ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 2 (1984), p. 257-270

J. KACPRZYK and W. STAŃCZAK (Warszawa)

A DISCRETE APPROXIMATION OF THE WEBER PROBLEM WITH EUCLIDEAN DISTANCE

1. Introduction. The *Weber problem*, known also as the *1-median problem*, is some specific location problem. It may be summarized as follows. In the plane there are given some points whose number and locations are fixed and specified in advance. The problem is to find a point in this plane such that the sum of weighted distances between this point and the previous ones is minimized.

The roots of the problem are very old. Its ancestor is the problem formulated first by Fermat in the early 1600's: "Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is as small as possible." The problem was solved geometrically around 1640 by Torricelli. In 1750 Simpson generalized this problem by including unequal weights for the respective distances. In 1909 Weber used this generalized model to determine the optimal location of a factory which produces one product with two given distinct sources of raw material and supplies one customer. By assumption, the three above points are not collinear, i.e. they are not all on the same straight line. The optimum was meant in terms of minimizing the sum of weighted distances between the factory and the raw material sources (suppliers) and the customer. The distance may be defined in various ways. The most popular are the rectilinear and Euclidean (quadratic) distances which are also the most important in practice.

Evidently, the above problem may further be extended to the case of more than three suppliers and customers. In fact, by the Weber problem we mean now this extended case.

The Weber problem is an adequate model of many practical problems, for example in spatial planning (e.g., facility location), telecommunication (the "copper centre" problem for locating a separate telephone exchange in a zone area), designing computer and/or terminal networks (e.g., the location of concentrators and communication computers), etc.

As was mentioned above, the most popular distances used are rectilinear and Euclidean ones. The first is more adequate for the case of highly urbanized areas, in which, e.g., the roads must follow the street configurations, while the second one is more adequate to non-urbanized areas, in which the roads may go more or less arbitrarily, mostly along the shortest path between two points of the plane.

Previous works concerned mostly the Weber problem with the Euclidean distance and in the continuous form, i.e., the point to be determined could be located at any point in the plane.

For the above distances, as well as for many others, the objective function is strictly convex (or convex when the points are collinear). Hence, a local minimum, at which the gradient of the objective function vanishes, is also a global optimum. However, the equations resulting from the zeroing the gradient are non-linear and have not been solved analytically yet. Hence, a numerical (recursive) solution was proposed, e.g., by Kuhn [5]. Unfortunately, this method failed in the case where a successive point coincided with some of the fixed points. Thus, to overcome this difficulty, some approaches appeared, e.g., [6] and [1]. Anyway, the above case of coincidence was shown to be very unlikely [5]. Moreover, to overcome the non-differentiability a parabolic approximation of the objective function was applied [3].

Another approach was presented by Łukaszewicz and Steinhaus [7] who proposed a geometric procedure. Although the proof of convergence was not given, the method was practically convergent.

Francis and White [2] also tried to use the Newton method for the minimization of the objective function. It converged fast if it did not oscillate.

As an extension, other problems were also considered, e.g., the Weber problem in R^m, $m \geqslant 2$ (see [10]).

It may easily be noticed that the solution of the problem is difficult. In particular, in real life situations (e.g., in telephone networks) the set of fixed and specified points consists of many thousands of elements. In such a case all the methods mentioned previously fail. Thus, in the paper we propose a practical method for some approximation of the Weber problem. The initial continuous problem is transformed into a discrete one. This is done by partitioning the area under consideration into squares by horizontal and vertical stripes. The partitioning of the area results in a considerable simplification of the analysis. An algorithm for optimization — simple and efficient — is presented.

Section 2 is devoted to the description of the model considered. In Section 3 we deal with some specific features of the problem. These considerations allow us to derive an optimal procedure for solving the model, which is presented in Section 4. In Section 5 a numerical example is shown. In Section 6 the efficiency of the algorithm is discussed. Section 7 includes concluding remarks.

2. Formulation of the problem. Let us consider a finite set of points W in a plane. Their Cartesian coordinates are given as $\langle x_j, y_j \rangle$ and there are assigned positive weights w_j to them, $j \in W$. We seek for a point q with coordinates $\langle x_q, y_q \rangle$ such that the sum of weighted distances $w_j d_{qj}$ is as small as possible. The distance d_{qj} between the q-th and the j-th points is here assumed to be Euclidean, i.e.,

$$d_{qj} = \sqrt{(x_q - x_j)^2 + (y_q - y_j)^2}.$$

The above-described problem is the well-known Weber problem. Now, we assume that the planar map of the area we consider is a rectangle. This assumption can be easily realized by drawing in the plane a rectangle such that each point belonging to W is included in it. Further, we divide this rectangle by lines parallel to its sides into m vertical and n horizontal stripes of equal dimensions, i.e., into mn squares. We assume that for $i = 1$ and $i = n$ (the numbers of horizontal stripes) there exists some point $j \in W$ which lies in this stripe. Otherwise, we can renumber the horizontal stripes, and then we obtain a smaller problem. The same refers to the vertical stripes (i.e., the 1-st and the m-th ones). Moreover, we assume that elements of the set W are placed in the middles of squares. This is a simplification, but in practice we can easily achieve a partitioning of the above-mentioned rectangle such that this assumption is fulfilled with a negligible error [9] (there appears here another optimization problem which is not considered in this paper). A simple example of such a partitioning is shown in Fig. 1.

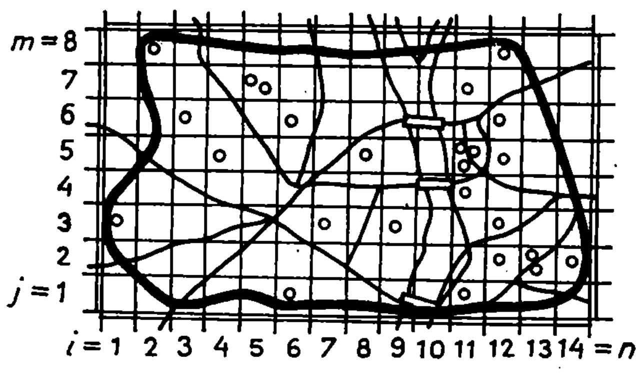

Fig. 1. The idea of discrete approximation of the area in the Weber problem. The circles are the locations of customers, double lines are the sides of rectangles, and the heavy line is the boundary of the area considered

Let us put $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. To each square with coordinates i and j we assign a non-negative weight a_{ij}, $i \in N$,

$j \in M$. It is defined as the sum of w_r's taken over all the points r $(r \in W)$ placed in the (i, j)-th square [4]. Evidently, if for some square there is no r $(r \in W)$ placed in it, then the corresponding a_{ij} is equal to 0. Then, we can formulate the problem as to minimize the value of

$$K(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\sqrt{(i-x)^2+(j-y)^2}$$

subject to

$$a_{ij} \geqslant 0 \text{ for each } i \in N,\ j \in M, \qquad A = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} > 0,$$

where x and y are integers. It must be pointed out that we assume neither $x \in N$ nor $y \in M$.

3. Some properties of the problem. Now, we deal with some properties of $K(x, y)$ defined in Section 2. For brevity, the problem of minimizing $K(x, y)$ described in the previous section is said to be here the *K-problem.* First, we present some specific properties of $K(x, y)$ which simplify – to a great extent – the minimization problem. At the very beginning we note that the properties of $K(x, y)$ for a fixed and specified y are the same as the ones for a fixed and specified x. In other words, the K-problem has a quasi-symmetry (full symmetry is obtained for $m = n$). Then, we can restrict our considerations to the case where y is fixed and specified. We put [8]

$$(1) \quad D(x_1, x_2-x_1) = K(x_1, y)-K(x_2, y)$$
$$= (x_2-x_1)\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\frac{2i-(x_1+x_2)}{\sqrt{(x_1-i)^2+(y-j)^2}+\sqrt{(x_2-i)^2+(y-j)^2}},$$

where x_1 and x_2 are assumed to be two distinct integer numbers. The values of x_1 and x_2 must be distinct, because otherwise for $i = x_1 = x_2$ and $j = y$ the denominator on the right-hand side of (1) becomes zero. (Note that it is inconvenient to make this assumption in the case of a continuous model, where we use derivatives instead of differences, and it is the source of difficulties described, e.g., by Kuhn [5] and mentioned in the Introduction.) For a specific case, which plays a crucial role in further considerations, we have

$$D(x_1, 1) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\frac{2i-(2x_1+1)}{\sqrt{(x_1-i)^2+(y-j)^2}+\sqrt{(x_1+1-i)^2+(y-j)^2}}.$$

Evidently, in our discrete model, $D(x, 1)$ plays the role of a derivative in a continuous model and is often used in further proofs and formulations. Now, we state the first property of the K-problem:

LEMMA 1. *If $\langle x^0, y^0\rangle$ is the solution of the K-problem, then $x^0 \in N$ and $y^0 \in M$.*

Proof. Due to the above remarks we consider only the case in which y is fixed and specified (later on, this is omitted). Let $x_1 \leqslant 0$. Then we have $2i-(2x_1+1) \geqslant 2i-1$ for each $i \in N$, i.e., $D(x_1, 1) > 0$ and $K(x_1, y) > K(x_1+1, y)$, due to (1). This means that $x^0 \geqslant 1$. Now, we assume that $x_1 \geqslant n$. Hence $2i-(2x_1+1) \leqslant 2i-(2n+1) \leqslant -1$ for each $i \in N$, i.e., $D(x_1, 1) < 0$, and then $K(x_1, y) < K(x_1+1, y)$. This means that $x^0 \leqslant n$, and thus the proof is completed.

Lemma 1 results in important consequences. Namely, the solution must lie in some square in the area, i.e., in the Cartesian product $N \times M$, which has a finite number of elements. Hence the K-problem has a solution which can be obtained in a finite number of steps. The method is to seek the solution of the K-problem by a full enumeration of all the squares of the rectangle mentioned in Section 2. This method, however, is evidently inefficient both from the theoretical and the practical point of view. Hence, we derive further properties of the model.

Let $x_1 \in N$. We introduce the following symbols:

$$D_1(x_1) = \sum_{i=1}^{x_1} \sum_{j=1}^{m} a_{ij} \frac{2i-(2x_1+1)}{\sqrt{(x_1-i)^2+(y-j)^2}+\sqrt{(x_1+1-i)^2+(y-j)^2}},$$

$$D_2(x_1) = \sum_{i=x_1+1}^{n} \sum_{j=1}^{m} a_{ij} \frac{2i-(2x_1+1)}{\sqrt{(x_1-i)^2+(y-j)^2}+\sqrt{(x_1+1-i)^2+(y-j)^2}}.$$

It is easy to see that $D_1(x_1) \leqslant 0 \leqslant D_2(x_1)$ for each $x_1 \in N$. Moreover

$$D_1(x_1)+D_2(x_1) = D(x_1, 1). \tag{2}$$

Now, we can formulate and prove the next property:

LEMMA 2. *If $D(x_1, 1) \leqslant 0$, then $D(x_1+1, 1) \leqslant 0$ for each $x_1 \in N$.*

Proof. The assumption $D(x_1, 1) \leqslant 0$ is equivalent to the inequality $D_2(x_1) \leqslant |D_1(x_1)|$. Consequently, we have to prove the following two relations:

$$D_2(x_1+1) \leqslant D_2(x_1) \tag{3}$$

and

$$|D_1(x_1)| \leqslant |D_1(x_1+1)|. \tag{4}$$

First, let us consider

$$Q = \frac{[2i-(2x+3)]\left[\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x+1-i)^2+(y-j)^2}\right]}{[2i-(2x+1)]\left[\sqrt{(x+1-i)^2+(y-j)^2}+\sqrt{(x+2-i)^2+(y-j)^2}\right]}.$$

After some straightforward calculations we obtain

$$Q = \frac{\sqrt{(x+1-i)^2+(y-j)^2}-\sqrt{(x+2-i)^2+(y-j)^2}}{\sqrt{(x-i)^2+(y-j)^2}-\sqrt{(x+1-i)^2+(y-j)^2}}.$$

We assume that $x+2 \leqslant i \leqslant n$. For brevity, we put $x+1-i = a$ and $y-j = b$. Hence $a \leqslant -1$. It is easy to derive that in the considered case we get $\sqrt{(a-1)^2+b^2} > \sqrt{a^2+b^2} \geqslant 1$. Assume that $Q > 1$. Thus, we have

$$\sqrt{(a-1)^2+b^2}-\sqrt{a^2+b^2} < \sqrt{a^2+b^2}-\sqrt{(a+1)^2+b^2}. \tag{5}$$

After some additional calculations we infer that (5) is equivalent to the relation $b^2 < -b^2$, where b is some real number, i.e., we get a contradiction. Therefore, $Q \leqslant 1$. In other words, by the definition of Q, the inequality

$$a_{ij}\frac{2i-(2x+3)}{\sqrt{(x+1-i)^2+(y-j)^2}+\sqrt{(x+2-i)^2+(y-j)^2}} \leqslant a_{ij}\frac{2i-(2x+1)}{\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x+1-i)^2+(y-j)^2}}$$

holds for each i $(x+2 \leqslant i \leqslant n)$. Then, we have

$$D_2(x_1+1) \leqslant \sum_{i=x_1+2}^{n}\sum_{j=1}^{m} a_{ij}\frac{2i-(2x_1+1)}{\sqrt{(x_1-i)^2+(y-j)^2}+\sqrt{(x_1+1-i)^2+(y-j)^2}} \leqslant D_2(x_1)$$

for each $x_1 \leqslant n-2$, i.e., relation (3) is fulfilled. In an analogous way we prove that (4) holds. According to (2)-(4) we obtain $D(x_1+1, 1) \leqslant D(x_1, 1)$. By assumption and Lemma 1 we have $D(x_1+1, 1) \leqslant 0$ for each $x_1 \in N$, which completes the proof.

In a similar way we can prove

LEMMA 3. *If* $D(x_1, 1) \geqslant 0$, *then* $D(x_1-1, 1) \geqslant 0$ *for each* $x_1 \in N$.

Lemmas 2 and 3 imply the following theorem:

THEOREM 1. *First, let x be fixed. If $K(x, y_1)-K(x, y_1+1) < 0$, then any pair $\langle x, y\rangle$, $y \geqslant y_1+1$, cannot be the solution of the K-problem. If $K(x, y_1)-K(x, y_1+1) > 0$, then any pair $\langle x, y\rangle$, $y \leqslant y_1$, cannot be the solution of the K-problem.*

Second, let y be fixed. If $K(x_1, y)-K(x_1+1, y) < 0$, then any pair $\langle x, y\rangle$, $x \geqslant x_1+1$, cannot be the solution of the K-problem. If $K(x_1, y)-$

$-K(x_1+1, y) > 0$, *then any pair* $\langle x, y\rangle$, $x \leqslant x_1$, *cannot be the solution of the K-problem.*

In some specific situations we have non-zero a_{ij}'s only for $i = 1$, $j = 1$ and $i = n$, $j = m$ or $i = 1$, $j = m$ and $i = n$, $j = 1$. The non-zero a_{ij}'s are sometimes placed only on some diagonal of the matrix created by the squares of the rectangle considered. Both these cases mentioned above are called the *degenerate K-problem*, and further on the non-degenerate K-problem is mainly concerned. Now, we can formulate the next property:

THEOREM 2. *Let* $\langle x^0, y^0\rangle$ *and* $\langle \hat{x}, \hat{y}\rangle$ *be some optimal solutions of the non-degenerate K-problem. If* $\langle x^0, y^0\rangle \neq \langle \hat{x}, \hat{y}\rangle$, *then there exists a sequence of pairs* $(\langle x_i, y_i\rangle\colon i = 1, 2, \ldots, h)$ *such that* $\langle x_1, y_1\rangle = \langle x^0, y^0\rangle$, $\langle x_h, y_h\rangle = \langle \hat{x}, \hat{y}\rangle$, *and one of the four conditions*

$$\langle x_{i+1}, y_{i+1}\rangle = \langle x_i+1, y_i\rangle, \quad \langle x_{i+1}, y_{i+1}\rangle = \langle x_i-1, y_i\rangle,$$

$$\langle x_{i+1}, y_{i+1}\rangle = \langle x_i, y_i+1\rangle, \quad \langle x_{i+1}, y_{i+1}\rangle = \langle x_i, y_i-1\rangle$$

holds for each $i = 1, 2, \ldots, h-1$ *and each* $\langle x_i, y_i\rangle$ *for* $i = 1, 2, \ldots, h$ *is also the solution of the non-degenerate K-problem.*

Theorem 2 follows immediately from Theorem 1. Moreover, we have to notice that in the case of the degenerate K-problem Theorem 2 fails. This can be shown, by a simple counterexample, as follows. Let $a_{11} = 1$, $a_{nm} = 1$, and let all the a_{ij}'s be zeros for other squares. Moreover, we assume that $n = m$. It is easy to find that each ordered pair $\langle i, i\rangle$, $i = 1, 2, \ldots, n$, is the optimal solution of the K-problem considered. Furthermore, $K(i, i) = \sqrt{2}(n-1)$. For simplicity, let $n = 4$. Hence $K(1, 1) = 3\sqrt{2}$ and $K(2, 1) = K(1, 2) = 1+\sqrt{13}$. Thus $K(1, 1) < K(1, 2) = K(2, 1)$, which completes our counterexample.

In other words, Theorem 1 (and – in the case of the non-degenerate K-problem – also Theorem 2) states that the K-problem is convex. From Theorem 2 it follows that any solution of the non-degenerate K-problem is not separated from other solutions, if they exist. Thus, they are grouped together.

In [8] there were given some heuristic ideas for the solution of the K-problem. They give good results in practice. The coordinates of the suboptimal point proposed in [8] are defined as follows:

$$\hat{x} = \text{entier}\left(\frac{1}{A}\sum_{i=1}^{n}\sum_{j=1}^{m} i a_{ij}\right), \tag{6}$$

$$\hat{y} = \text{entier}\left(\frac{1}{A}\sum_{i=1}^{n}\sum_{j=1}^{m} j a_{ij}\right). \tag{7}$$

However, for some reason it may happen in practice that the single optimal solution of the K-problem is not preferable. Thus, we may be interested in another optimal solution (if it exists) or even in a suboptimal solution. In the next section we give an algorithm for determining such solutions.

4. Algorithm for determining all the (sub)optimal solutions. The preceding section gives the basis for a simple technique determining the optimal solution of the non-degenerate K-problem. The idea of the algorithm is based on the convexity of the problem (Theorem 1) and on Theorem 2. First, we choose the starting point. Its coordinates can be determined by using formulae (6) and (7). Second, we check whether for any point in the neighbourhood of the current point the value of the objective function is not greater than that for the current point mentioned. If so, then this point becomes the new current point, etc. Otherwise, the current point is optimal.

However, there occur situations when the problem mentioned has more than a single solution. A simple example for the degenerate K-problem is given in the previous section. Let us now consider a particular case of the non-degenerate K-problem. We assume that $a_{11} = a_{1m} = a_{n1} = a_{nm} = 1$ and that all the remaining a_{ij}'s are equal to 0. Let $n = m = 2k$, where k is some positive integer, $k > 1$. It is easy to prove that the points $\langle k, k\rangle$, $\langle k, k+1\rangle$, $\langle k+1, k\rangle$, and $\langle k+1, k+1\rangle$ are optimal (the value of the objective function is here $2k\sqrt{2}+2\sqrt{(k-1)^2+k^2}-\sqrt{2}$). Moreover, in many practical cases we are interested not only in the optimal solutions but also in some suboptimal solutions; for instance, if we want to locate a factory or a telephone exchange, but the optimal solutions are not realizable. This non-realizability can follow, e.g., from the fact that the optimal solution coincides with some lake, some highly urbanized area, etc. Then we must seek some suboptimal solutions, which are both the nearest to the optimal one (in terms of the value of $K(\cdot, \cdot)$) and realizable in practice. In other words, we are interested in solutions such that their value of the objective function is not greater than $(1+r)K(x^0, y^0)$, where $\langle x^0, y^0\rangle$ is the optimal point and r is some positive constant, usually $r \ll 1$. Obviously, by using $r = 0$ we obtain all the optimal solutions of the non-degenerate K-problem only. The subsequent steps of the algorithm are the following:

1. START.
2. Define the set of feasible solutions $Q \leftarrow N \times M$.
3. Compute $\hat{x}$ and $\hat{y}$ by (6) and (7), respectively.
4. $\langle x, y\rangle \leftarrow \langle \hat{x}, \hat{y}\rangle$.
5. Compute $K(x, y)$.
6. $K \leftarrow K(x, y)$.

7. Initiate the set of results $Z \leftarrow \{\langle x, y\rangle\}$.
8. Initiate the set of feasible starting points $V \leftarrow Z$.
9. Take any point $\langle x, y\rangle \in V$.
10. If $\langle x+1, y\rangle \notin Q-Z$, then go to Step 14.
11. $\langle x_1, y_1\rangle \leftarrow \langle x+1, y\rangle$.
12. $p \leftarrow 0$ (p is the control parameter).
13. Go to Step 17.
14. If $\langle x-1, y\rangle \notin Q-Z$, then go to Step 20.
15. $\langle x_1, y_1\rangle \leftarrow \langle x-1, y\rangle$.
16. $p \leftarrow 1$.
17. $T \leftarrow \{\langle a, b\rangle\colon a \leqslant x-1, a \in N, b = y\}$.
18. $W \leftarrow \{\langle a, b\rangle\colon a \geqslant x+1, a \in N, b = y\}$.
19. Go to Step 29.
20. If $\langle x, y+1\rangle \notin Q-Z$, then go to Step 24.
21. $\langle x_1, y_1\rangle \leftarrow \langle x, y+1\rangle$.
22. $p \leftarrow 0$.
23. Go to Step 27.
24. If $\langle x, y-1\rangle \notin Q-Z$, then go to Step 45.
25. $\langle x_1, y_1\rangle \leftarrow \langle x, y-1\rangle$.
26. $p \leftarrow 1$.
27. $T \leftarrow \{\langle a, b\rangle\colon a = x, b \leqslant y-1, b \in M\}$.
28. $W \leftarrow \{\langle a, b\rangle\colon a = x, b \geqslant y+1, b \in M\}$.
29. Compute $K(x_1, y_1)$ (the value of the objective function at the new point).
30. $K_1 \leftarrow K(x_1, y_1)$.
31. If $K_1 > K$ (the new point cannot be optimal), then go to Step 39.
32. $\langle x, y\rangle \leftarrow \langle x_1, y_1\rangle$ (changing the current point).
33. If $K_1 = K$, then go to Step 42.
34. $K \leftarrow K_1$ (the better solution).
35. $Q \leftarrow Q-Z$ (reducing the set of feasible solutions).
36. If $p = 0$, then $Q \leftarrow Q-T$.
37. If $p = 1$, then $Q \leftarrow Q-W$.
38. Go to Step 7.
39. If $p = 0$, then $Q \leftarrow Q-W$.
40. If $p = 1$, then $Q \leftarrow Q-T$.
41. Go to Step 10.
42. $Z \leftarrow Z \cup \{\langle x, y\rangle\}$ (enlarging the set of current "optimal" solutions).
43. $V \leftarrow V \cup \{\langle x, y\rangle\}$ (enlarging the set of feasible starting points).
44. Go to Step 10.
45. $V \leftarrow V - \{\langle x, y\rangle\}$ (diminishing the set of feasible starting points).
46. If $V \neq \emptyset$, then go to Step 9.
47. The minimal value of the objective function is equal to K. In the case of the non-degenerate K-problem the set Z contains all the optimal

solutions. In the case of the degenerate K-problem the set Z consists of some optimal solutions (there may be optimal solutions which do not belong to Z).

48. If $r = 0$, then go to Step 80.
49. $K \leftarrow (1+r)K$.
50. Define the set of the new feasible solutions $Q \leftarrow N \times M - Z$.
51. Initiate the set of feasible starting points $V \leftarrow Z$.
52. Take any starting point $\langle x, y\rangle \in V$.
53. If $\langle x+1, y\rangle \notin Q-Z$, then go to Step 57.
54. $\langle x_1, y_1\rangle \leftarrow \langle x+1, y\rangle$.
55. $T \leftarrow \{\langle a, b\rangle : a \geqslant x+1, a \in N, b = y\}$.
56. Go to Step 68.
57. If $\langle x-1, y\rangle \notin Q-Z$, then go to Step 61.
58. $\langle x_1, y_1\rangle \leftarrow \langle x-1, y\rangle$.
59. $T \leftarrow \{\langle a, b\rangle : a \leqslant x-1, a \in N, b = y\}$.
60. Go to Step 68.
61. If $\langle x, y+1\rangle \notin Q-Z$, then go to Step 65.
62. $\langle x_1, y_1\rangle \leftarrow \langle x, y+1\rangle$.
63. $T \leftarrow \{\langle a, b\rangle : a = x, b \geqslant y+1, b \in M\}$.
64. Go to Step 68.
65. If $\langle x, y-1\rangle \notin Q-Z$, then go to Step 77.
66. $\langle x_1, y_1\rangle \leftarrow \langle x, y-1\rangle$.
67. $T \leftarrow \{\langle a, b\rangle : a = x, b \leqslant y-1, b \in M\}$.
68. Compute $K(x_1, y_1)$.
69. $K_1 \leftarrow K(x_1, y_1)$.
70. If $K_1 \leqslant K$ (the new point is suboptimal), then go to Step 73.
71. $Q \leftarrow Q-T$ (diminishing the set of feasible solutions).
72. Go to Step 53.
73. $\langle x, y\rangle \leftarrow \langle x_1, y_1\rangle$.
74. $Z \leftarrow Z \cup \{\langle x, y\rangle\}$ (enlarging the set of suboptimal points).
75. $V \leftarrow V \cup \{\langle x, y\rangle\}$ (enlarging the set of feasible starting points).
76. Go to Step 53.
77. $V \leftarrow V - \{\langle x, y\rangle\}$ (diminishing the set of feasible starting points).
78. If $V \neq \emptyset$, then go to Step 52.
79. In the case of the non-degenerate K-problem the set Z contains all suboptimal solutions $\langle x, y\rangle$ such that $K(x, y) \leqslant K(x^0, y^0)(1+r)$, where $\langle x^0, y^0\rangle$ denotes the optimal point. In the case of the degenerate K-problem the set Z consists of some suboptimal solutions (but not necessarily all of them).
80. STOP.

5. Numerical example. To show the example of the solution of some non-degenerate K-problem we consider the situation demonstrated in Fig. 2. The map of the area considered is embedded into the rectangle which is divided into $n = 20$ vertical and $m = 20$ horizontal stripes, i.e., 400 equal squares. The number located in each square is equal to the corresponding weight a_{ij} and there exists only one optimal point which has the coordinates $x = 11$ and $y = 10$. The solution was obtained on the ODRA 1305 computer with computational time less than 1 sec. after the inspection of 9 possible locations.

20	0	0	0	0	0	0	0	0	0	0	9	23	53	9	26	16	9	120	16	10
19	0	0	0	0	0	0	0	0	0	0	27	4	2	1	5	2	44	8	254	33
18	0	0	0	0	0	0	0	0	0	0	3	4	29	4	36	9	4	3	7	3
17	0	0	0	0	0	19	0	51	3	364	4	9	4	10	3	174	17	47	91	24
16	123	56	26	293	25	0	28	37	0	24	581	77	4	411	4	10	4	10	3	3
15	4	37	6	37	8	5	4	8	3	32	4	2	18	6	3	38	11	31	25	13
14	31	71	11	18	16	10	139	25	3	19	299	11	0	6	8	6	955	12	22	12
13	29	6	8	12	61	0	0	27	125	22	14	2	139	26	5	2	39	10	16	11
12	3	13	3	4	4	3	12	4	3	2	23	3	1	12	4	2	32	10	316	24
11	13	28	124	9	14	15	10	12	1	1	0	335	3	5	2	121	19	114	27	16
10	12	25	9	91	13	77	737	312	394	6	838	9	22	10	26	5	9	59	8	88
9	25	8	6	14	22	182	39	21	197	22	143	18	379	750	49	15	171	14	478	80
8	15	45	53	8	120	34	8	7	17	8	68	10	5	172	15	8	2	35	3	3
7	4	7	20	3	5	28	5	13	18	188	0	13	3	3	4	2	6	16	62	29
6	7	3	11	177	0	243	953	4	283	5	18	3	24	56	78	45	16	54	15	19
5	209	77	264	857	51	46	13	154	15	17	16	33	88	2	3	27	21	58	0	104
4	14	17	15	0	0	0	3	3	5	10	30	8	21	24	4	30	29	77	5	22
3	2	180	74	5	1	40	7	16	12	3	15	2	2	12	8	0	140	20	5	2
2	29	2	3	3	6	19	3	3	7	0	27	35	73	6	15	2	3	3	5	8
$j=1$	22	6	11	15	3	12	6	53	8	3	33	11	4	164	15	559	0	0	0	0
	$i=1$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Fig. 2. An example of the discrete Weber problem. The optimal square is shaded

6. Efficiency of the algorithm. Let us now consider the efficiency of the algorithm. It is obvious that the efficiency depends upon the number of optimal (or suboptimal, as defined in Section 4) points, which is difficult to estimate. Moreover, it depends also on the distance between the starting point $\langle \hat{x}, \hat{y} \rangle$ and the optimal point $\langle x^0, y^0 \rangle$ defined in the rectilinear way, i.e., $|\hat{x} - x^0| + |\hat{y} - y^0|$. To evaluate this distance we need the following lemma:

LEMMA 4. *The point $\langle x^0, y^0\rangle$ is the optimal solution of the K-problem if and only if*

$$f(x^0, y^0)-\tfrac{1}{2}\leqslant x^0\leqslant f(x^0-1, y^0)+\tfrac{1}{2}, \tag{8}$$
$$g(x^0, y^0)-\tfrac{1}{2}\leqslant y^0\leqslant g(x^0, y^0-1)+\tfrac{1}{2},$$

where

$$f(x,y)=\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} i a_{ij}\left(\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x+1-i)^2+(y-j)^2}\right)^{-1}}{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\left(\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x+1-i)^2+(y-j)^2}\right)^{-1}},$$

$$g(x,y)=\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} j a_{ij}\left(\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x-i)^2+(y+1-j)^2}\right)^{-1}}{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\left(\sqrt{(x-i)^2+(y-j)^2}+\sqrt{(x-i)^2+(y+1-y)^2}\right)^{-1}}.$$

The proof of this lemma is obvious and is based on the convexity of $K(\cdot,\cdot)$. For instance, if $\langle x^0, y^0\rangle$ is the optimal point, then $D(x^0, 1)\leqslant 0$ and $D(x^0-1, 1)\geqslant 0$, which implies (8), etc. (Lemma 4 was not stated before because it had not any influence on the method of seeking the optimal solution, as was evident in Section 4).

If we assume that the starting point is determined by using (6) and (7), then we can estimate

$$\bar{k}=|\hat{x}-x^0|+|\hat{y}-y^0|\approx\left|\frac{f(x^0,y^0)+f(x^0-1,y^0)}{2}-\hat{x}\right|+\left|\frac{g(x^0,y^0)+g(x^0,y^0-1)}{2}-\hat{y}\right|,$$

and we take $k=\operatorname{entier}(\bar{k}+0.5)$. But the above estimation is rather of less practical importance because in practice [8] we have $k\approx 2$.

Now, we restrict our considerations to the case $r=0$, i.e., to seeking optimal points only. We need $3mn$ additions, $m+n+2$ multiplications, and 1 division for defining the starting point by (6) and (7). To compute the value of the objective function we have to perform $n(m+1)$ subtractions, $2mn$ additions, $n(2m+1)$ multiplications, and mn extractions of a square root. Since we check h points, we need $mn(2h+3)$ additions, $hn(m+1)$ subtractions, $hn(2m+1)+n+m+2$ multiplications, 1 division, and hmn extractions of a square root. To evaluate the value of the parameter h we notice that the number of optimal solutions is not greater than 4 in the case of the non-degenerate K-problem. This follows from the strict convexity of the problem and the example presented in Section 4.

Thus $h = 12-2+k \approx 12$ (we have here 4 optimal points and 8 points in their neighbourhood, which we must check). But 2 of them were inspected during the passage from $\langle \hat{x}, \hat{y} \rangle$ to the nearest optimal solution, which can be easily shown in a simple diagram). This means that the algorithm is of type $O(p^2)$, where $p = \max\{m, n\}$.

7. Concluding remarks. The purpose of the paper was mainly to present some discrete approximation technique for the continuous problem. We do this by giving a discrete approximation of the Weber problem with Euclidean distance. It was shown that this approximation results in a particularly simple and efficient procedure for solving the problem. Experience gained during the practical solving of many large-scale Weber-type problems within structuring, e.g., telecommunication and computer networks has fully justified the approach. It is probably the right way for solving all the large-scale practical combinatorial problems. Namely, more emphasis should be done on obtaining a relatively simple and quick way for solving some approximations of the real combinatorial problems. Moreover, the methods should be constructed in a way which eventually could allow us to repeat the solution process under a changeable parameter to arrive finally at a satisfactory solution, rather than to handle "accurately" the problem. In the case considered in the paper we can proceed as follows. First, we obtain the solution of the K-problem. Then, we rearrange the previous problem by taking $n' = nt$ and $m' = mt$, where t is an integer (for convenience, we can assume that t is an odd number, and then the previous centres of squares are remained as the middles of some new squares, which results in some computational improvements), $t > 1$, and by respective changing the a_{ij}'s. We solve this new K-problem, on the basis of convexity, starting from the previous optimal solution, etc. We repeat this procedure to arrive at a satisfactory solution where the error can be estimated as the length of the square side.

It must be pointed out that there may occur some differences between the continuous origin and its discrete approximation. First, the techniques for solving the problems are distinct. Some proofs are easier in the continuous version (e.g., the proof of convexity of the Weber problem) and some are easier and more obvious in the discrete one (e.g., the convergence of the algorithm and its efficiency). Moreover, there sometimes occur distinct features of the problems. For instance, the strictly convex nondegenerate K-problem can have 4 optimal solutions (see, e.g., Section 4), whilst its continuous form has a unique solution. Moreover, the optimal solutions of the degenerate K-problem can be separated one from another (see, e.g., Section 3), whereas this is impossible for the continuous weakly convex Weber problem.

References

[1] F. Cordellier, J. C. Fiorot and S. K. Jacobsen, *An algorithm for the generalized Fermat-Weber problem*, IMSOR, Technical University of Denmark 1976.
[2] R. L. Francis and J. A. White, *Facility layout and location: an analytical approach*, Prentice-Hall, Englewood Cliffs 1974.
[3] S. K. Jacobsen, *Weber rides again*, IMSOR, Technical University of Denmark 1974.
[4] J. Kacprzyk and W. Stańczak, *Partitioning a computer network into subnetworks and allocation of distributed data bases*, p. 464-472 in: *Proc. 8th IFIP Conf. Optimization Techn.*, Würzburg 1977, Springer, Berlin 1978.
[5] H. W. Kuhn, *A note on Fermat's problem*, Math. Programming 4 (1973), p. 98-107.
[6] — and R. E. Kuenne, *An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics*, J. Reg. Sci. 4 (1962), p. 21-33.
[7] J. Łukaszewicz and H. Steinhaus, *O wyznaczaniu środka miedzi sieci telefonicznej*, Zastos. Mat. 1 (1954), p. 299-307.
[8] T. Nowicki and W. Stańczak, *Wybrane aspekty konfiguracji telefonicznej sieci strefowej*, Rozprawy Elektrotech. 23 (1977), p. 237-251.
[9] — *O metodzie wstępnego projektowania struktur topologicznych i konfiguracji sieci teleprzetwarzania*, PWN, Warszawa 1980.
[10] M. K. Schaefer and A. P. Hureter, Jr., *An algorithm for the solution of a location problem with metric constraints*, Naval Res. Logist. Quart. 21 (1974), p. 625-636.

SYSTEM RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES
UL. NEWELSKA 6
01-447 WARSZAWA

INSTITUTE OF COMPUTER SCIENCE
POLISH ACADEMY OF SCIENCES
PKiN, SKR. POCZT. 22
00-901 WARSZAWA

Received on 17. 3. 1981