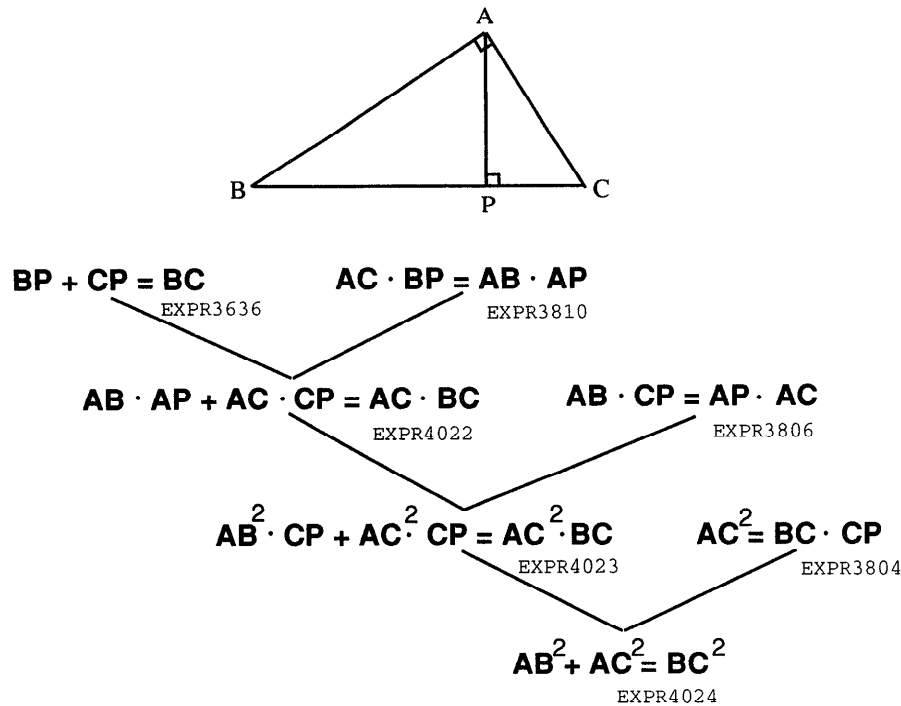


Figure 4: Discovery of the Pythagorean theorem

MFT combines theorems which are discovered from different figures to generate new different theorems. From the first and the second theorems listed above, a theorem “ $\cos \theta = \sin(90^\circ - \theta)$ ” will be generated.

## Discussion

As Zytow and Baker pointed out (Zytow & Baker 1991), experimentation has a number of advantages over mere manipulation of given data, since an abundance of data are provided, the quality of data is improved, important data for constructing theorems can be obtained, and various situations are created so that regularities of data are easy to discover. Manipulation on figures to get useful data corresponds to experimentation in a plane geometry domain. DST autonomously manipulates its environment and acquires data in order to discover with little initial knowledge.

To evaluate the generality of discovered laws or theorems enhances their utility. ABACUS (Falkenhainer & Michalski 1986) employs discriminant descriptions of classifying observed data into classes in generated numeric laws. FAHRENHEIT (Zytow 1987) specifies the scope of discovered laws as extended numeric laws. Our DST analyzes and specializes figures under geometrical constraints. The scope of theorems discovered in DST, therefore, is given by such constraints of analyzed figures in the specialization process in a plane

geometry.

In order to evaluate the utility of concepts and to restrict the search space of discovery process, AM uses “interestingness.” The criteria are, however, fixed by the initial definitions of concepts. The complexity of expressions in DST is evaluated by the number of variables, dimensions of formulas, and the number of subproducts. Though DST has no similar criteria to AM, elimination of subproducts turns out to be an effective approach for discovering useful theorems. Also the elimination avoids the combinational explosion of formulas since it restricts the search space of formula transformation.

Sometimes, however, we encounter the case where theorems employing subproducts are important in a target domain. To find such theorem, formulas obtained in PGA should not be eliminated. For example, to find theorems about a center of gravity in a triangle, additional lines to a triangle are required. It is important, therefore, to discriminate geometrically important subproducts from mere auxiliary ones.

## Concluding Remarks

We have developed a discovery system for trigonometric functions in a plane geometry. Our system rediscovers many theorems, including the Pythagorean theorem, with a simple method. It should be noted that

DST rediscovered from quite little knowledge. In a similar manner of human problem-solving in a plane geometry domain, PGA acquires relations among basic elements under various conditions. MFT generates 10,000 formulas to discover the theorems described above. Omitting useless formulas during its transformation avoids the generation of explosive number of formulas.

One direction to extend DST is to consider new basic elements, such as circles and four-sided figures. If a circle is added as a basic element, relations about an inscribed circle and a circle circumscribing a triangle will be extracted by PGA.

## Acknowledgments

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