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A discrete analogue of the inequality of Lyapunov

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1. This paper is concerned with the difference equation

$$\Delta^2 x(k-1)+p(k)\,x(k)=0 \tag{1}$$

where $p(k)$ is a real valued function defined on a set of consecutive integers to be specified later. Our work is motivated by a classical result of Lyapunov [2] which states that if $x(t)$ is a nontrivial solution of the differential system

$$x''(t)+p(t)\,x(t)=0 \qquad a\leq t\leq b \tag{2}$$

$$x(a)=0, \quad x(b)=0 \tag{3}$$

where $p(t)$ is a continuous and non-ngeative function defined on $[a, b]$, then

$$(b-a)\int_a^b p(s)\,ds>4 \tag{4}$$

and the inequality is sharp. In view of the obvious similarity between the equations (1) and (2), we expect to find discrete analogoues of (4) which are necessary for the existence of a non-trivial solution of (1) satisfying certain boundary conditions.

In the next section, we shall assume that $p(k)$ is a non-negative function defined on the set $\{1, 2, \cdots, N\}$ and derive a condition which is necessary for (1) to have a non-trivial solution $x(k)$ satisfying $x(0)=0$ and $x(N+1)=0$. Under the same assumption on $p(k)$, we then derive in the third section a more general condition which is necessary for the same equation to have a nontrivial solution $x(k)$ satisfying $x(0)+\sigma x(1)=0$ and $x(N+1)+\lambda x(N)=0$ where σ and λ are non-negative real numbers. We could have omitted the next section entirely but include it here for constrasting the principles and computations involved. In the final section, we use a comparison theorem to deal with the case when $p(k)$ can take on nonpositive values.

2. In the sequel, the smallest integer which is larger than or equal to the real number t will be denoted by t^+. Let $A_n=(a(i,j))$ be the n by n tridiagonal matrix defined by

$$a(i,j)=\begin{cases}2 & i=j\\ -1 & |i-j|=1\\ 0 & \text{otherwise.}\end{cases}$$

Let $G_n=(g(i,j))$ be the n by n matrix defined by

$$g(i,j)=\begin{cases}(n-i+1)j & 1\le j\le i\\ (n-j+1)i & i\le j\le n.\end{cases}$$

The elements of G_n are clearly positive. Furthermore, we may easily verified that

$$\max_{1\le i,\, j\le n}\left\{g(i,j)\right\}=g\left((n/2)^+,(n/2)^+\right)=\left(n-(n/2)^++1\right)(n/2)^+, \tag{5}$$

and that $(n+1)^{-1}G_n$ is the inverse of A_n.

If $p(k)$ is a non-negative function defined on $\{1,2,\cdots,N\}$ and if $x(k)$, $0\le k\le N+1$, is a nontrivial solution of (1) satisfying $x(0)=0$ and $x(N+1)=0$, then the vector

$$\underset{\sim}{x}=\mathrm{col}\left(x(1),x(2),\cdots,x(N)\right)$$

satisfies the matrix equation

$$A_N\underset{\sim}{x}-\mathrm{diag}\left(p(1),p(2),\cdots,p(N)\right)\underset{\sim}{x}=0. \tag{6}$$

Multiplying the above equation by $(N+1)^{-1}G_N$, we obtain

$$\underset{\sim}{x}=(N+1)^{-1}G_N\,\mathrm{diag}\left(p(1),\cdots,p(N)\right)\underset{\sim}{x}.$$

Let i be the integer such that

$$\left|x(i)\right|=\max_{1\le j\le N}\left|x(j)\right|,$$

then

$$\begin{aligned}\left|x(i)\right| &\le (N+1)^{-1}\sum_{j=1}^{N} g(i,j)\,p(j)\left|x(j)\right|\\ &\le \left|x(i)\right|(N+1)^{-1}\max G_N\sum_{j=1}^{N}p(j)\end{aligned}$$

or

$$p(1)+p(2)+\cdots+p(N)\ge\mu(N) \tag{7}$$

where

$$\begin{aligned}\mu(N)&=(N+1)/\left(N-(N/2)^++1\right)(N/2)^+\\ &=\begin{cases}(2m+1)/m(m+1) & \text{if } N=2m\\ 2/(m+1) & \text{if } N=2m+1.\end{cases}\end{aligned} \tag{8}$$

The inequality (7) is best possible in the sense that for any N, we can find non-negative $p(k)$ and nontrivial $x(k)$ such that $p(1)+\cdots+p(N)=\mu(N)$ and

$x(k)$ is a solution of (1) for $0 \le k \le N+1$. To see this, we first suppose $N=2m+1$. Let

$$x(k)=\begin{cases} k & 0\le k\le m+1 \\ 2m-k+2 & m+1\le k\le 2m+2 \end{cases}$$

and let

$$p(k)=-\Delta^2 x(k-1)/x(k)$$

for $1\le k\le 2m+1$. Then $x(k)$ satisfies (1) for $1\le k\le N$, $x(0)=0$, $x(N+1)=0$ and

$$p(k)=\begin{cases} 0 & 1\le k\le m \\ 2/(m+1) & k=m+1 \\ 0 & m+2\le k\le 2m+1 \end{cases}$$

as required. If $N=2m$, let

$$x(k)=\begin{cases} k & 0\le k\le m+1 \\ (m+1)(2m-k+1)/m & m+1\le k\le 2m+1 \end{cases}$$

and $p(k)=-\Delta^2 x(k-1)/x(k)$ for $1\le k\le 2m$. It can similarly be verified that $x(k)$ and $p(k)$ are the desired functions.

After a change of the independent variable k in (1), the above conclusions can be summarized in the following

PROPOSITION 1. *If $p(k)$ is a non-negative function defined on the set of consecutive integers $\{a, a+1, \cdots, b\}$ and if*

$$\Delta^2 x(k-1)+p(k)\,x(k)=0, \qquad a\le k\le b \tag{9}$$

has a nontrivial solution $x(k)$ which satisfies $x(a-1)=0$ and $x(b+1)=0$, then

$$p(a)+\cdots+p(b)\ge \mu(b-a+1)$$

and the inequality is sharp.

Note that $\mu(N)$ is a strictly decreasing function of N. It follows from Proposition 1 that if

$$p(a)+p(a+1)+\cdots+p(b)<\mu(b-a+1)$$

then (9) cannot have a nontrivial solution $x(k)$ which satisfies $x(c-1)=0$ and $x(d+1)=0$, where $a-1\le c-1\le d+1\le b+1$. For otherwise

$$\sum_{j=a}^{b} p(j)\ge \sum_{j=c}^{d} p(j)\ge \mu(d-c+1)\ge \mu(b-a+1)>\sum_{j=a}^{b} p(j)$$

which is a contradiction.

3. The principle used in the previous section can be applied to the more general system

$$\Delta^2 x(k-1)+p(k)\,x(k)=0, \qquad k=1,2,\cdots,N \tag{10}$$

$$x(0)+\sigma x(1)=0, \qquad \sigma\in R,\ \sigma\geq 0 \tag{11}$$

$$x(N+1)+\lambda x(N)=0, \qquad \lambda\in R,\ \lambda\geq 0 \tag{12}$$

where $p(k)$ is a non-negative function defined on $\{1,2,\cdots,N\}$. If $x(k)$, $0\leq k\leq N+1$, is a nontrivial solution of (10–12), then $\underset{\sim}{x}=\mathrm{col}\,(x(1),x(2),\cdots,x(N))$ satisfies

$$B_N\underset{\sim}{x}-\mathrm{diag}\left(p(1),\cdots,p(N)\right)\underset{\sim}{x}=0,$$

where $B_N=(b(i,j))$ is the matrix defined by

$$b(i,j)=\begin{cases}2+\sigma & i=j=1\\ 2+\lambda & i=j=N\\ a(i,j) & \text{otherwise.}\end{cases}$$

It can be verified that the inverse of B_N is the matrix

$$\left\{N+1+N\sigma+N\lambda+(N-1)\,\sigma\lambda\right\}^{-1}H_N$$

where $H_N=(h\,(i,j))$ is defined by

$$\begin{aligned}
&h(1,j)=(N-j+1)+(N-j)\lambda && 1\leq j\leq N\\
&h(N,j)=j+(j-1)\,\sigma && 1\leq j\leq N\\
&h(i,1)=(N-i+1)+(N-i)\lambda && 1\leq i\leq N\\
&h(i,j)=i+(i-1)\,\sigma && 1\leq i\leq N\\
&h(i,j)=h(i,1)\,h(N,j) && 1\leq j\leq i\leq N\\
&h(i,j)=h(i,N)\,h(1,j) && 1\leq i\leq j\leq N.
\end{aligned}$$

Note that the matrix H_N has positive components and that its "interior" components are products of "boundary" components. With these observations, we may arrive at the fact that

$$\max H_N=\begin{cases}h(m+1,m+1) & N=2m+1\\ \max\left\{h(m,m),\ h(m+1,m+1)\right\} & N=2m\end{cases} \tag{13}$$

$$=\begin{cases}(m+1+m\lambda)\,(m+1+m\sigma) & N=2m+1\\ \max\left\{(m+1+m\lambda)\,(m+(m-1)\sigma),\ (m+1+m\sigma)(m+(m-1)\lambda)\right\} & N=2m.\end{cases}$$

We may now proceed as in Section 1 to obtain the inequality

(14) $\qquad p(1)+p(2)+\cdots+p(N)\geq\mu(N,\sigma,\lambda)$

where

(15) $\qquad \mu(N,\sigma,\lambda)=\{N+1+N\sigma+N\lambda+(N-1)\,\sigma\lambda\}/\max H_N\,.$

The inequality (14) is sharp. To see this, we first assume that $N=2m+1$. Let $x_1^*(t)$ be the linear function whose graph passes through the points $(0,-\sigma)$ and $(1,1)$. Let $x(k)=x_1^*(k)$ for $0\leq k\leq m+1$. The linear function whose graph passes through the points $(N,1)$ and $(N+1,-\lambda)$ has a zero $N+1/(1+\lambda)$ in $(N,N+1]$. Let $x_2^*(t)$ be the linear function whose graph passes through the points $(m+1,x(m+1))$ and $(N+1/(1+\lambda),0)$ and let $x(k)=x_2^*(k)$ for $m+1\leq k\leq 2m+2$. If we now set $p(k)=-\Delta^2x(k-1)/x(k)$ for $1\leq k\leq N$, then clearly $x(k)$ satisfies (10–12) and $p(k)=0$ for $1\leq k\leq m$ and $m+2\leq k\leq 2m+1$. Moreover, $p(m+1)$ is equal to

$$\begin{aligned}
&\frac{\text{slope of } x_1^*(t)}{x(m+1)}-\frac{\text{slope of } x_2^*(t)}{x(m+1)}\\
&=\frac{1+\sigma}{(1+\sigma)(m+1)-\sigma}-\frac{\sigma-(1+\sigma)(m+1)}{\big(m+1/(1+\lambda)\big)\big((1+\sigma)(m+1)-\sigma\big)}\\
&=\frac{2m+2+(2m+1)\,\lambda+(2m+1)\,\sigma+2m\,\sigma\lambda}{(m+1+m\sigma)\,(m+1+m\lambda)}\\
&=\mu(2m+1,\sigma,\lambda)\,.
\end{aligned}$$

Next we suppose $N=2m$ and that $\lambda\geq\sigma$. Let $x_1^*(t)$ be the linear function whose graph passes through the points $(0,-\sigma)$ and $(1,1)$ and let $x_1^*(k)=x(k)$ for $0\leq k\leq m$. Let $x_2^*(t)$ be the linear function whose graph passes through the points $(m,x(m))$ and $(2m+1/(1+\lambda),0)$. Let $x(k)=x_2^*(k)$ for $m\leq k\leq 2m+1$ and let $p(k)=-\Delta^2x(k-1)/x(k)$ for $1\leq k\leq 2m$. Clearly, $x(k)$ satisfies (10–12) and $p(k)=0$ for $1\leq k\leq m-1$ and $m+1\leq k\leq 2m$. Furthermore,

$$\begin{aligned}
p(m)&=\frac{1+\sigma}{m(1+\sigma)-\sigma}-\frac{\sigma-m\,(1+\sigma)}{\big(m\,(1+\sigma)-\sigma\big)\big(2m+1/(1+\lambda)\big)}\\
&=\frac{2m+1+2m\,\sigma+2m\,\lambda+(2m-1)\,\sigma\lambda}{(m+1+m\lambda)\big(m+(m-1)\,\sigma\big)}\\
&=\mu\,(2m,\sigma,\lambda)\,.
\end{aligned}$$

The case $N=2m$ and $\sigma\geq\lambda$ can be dealt with similarly. We summarize our results as follows.

PROPOSITION 2. *If $p(k)$ is a non-negative function defined on the set* *of consecutive integers* $\{a, a+1, \cdots, b\}$ *and if the system*

(16) $\Delta^2 x(k-1)+p(k)\,x(k)=0 \qquad a\leq k\leq b$

(17) $x(a-1)+\sigma x(a)=0 \qquad \sigma\in R,\ \sigma\geq 0$

(18) $x(b-1)+\lambda x(b)=0 \qquad \lambda\in R,\ \lambda\geq 0$

has a nontrivial solution, then

(19) $p(a)+p(a+1)+\cdots+p(b)\geq\mu(b-a+1, \sigma, \lambda)$

and the inequality is sharp.

The function $\mu(N, \sigma, \lambda)$ has several important properties which we shall use later:

(i) $\mu(N, 0, 0)=\mu(N)$

(ii) if $\sigma'\geq\sigma$ and $\lambda'\geq\lambda$, then $\mu(N, \sigma', \lambda')\geq\mu(N, \sigma, \lambda)$,

(iii) if $\sigma\to\infty$, then $\mu(N, \sigma, \lambda)\to\mu(N-1, 0, \lambda)$,

(iv) if $\lambda\to\infty$, then $\mu(N, \sigma, \lambda)\to\mu(N-1, \sigma, 0)$,

and (v) if σ, $\lambda\to\infty$, then $\mu(N, \sigma, \lambda)\to\mu(N-2, 0, 0)$.

The verifications of these properties are straightforward and thus omitted.

4. Let $f(k)$ be a real function defined on a set of consecutive integers $\{a, a+1, \cdots, b\}$. If the points $(k, f(k))$ for $a\leq k\leq b$ are joined by straight line segments to form a broken line, then this broken line gives a representation of a continuous function, henceforth denoted by $f^*(t)$, such that $f(k)=f^*(k)$ for $a\leq k\leq b$. The zeros of $f^*(t)$ are called the nodes of $f(k)$. Note that $x(k)$ is a nontrivial solution of (16–18) if and only if $x(k)$ is a nontrivial solution of (16) with nodes $a-1/(1+\sigma)$ and $b+1/(1+\lambda)$. Note also that if α and β are consecutive nodes of a nontrivial solution of (16), then $\beta^+>\alpha+1$.

PROPOSITION 3. *Let σ and λ be two non-negative real numbers. Let $p(k)$ be a non-negative function defined on the set of consecutive integers $\{a, a+1, \cdots, b\}$. If*

$$p(a)+p(a+1)+\cdots+p(b)<\mu(b-a+1, \sigma, \lambda)$$

then (16) *cannot have a nontrivial solution which has nodes ξ and δ satisfying*

$$a-1/(1+\sigma)\leq\xi<\delta\leq b+1/(1+\lambda).$$

PROOF. Suppose our assertion is false. If $\xi=a-1/(1+\sigma')$ or $\delta=b+1/(1+\lambda')$ where σ' and λ' are non-negative real numbers satisfying $\sigma'\geq\sigma$ and

$\lambda' \geq \lambda$, then by Proposition 2, we would have

$$\sum_{j=a}^{b} p(j) \geq \mu(b-a+1, \sigma', \lambda') \geq \mu(b-a+1, \sigma, \lambda) > \sum_{j=a}^{b} p(j).$$

If $\xi = a$, $\delta = b + 1/(1+\lambda')$, we would have

$$\begin{aligned} p(a+1) + \cdots + p(b) &\geq \mu(b-a, 0, \lambda') \\ &= \lim_{\sigma' \geq \sigma, \lambda' \geq \lambda, \sigma' \to \infty} \mu(b-a+1, \sigma' \lambda') \\ &\geq \mu(b-a+1, \sigma, \lambda) \\ &> p(a) + \cdots + p(b). \end{aligned}$$

If $\xi = a$ and $\delta = b$, we would have

$$\begin{aligned} p(a+1) + \cdots + p(b-1) &\geq \mu(b-a-1, 0, 0) \\ &= \lim_{\sigma' \geqslant \sigma, \lambda' \geqslant \lambda;\ \sigma', \lambda' \to \infty} \mu(\mathrm{b}-\mathrm{a}+1, \sigma', \lambda') \\ &\geq \mu(b-a+1, \sigma, \lambda) \\ &> p(a) + \cdots + p(b). \end{aligned}$$

Similarly, we can show that the other cases are also impossible.

The following comparison theorem shall be needed in proving our discrete analogue of the inequality of Lyapunov.

THEOREM ([1, Lemma 2]). *Suppose $x(k)$ and $y(k)$, $a-1 \leq k \leq b+1$, are respectively nontrivial solutions of the equations*

$$\Delta^2 x(k-1) + f(k)\, x(k) = 0, \qquad a \leq x \leq b$$

and

$$\Delta^2 y(k-1) + g(k)\, y(k) = 0, \qquad a \leq x \leq b.$$

If $x(k)$ has two consecutive nodes α and β in $[a-1, b+1]$ and if $g(k) \geq f(k)$ for $a \leq k \leq b$, then $y(k)$ has a node in $(\alpha, \beta]$.

THEOREM. *Let σ and λ be two non-negative real numbers. Let $p(k)$ be a real function defined on the set of consecutive integers $\{a, a+1, \cdots, b\}$. If*

$$\sum_{k=a}^{b} \max\left\{p(k), 0\right\} < \mu(b-a+1, \sigma, \lambda), \tag{20}$$

then the equation (16) cannot have a nontrivial solution which has two distinct nodes in $[a-1/(1+\sigma), b+1/(1+\lambda)]$. The inequality (20) is sharp.

PROOF. Assume to the contrary that $x(k)$ is a nontrivial solution of

(16) which has two consecutive nodes ξ and β in $[a-1/(1+\sigma),\ b+1/(1+\lambda)]$. Then the Comparison Theorem asserts that the system

$$\begin{aligned} &\Delta^2 y(k-1)+\max\{p(k), 0\}\, y(k)=0, \qquad a\leq k\leq b \\ &y^*(\xi)=0 \end{aligned}$$

has a nontrivial solution $y(k)$ which has a node δ in $(\xi, \beta]$. Since

$$a-1/(1+\sigma)\leq\xi\leq\delta\leq\beta\leq b+\ /(1+\lambda),$$

by Proposition 3,

$$\sum_{k=a}^{b} \max\{p(k), 0\}\geq\mu(b-a+1, \sigma, \lambda)>\sum_{k=a}^{b} \max\{p(k), 0\}$$

which is the desired contradiction. The sharpness of inequality (20) has been shown in Section 3. Q. E. D.

References

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