A discrete limit theorem for the periodic Hurwitz zeta-function

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Abstract. In the paper, we prove a limit theorem of discrete type on the weak convergence of probability measures on the complex plane for the periodic Hurwitz zeta-function.

Keywords: Hurwitz zeta-function, limit theorem, probability measure, weak convergence.

Let $s = \sigma + it$ be a complex number, $0 < \alpha \leq 1$, be a fixed parameter, and let $a = \{a_m; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; a)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m (m+\alpha)}{(m+\alpha)^s},$$

and, by using the equality,

$$\zeta(s, \alpha; a) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha + l}{k}\right),$$

where $\zeta(s, \alpha)$ is the classical Hurwitz zeta-function, can be meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$ with residue

$$a \overset{\text{def}}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l.$$

If $a = 0$, then the function $\zeta(s, \alpha; a)$ is entire one.

In $[2, 4, 6]$ and $[7]$, limit theorem on the weak convergence of probability measures on the complex plane $\mathbb{C}$ for the function $\zeta(s, \alpha; a)$ with parameter $\alpha$ of various arithmetical types were obtained. In these works, the weak convergence for

$$\frac{1}{T} \text{meas}\{t \in [0, T]; \zeta(\sigma + it, \alpha; a) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

where $\mathcal{B}(S)$ denotes the Borel $\sigma$-field of the space $S$, and $\text{meas}A$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, was considered.
The function $\zeta$ takes arbitrary real values. The aim of this note is to prove a discrete limit theorem for the function $\zeta(s, \alpha; \omega)$ when $t$ takes values from the set $\{h_m: m \in \mathbb{N}_0\}$, where $h > 0$ is a fixed number. Define the set

$$L(\alpha, h, \pi) = \left\{(\log(m + \alpha); m \in \mathbb{N}_0, \frac{\pi}{h}\right\},$$

and the torus

$$\Omega = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_m$ is the unit circle $\{s \in \mathbb{C}: |s| = 1\}$ for all $m \in \mathbb{N}_0$. The torus $\Omega$ is a compact topological group, therefore, on $(\Omega, B(\Omega))$, the probability Haar measure $m_H$ can be defined. This gives the probability space $(\Omega, B(\Omega), m_H)$.

Denote by $\omega(m)$ the projection of an element $\omega \in \Omega$ to the coordinate space $\gamma_m$, $m \in \mathbb{N}_0$, and, on the probability space $(\Omega, B(\Omega), m_H)$, define the complex-valued random element $\zeta(\sigma, \alpha; \omega; a)$ by the formula

$$\zeta(\sigma, \alpha, \omega; a) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m + \alpha)^{\sigma}}, \quad \sigma > \frac{1}{2},$$

and denote by $P_\chi$ the distribution of $\zeta(\sigma, \alpha, \omega; a)$, i.e.,

$$P_\chi(A) = m_H(\omega \in \Omega: \zeta(\sigma, \alpha, \omega; a) \in A), \quad A \in B(\mathbb{C}).$$

**Theorem 1.** Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and that $\sigma > \frac{1}{2}$. Then the probability measure

$$P_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \#\{0 \leq m \leq N: \zeta(\sigma + \imath m h, \alpha; a) \in A\}, \quad A \in B(\mathbb{C}),$$

converges weakly to the measure $P_\chi$ as $N \to \infty$.

For the proof of Theorem 1, the following two lemmas involving the set $L(\alpha, h, \pi)$ are applied. Let

$$Q_N(A) \overset{\text{def}}{=} \frac{1}{N+1} \#\{0 \leq m \leq N: ((m + \alpha)^{-\imath h}: m \in \mathbb{N}_0) \in A\}, \quad A \in B(\Omega).$$

**Lemma 1.** Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $Q_N$ converges weakly to the Haar measure $m_H$ as $N \to \infty$.

Proof of the lemma is given in [3, Lemma 2.3].

For $a_h = ((m + \alpha)^{-\imath h}: m \in \mathbb{N}_0)$, on the probability space $(\Omega, B(\Omega), m_H)$, define the transformation $\psi_h$ by $\psi_h(\omega) = a_h \omega$, $\omega \in \Omega$. Then $\psi_h$ is a measurable measure preserving transformation.

**Lemma 2.** Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then the transformation $\psi_h$ is ergodic, i.e., if $A \in B(\Omega)$ and $A_h = \psi_h(A)$ differ from other at most by $m_H$-measure zero, then $m_H(A) = 0$ or $m_H(A) = 1$.
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Proof of the lemma is given in [3, Lemma 2.8]. The further proof of Theorem 1 can be divided into following parts: limit theorems for absolutely convergent Dirichlet series, approximation of the function $\zeta(\sigma, \alpha; a)$ in the mean by absolutely convergent Dirichlet series, limit theorems for $\zeta(\sigma, \alpha; a)$ and $\zeta(\sigma, \alpha, \omega; a)$, and identification of the limit measure.

For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, define $\nu_n(m, \alpha) = \exp\left\{ -\left(\frac{m+n}{n+\alpha}\right)^{\sigma_1}\right\}$, where $\sigma_1 > \frac{1}{2}$ is a fixed number, and set

$$\zeta_n(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m \nu_n(m, \alpha)}{(m + \alpha)^s}$$

and

$$\zeta_n(s, \alpha, \omega; a) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) \nu_n(m, \alpha)}{(m + \alpha)^s}.$$ 

Then the latter series both are absolutely convergent for $\sigma > \frac{1}{2}$. Moreover, for $A \in B(\mathbb{C})$, let

$$P_{N,h}(A) \overset{\text{def}}{=} \frac{1}{N+1} \#\left\{0 \leq m \leq N: \zeta_n(\sigma + imh, \alpha; a) \in A\right\},$$

and

$$P_{N,h,\omega}(A) \overset{\text{def}}{=} \frac{1}{N+1} \#\left\{0 \leq m \leq N: \zeta_n(\sigma + imh, \alpha, \omega; a) \in A\right\}.$$ 

**Lemma 3.** Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$ and that $\sigma > \frac{1}{2}$. Then $P_{N,h}$ and $P_{N,h,\omega}$ both converge weakly to the same probability measure $P_n$ on $(\mathbb{C}, B(\mathbb{C}))$ as $N \to \infty$.

**Proof.** The lemma is a result of the application of Lemma 1, Theorem 5.1 of [1], and the invariance of the Haar measure.

Lemma 2 is applied to show that, for almost all $\omega \in \Omega$, the estimate

$$\int_0^T |\zeta(\sigma + it, \omega; a)|^2 = O(T), \quad T \to \infty,$$

is valid for $\sigma > \frac{1}{2}$. From this, using the Gallagher lemma, Lemma 1.4 of [5], we deduce that, for almost all $\omega \in \Omega$, the estimate

$$\frac{1}{N+1} \sum_{m=0}^{N} |\zeta(\sigma + imh, \alpha, \omega; a)|^2 = O(1), \quad T \to \infty,$$

is valid for $\sigma > \frac{1}{2}$. Analogically, we obtain, for $\sigma > \frac{1}{2}$, the bound

$$\frac{1}{N+1} \sum_{m=0}^{N} |\zeta(\sigma + imh, \alpha; a)|^2 = O(1), \quad T \to \infty.$$ 

Using the latter estimates and contour integration, we arrive to the following assertion.
Lemma 4. Suppose that the set \( L(\alpha, h, \pi) \) is linearly independent over \( \mathbb{Q} \) and that \( \sigma > \frac{1}{2} \). Then
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N + 1} \sum_{m=0}^{N} |\zeta(\sigma + imh, \alpha; a) - \zeta_n(\sigma + imh, \alpha; a)| = 0.
\]
and, for almost all \( \omega \in \Omega \),
\[
\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N + 1} \sum_{m=0}^{N} |\zeta(\sigma + imh, \alpha, \omega; a) - \zeta_n(\sigma + imh, \alpha, \omega; a)| = 0.
\]

Let, for \( A \in \mathcal{B}(\mathbb{C}) \),
\[
P_{N, \omega}(A) = \frac{1}{N + 1} \# \{ 0 \leq m \leq N : \zeta(\sigma + imh, \alpha, \omega; a) \in A \}.
\]

Lemma 5. Suppose that the set \( L(\alpha, h, \pi) \) is linearly independent over \( \mathbb{Q} \) and that \( \sigma > \frac{1}{2} \). Then \( P_N \) and \( P_{N, \omega} \) both converge weakly to the same probability measure \( P \) on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) as \( N \to \infty \).

Proof. First we show that the family of probability measures \( \{ P_n : n \in \mathbb{N} \} \) tight. Therefore, by the Prokhorov theorem [1], this family is relatively compact. Hence, there exists a sequence \( \{ P_{n_k} \} \subset \{ P_n \} \) such that \( P_{n_k} \) converges weakly to a certain probability measure \( P \) on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) as \( k \to \infty \). This, Lemmas 3 and 4, and Theorem 4.2 of [1] prove the lemma.

Proof of Theorem 1. In virtue of Lemma 5, it suffices to show that the measure \( P \) in Lemma 5 coincides with \( P_c \).

Let \( A \) be an arbitrary continuity set of the measure \( P \), i.e., \( P(\partial A) = 0 \), where \( \partial A \) is the boundary of \( A \). Then Lemma 5 and the equivalent of weak convergence of probability measures in terms of continuity sets, Theorem 2.1 of [1], imply that
\[
\lim_{n \to \infty} \frac{1}{N + 1} \# \{ 0 \leq m \leq N : \zeta(\sigma + imh, \alpha, \omega; a) \in A \} = P(A).
\]  
(1)

Now, on the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \), define the random variable \( \theta \) by the formula
\[
\theta(\omega) = \begin{cases} 
1 & \text{if } \zeta(s, \alpha, \omega; a) \in A, \\
0 & \text{otherwise}.
\end{cases}
\]

Obviously, the expectation \( E\theta \) of the random variable \( \theta \) is given by
\[
E\theta = \int_{\Omega} \theta \, dm_H = m_H(\omega \in \Omega: \zeta(s, \alpha, \omega; a) \in A) = P_c(A).
\]  
(2)

Now we apply Lemma 1, and obtain by the classical Birkhoff–Khintchine ergodicity theorem that
\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{m=0}^{N} \theta(\psi_m^c(\omega)) = E\theta
\]  
(3)

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for almost all $\omega \in \Omega$. On the other hand, the definitions of the random variable $\theta$ and the transformation $\psi_h$ yield the equality

$$\frac{1}{N+1} \sum_{m=0}^{N} \theta(\psi_h^m(\omega)) = \frac{1}{N+1} \#\{0 \leq m \leq N: \zeta(\sigma + imh, \alpha, \omega; a) \in A\}. $$

This together with (2) and (3) shows that, for almost all $\omega \in \Omega$,

$$\lim_{N \to \infty} \frac{1}{N+1} \#\{0 \leq m \leq N: \zeta(\sigma + imh, \alpha, \omega; a) \in A\} = P_\zeta(A).$$

Hence, in view of (1), we obtain that $P(A) = P_\zeta(A)$. Since the set $A$ was arbitrary, we have that $P(A) = P_\zeta(A)$ for all continuity sets of the measure $P$. However, the continuity sets constitute the determining class, therefore, $P(A) = P_\zeta(A)$ for all $A \in B(\mathbb{C})$.

References


REZIUMĖ

Diskrečioji ribinė teorema periodinei Hurvico dzeta funkcijai

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Straipsnyje jrodys diskretaus tipo ribinę teoremą, silpniojo tikimybinių matų konvergavimo prasme, periodinei Hurvico dzeta funkcijai kompleksineje plokštumoje.

Raktiniai žodžiai: Hurvico dzeta funkcija, ribinę teorema, silpnasis konvergavimas, tikimybiniis matas.