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A discussion concerning the existence results for the Sobolev-type Hilfer fractional delay integro-differential systems

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Abstract

The goal of this study is to propose the existence results for the Sobolev-type Hilfer fractional integro-differential systems with infinite delay. We intend to implement the outcomes and realities of fractional theory to obtain the main results by Monch's fixed point technique. Moreover, we show the existence and controllability of the thought about the fractional system with the nonlocal condition. In addition, an application to illustrate the outcomes is also included.

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1 Introduction

In recent years, mathematical modeling has been upheld by fractional calculus, with a few outcomes, and fractional operators were demonstrated to be a fantastic instrument to depict the hereditary characteristics of different patterns. As of late, this blend has acquired a lot of significance, basically because fractional differential equations have become amazing assets for displaying a few complex wonders in various assorted and boundless fields of science and engineering; readers are referred to [1–20] and articles [21–37]. Hilfer [38] initiated another kind of derivative, along with Riemann–Liouville and Caputo fractional derivative. Motivated by the monograph, nowadays, several authors focus on these Hilfer fractional differential equations, and we refer to [24, 39–48]. Singh et al. [49] discussed the existence and Ulam stability of solutions for a class of boundary value problems for Hilfer-type nonlinear implicit fractional differential equations with instantaneous impulses in Banach spaces.

The differential system with Sobolev-type is frequently evident in the mathematical structure of several physical events similar to the flowing of fluids through fractured rocks, thermodynamics. The readers may refer to [50–56]. Many authors discussed the relations between the asymptotic stability of the zero solution for retarded differential equations

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and real parts of all characteristic roots of characteristic equations. In [57] the author investigated the asymptotic stability of the zero solution for Caputo–Hadamard fractional dynamic equations on a time scale. These equations guarantee the effectiveness of the zero solution, and several authors reported interesting fixed point results in the framework of complete b-metric spaces, recently, Lazreg et al. [58] established some impulsive Caputo–Fabrizio fractional differential equations in b-metric spaces.

Control hypothesis is a significant region of usage arranged in mathematics which deals with the design and assessment of control structures. The development of modern mathematical control theory is heavily influenced by controllability. The problem of controllability of dynamical systems is commonly employed in control system analysis and design. Fractional-order control systems defined by fractional-order differential equations have gotten a lot of interest in recent years, a wide list of these distributions can be found in [25, 26, 28, 29, 40, 43, 48, 51, 56, 59–62]. The controllability of impulsive fractional evolution inclusions with state-dependent delay is demonstrated in [63], which employs a fixed point theorem for condensing maps.

From the above literature survey, to our knowledge the existence and exact controllability of the fractional system have not been studied fully. Motivated by this fact, we consider the Sobolev-type Hilfer fractional integro-differential system of the form

$$D_{0^+}^{\alpha,\beta}[\mathcal{J}z(t)] = Az(t) + f\left(t, z_t, \int_0^t e(t, s, z_s) ds\right), \quad t \in \mathbb{N} = (0, b], \tag{1.1}$$

$$\mathcal{I}_{0^+}^{(1-\alpha)(1-\beta)}z(0) = \phi \in \mathcal{B}_l, \tag{1.2}$$

and assume that the system with control has the following form:

$$D_{0^+}^{\alpha,\beta}[\mathcal{J}z(t)] = Az(t) + f\left(t, z_t, \int_0^t e(t, s, z_s) ds\right) + Bu(t), \quad t \in \mathbb{N} = (0, b], \tag{1.3}$$

$$\mathcal{I}_{0^+}^{(1-\alpha)(1-\beta)}z(0) = \phi \in \mathcal{B}_l, \tag{1.4}$$

where $D_{0^+}^{\alpha,\beta}$ stands for Hilfer fractional derivative of type $\frac{1}{2} < \beta < 1$, order $0 \leq \alpha \leq 1$. The state $z(\cdot)$ takes values in a Banach space along with the norm $\|\cdot\|$, A is the infinitesimal generator of a C_0 -semigroup. The control function $u(\cdot) \in L^2(\mathbb{N}, \mathbb{U})$. The histories $z_t : (-\infty, 0] \rightarrow \mathcal{B}_l$, $z_t(s) = z(t + s)$, $s \leq 0$ are associated with phase space \mathcal{B}_l . Additionally, a bounded linear operator $B : \mathbb{U} \rightarrow Z$, $\mathbb{U} \in Z$, $f : \mathbb{N} \times \mathcal{B}_l \times Z \rightarrow Z$ and $e : \mathbb{N} \times \mathbb{N} \times \mathcal{B}_l \rightarrow Z$ are given functions.

We organize the remaining part of our article as follows: Some new notations, important facts, lemmas, vital definitions, and theoretical results are recalled in Sect. 2. Section 3 provides the existence of fractional system (1.1)–(1.2) which is proven by Monch’s fixed point theorem. We extended the study to deal with the exact controllability for (1.3)–(1.4) in Sect. 4. In Sect. 5, we discuss the system with nonlocal conditions. Finally, we end with Sect. 6, which presents our conclusions.

2 Preliminaries

We review the essential hypothesis which is utilized all through the work in request to acquire new outcomes. Let $\nu = \alpha + \beta - \alpha\beta$, we have $(1 - \nu) = (1 - \alpha)(1 - \beta)$. We define

$\mathcal{C}_{1-\nu}(\mathbb{N}, Z) = \{z : t^{1-\nu}z(t) \in \mathcal{C}(\mathbb{N}, Z)\}$ with $\|\cdot\|_\nu$ defined by $\|z\|_\nu = \sup\{t^{1-\nu}\|z(t)\|, t \in \mathbb{N}\}$. Suppose $\mathcal{C}(\mathbb{N}, Z) : \mathbb{N} \rightarrow Z$ with $\|z\|_{\mathcal{C}} := \sup_{t \in \mathbb{N}} \|z(t)\|$ for $z \in \mathcal{C}$, and we introduce $A : D(A) \subset Z \rightarrow Z$, $\mathcal{J} : D(A) \subset Z \rightarrow Z$ is satisfied, refer to [53].

(F₁) The linear operators A and \mathcal{J} are closed.

(F₂) $D(\mathcal{J}) \subset D(A)$, \mathcal{J} is bijective.

(F₃) $\mathcal{J}^{-1} : Z \rightarrow D(\mathcal{J})$ is continuous.

In addition, from (F₁), (F₂), \mathcal{J}^{-1} is closed. Applying the closed graph theorem and (F₃), we obtain the boundedness of $A\mathcal{J}^{-1} : Z \rightarrow Z$. Designate $\|\mathcal{J}^{-1}\| = \tilde{\mathcal{J}}_m$ and $\|\mathcal{J}\| = \mathcal{J}_m$.

Definition 2.1 The fractional integral of order $\alpha \in (0, 1)$ of $f : [b, +\infty) \rightarrow \mathbb{R}$ is the function $I_{b^+}^\alpha f$ of the form

$$I_{b^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_b^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > b; \alpha > 0.$$

Definition 2.2 The Riemann–Liouville derivative of order $\alpha \in [m-1, m)$, $m \in \mathbb{Z}^+$ for $f : [b, +\infty) \rightarrow \mathbb{R}$, the function ${}^L D_{b^+}^\alpha f$ of the form

$${}^L D_{b^+}^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_b^t \frac{f(s)}{(t-s)^{\alpha+1-m}} ds, \quad t > b, m-1 \leq \alpha < m.$$

Definition 2.3 The Hilfer fractional derivative of order $0 \leq \alpha \leq 1$ and type $0 < \beta < 1$ for $f(t)$ of the form

$$D_{b^+}^{\alpha, \beta} f(t) = (I_{b^+}^{\alpha(1-\beta)} D(\mathcal{I}_{b^+}^{(1-\alpha)(1-\beta)} f))(t).$$

Remark 2.4

- (i) In case $\alpha = 0$, $b = 0$, the Hilfer fractional differential is identical to the classical Riemann–Liouville fractional derivative for f of the form

$$D_{0^+}^{0, \beta} f(t) = \frac{d}{dt} \mathcal{I}_{0^+}^{1-\beta} f(t) = {}^L D_{0^+}^\alpha f(t).$$

- (ii) In case $\alpha = 1$, $0 < \beta < 1$, and $b = 0$, the Hilfer fractional derivative is identical to the classical Caputo derivative for f of the form

$$D_{0^+}^{1, \beta} f(t) = \mathcal{I}_{0^+}^{1-\beta} \frac{d}{dt} f(t) = {}^c D_{0^+}^\beta f(t).$$

As of now, we characterize the abstract phase space \mathcal{B}_l , which is introduced in [51]. Let $g : (-\infty, 0] \rightarrow (0, +\infty)$ be a continuous function with $j = \int_{-\infty}^0 g(t) dt < +\infty$. For each $i > 0$, we define

$$\mathcal{B} = \{ \Phi : [-i, 0] \rightarrow Z, \Phi(t) \text{ is bounded and measurable} \},$$

and provide

$$\|\Phi\|_{[-i, 0]} = \sup_{\gamma \in [-i, 0]} |\Phi(\gamma)| \quad \text{for every } \Phi \in \mathcal{B}.$$

Define

$$\mathcal{B}_l = \left\{ \Phi : (-\infty, 0] \rightarrow Z; \text{ For every } i > 0, \Phi|_{[-i,0]} \in \mathcal{B}, \int_{-\infty}^0 g(t) \|\Phi\|_{[t,0]} dt < +\infty \right\}$$

and

$$\|\Phi\|_{\mathcal{B}_l} = \int_{-\infty}^0 g(t) \|\Phi\|_{[t,0]} dt \quad \text{for every } \Phi \in \mathcal{B}_l.$$

Hence $(\mathcal{B}_l, \|\cdot\|_{\mathcal{B}_l})$ is a Banach space. Suppose

$$\mathcal{B}'_l = \left\{ \begin{array}{l} z : (-\infty, b] \rightarrow Z; \\ z|_{\mathbb{N}} \in \mathcal{C}(\mathbb{N}, Z), \quad z(0) = \phi \in \mathcal{B}_l, \end{array} \right.$$

fix $\|\cdot\|_b$ in \mathcal{B}'_l , and it is characterized by

$$\|z\|_b = \|\phi\|_{\mathcal{B}_l} + \sup\{|z(t)| : t \in (0, b]\}, \quad z \in \mathcal{B}'_l.$$

Lemma 2.5 ([64]) *If $z \in \mathcal{B}'_l$, then for $t \in \mathbb{N}$, $z_t \in \mathcal{B}_l$. Also,*

$$j|z(t)| \leq \|z_t\|_{\mathcal{B}_l} \leq \|\phi\|_{\mathcal{B}_l} + j \sup_{s \in [0,t]} |z(s)|,$$

where $j = \int_{-\infty}^0 g(t) dt < +\infty$.

Definition 2.6 ([65]) *$z : (-\infty, b] \rightarrow Z$ is a mild solution of (1.1)–(1.2) only if $z(0) = \phi \in \mathcal{B}_l$ on $(-\infty, 0]$ and satisfies*

$$\begin{aligned} z(t) &= \mathcal{J}^{-1} \mathcal{S}_{\alpha,\beta}(t) \mathcal{J} \phi \\ &+ \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, z_s, \int_0^s e(s,\tau, z_\tau) d\tau\right) ds, \quad t \in \mathbb{N}, \end{aligned} \tag{2.1}$$

where

$$P_\beta(t) = t^{\beta-1} Q_\beta(t), \quad Q_\beta(t) = \int_0^\infty \beta \theta M_\beta(\theta) T(t^\beta \theta) d\theta.$$

Remark 2.7 We define the mild solution of (1.1)–(1.2) as follows:

$$M_\beta(\theta) = \sum_{k=1}^\infty \frac{(-\theta)^{k-1}}{(k-1)! \Gamma(1-\beta k)}, \quad 0 < \beta < 1, \theta \in \mathbb{C},$$

where $M_\beta(\theta)$ is a Wright function and satisfies

$$\int_0^\infty \theta^l M_\beta(\theta) d\theta = \frac{\Gamma(1+l)}{\Gamma(1+\beta l)} \quad \text{for } \theta \geq 0.$$

Lemma 2.8 ([65]) *The operators $\mathcal{S}_{\alpha,\beta}(t)$ and $Q_\beta(t)$ satisfy the following conditions:*

- For $t \geq 0$, the operators $\mathcal{S}_{\alpha,\beta}(t)$ and $Q_\beta(t)$ are linearly bounded, i.e., for every $z \in Z$,

$$\|\mathcal{S}_{\alpha,\beta}(t)z\| \leq \frac{Mt^{\mu-1}}{\Gamma(\alpha(1-\beta) + \beta)} \|z\|; \quad \|Q_\beta(t)z\| \leq \frac{M}{\Gamma(\beta)} \|z\|,$$

where $\mathcal{S}_{\alpha,\beta}(t) = I_{0^+}^{\alpha(1-\beta)} P_\beta(t)$.

- The operators $\{\mathcal{S}_{\alpha,\beta}(t)\}_{t \geq 0}$ and $\{Q_\beta(t)\}_{t \geq 0}$ are strongly continuous.

Lemma 2.9 *The strongly continuous operators $\{Q_\beta(t)\}_{t > 0}$ and $\{\mathcal{S}_{\alpha,\beta}(t)\}_{t > 0}$, $0 < t' < t'' \leq b$ are defined by*

$$\begin{aligned} \|(t')^{\beta-1} Q_\beta(t')z - (t'')^{\beta-1} Q_\beta(t'')z\| &\rightarrow 0 \quad \text{and} \\ \|\mathcal{S}_{\alpha,\beta}(t')z - \mathcal{S}_{\alpha,\beta}(t'')z\| &\rightarrow 0 \quad \text{as } t'' \rightarrow t'. \end{aligned}$$

Definition 2.10 ([60]) Assume F^+ of the Banach space $(F(\text{positive cone}), \leq)$. Define ϕ with values of F^+ , it is said to be a measure of noncompactness on Z iff $\phi(\overline{\text{co}}Y) = \phi(Y)$ for $Y \subseteq Z$, where $\overline{\text{co}}Y$ is a closed convex hull of Y .

The measure of noncompactness of ϕ is called:

- (1) Monotone if and only if $(Y_1 \subseteq Y_2) \Rightarrow (\phi(Y_1) \leq \phi(Y_2))$, Y_1, Y_2 are bounded subsets of Z ;
- (2) Nonsingular if and only if $\phi(\{a\} \cup Y) = \phi(Y)$ for every $a \in Z, Y \subset Z$;
- (3) Regular if and only if $\phi(Y) = 0, Y$ is relatively compact in Z .

The measure of noncompactness of Hausdorff $\widehat{\mu}$ is defined by

$$\widehat{\mu}(Y) = \inf \left\{ \epsilon > 0; Y \subset \bigcup_{k=1}^m N_k \text{ such that } \text{diam}(N_k) \leq \epsilon \right\}.$$

To know more information about the properties of MNC, the readers can refer to [66].

Now, for every Y, Y_1, Y_2 of Z ,

- (4) $\widehat{\mu}(Y_1 + Y_2) \leq \widehat{\mu}(Y_1) + \widehat{\mu}(Y_2)$, where $Y_1 + Y_2 = \{y_1 + y_2 : y_1 \in Y_1, y_2 \in Y_2\}$;
- (5) $\widehat{\mu}(Y_1 \cup Y_2) \leq \max\{\widehat{\mu}(Y_1), \widehat{\mu}(Y_2)\}$;
- (6) $\widehat{\mu}(\alpha Y) \leq |\alpha| \widehat{\mu}(Y)$ for any $\alpha \in \mathbb{R}$;
- (7) If $Q : D(Q) \subseteq Z \rightarrow Y$, then $\widehat{\mu}_Y(QY) \leq k \widehat{\mu}(Y), Y \subseteq D(Q)$, here Y is a Banach space and k is any constant.

Lemma 2.11 ([66]) *If $\mathcal{K} \subset \mathcal{C}(\mathbb{N}, Z)$ is bounded and equicontinuous, then $\widehat{\mu}(\mathcal{K}(t))$ is a continuous function for all $t \in \mathbb{N}$*

$$\widehat{\mu}(\mathcal{K}) = \sup_{t \in \mathbb{N}} \{\widehat{\mu}(\mathcal{K}(t)), t \in \mathbb{N}\}, \quad \text{where } \mathcal{K}(t) = \{z(t) : z \in \mathcal{K}\} \subseteq Z.$$

Theorem 2.12 ([62, 67]) *If $\{u_n : \mathbb{N} \rightarrow Z\}$ is Bochner's integrable function with $\|u_n(t)\| \leq \widehat{\mu}(t)$ a.e. for $t \in \mathbb{N}$ and for every $n \geq 1$, where $\widehat{\mu} \in L^1(\mathbb{N}, \mathbb{R})$, then $Y(t) = \widehat{\mu}(\{u_n(t) : n \geq 1\}) \in L^1(\mathbb{N}, \mathbb{R})$ and satisfies $\widehat{\mu}(\{\int_0^t u_n(s) ds : n \geq 1\}) \leq 2 \int_0^t \widehat{\mu}(s) ds$.*

Lemma 2.13 ([68]) *Let K be a closed convex subset of Z and $0 \in K$. If $F : K \rightarrow Z$ is a continuous map which satisfies Mönch's condition (i.e., $M \subseteq K$ is countable, $M \subseteq \overline{\text{co}}(\{0\} \cup F(M)) \Rightarrow \overline{M}$ is compact), then F has a fixed point in K .*

3 Existence

In this section, we mainly focus on the existence of (1.1)–(1.2), and in order to prove the main theorem, we have the following assumptions.

(A₀) For all $K \subset Z$, $\theta \in (0, \infty)$ and $z \in K$,

$$\|T(t_2^\beta \theta)z - T(t_1^\beta \theta)z\| \rightarrow 0, \quad \text{as } t_2 \rightarrow t_1.$$

(A₁) The function $f : \mathbb{N} \times \mathcal{B}_I \times Z \rightarrow Z$ satisfies the following:

- (i) $f(\cdot, s, z)$ is measurable for every $(s, z) \in \mathcal{B}_I \times Z$ and $f(t, \cdot, \cdot)$ is continuous, $t \in \mathbb{N}$ and $z \in \mathcal{B}_I$, $f(\cdot, \cdot, z) : (0, b] \rightarrow Z$ is strongly measurable.
- (ii) There exist $\beta_1 \in (0, \beta)$, $m_1 \in L^{\frac{1}{\beta_1}}(\mathbb{N}, \mathbb{R}^+)$ and $\Omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\|f(t, s, z)\| \leq m_1(t)\Omega(t^{1-\nu}\|s\|_{\mathcal{B}_I} + \|z\|)$$

for every $t, s, z \in \mathbb{N} \times \mathcal{B}_I \times Z$, where $\widehat{\mu}$ satisfies $\liminf_{j \rightarrow \infty} \frac{\widehat{\mu}(j)}{j} = 0$.

- (iii) There exists $\beta_2 \in (0, \beta)$, $m_2 \in L^{\frac{1}{\beta_2}}(\mathbb{N}, \mathbb{R}^+)$ such that, for any $K_1 \subset Z$ and $F_1 \subset \mathcal{B}_I$,

$$\widehat{\mu}(f(t, F_1, K_1)) \leq m_2(t) \left[\sup_{-\infty < \xi \leq 0} \widehat{\mu}(F_1(\xi)) + \widehat{\mu}(K_1) \right] \quad \text{for almost all } t \in \mathbb{N},$$

where $F_1(\xi) = \{\widehat{w}(\xi) : \widehat{w} \in K_1\}$, $\widehat{\mu}$ is the Hausdorff measure of noncompactness.

For $m_i \in L^{\frac{1}{\beta_i}}([0, b], \mathbb{R}^+)$, $\beta_i \in (0, \beta)$, $i = 1, 2$.

(A₂) The function $e : \mathbb{N} \times \mathbb{N} \times \mathcal{B}_I \rightarrow Z$ satisfies the following:

- (i) $e(\cdot, s, z)$ is measurable for all $(s, z) \in \mathcal{B}_I \times Z$.
- (ii) There exists $E_0 > 0$ such that $\|e(t, s, z)\| \leq E_0(1 + \|z\|_{\mathcal{B}_I})$ for every $t, z \in Z$, $s \in \mathcal{B}_I$.
- (iii) There exists $m_3 \in L^1(\mathbb{N}, \mathbb{R}^+)$ such that, for any $K_2 \subset Z$,

$$\widehat{\mu}(f(t, s, z)) \leq m_3(t, s) \left[\sup_{-\infty < \xi \leq 0} \widehat{\mu}(K_2(\xi)) \right] \quad \text{for almost all } t \in \mathbb{N},$$

with $m_3^* = \sup_{s \in \mathbb{N}} \int_0^s m_3(t, \tau) d\tau < \infty$.

For our convenience, we introduce

$$M_1 = k_1 \|m_1\|_{L^{\frac{1}{\beta_1}}(\mathbb{N}, \mathbb{R}^+)}, \quad M_2 = k_2 \|m_2\|_{L^{\frac{1}{\beta_2}}(\mathbb{N}, \mathbb{R}^+)},$$

$$k_i = \left[\left(\frac{1 - \beta_i}{\beta - \beta_i} \right) b^{\left(\frac{\beta - \beta_i}{1 - \beta_i} \right)} \right]^{1 - \beta_i}, \quad \beta_i \in (0, \beta), i = 1, 2.$$

Theorem 3.1 *Assume that (A₀)–(A₂) are satisfied, then (1.1)–(1.2) has at least one mild solution if*

$$P^* = \frac{2MM_2\widetilde{J}_m^{1-\nu}(1 + 2m_3^*)}{\Gamma(\beta)} < 1 \quad \text{for some } \frac{1}{2} < \beta < 1. \tag{3.1}$$

Proof We define the operator $\Upsilon : \mathcal{B}'_l \rightarrow \mathcal{B}'_l$ by

$$\Upsilon z(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{J}^{-1} \mathcal{S}_{\alpha, \beta}(t) \mathcal{J} \phi + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f(s, z_s, \int_0^s e(s, \tau, z_\tau) d\tau) ds, & t \in \mathbb{N}. \end{cases} \tag{3.2}$$

For $\phi \in \mathcal{B}_l$, we define $\hat{\eta}$ as follows:

$$\hat{\eta}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{J}^{-1} \mathcal{S}_{\alpha, \beta}(t) \mathcal{J} \phi, & t \in \mathbb{N}, \end{cases}$$

then $\hat{\eta} \in \mathcal{B}'_l$. Let $z(t) = g(t) + \hat{\eta}(t)$, $-\infty < t \leq b$. Clearly, z satisfies (2.1) if and only if g satisfies $g_0 = 0$ and

$$g(t) = \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau\right) ds.$$

Let $\mathcal{B}''_l = \{g \in \mathcal{B}'_l : g_0 = 0 \in \mathcal{B}_l\}$. For any $g \in \mathcal{B}''_l$,

$$\begin{aligned} \|g\|_b &= \|g_0\|_{\mathcal{B}_l} + \sup\{\|g(s)\| : 0 \leq s \leq b\} \\ &= \sup\{\|g(s)\| : 0 \leq s \leq b\}. \end{aligned}$$

Hence $(\mathcal{B}''_l, \|\cdot\|_b)$ is a Banach space. Now $\ell > 0$, we fix $F_\ell = \{g \in \mathcal{B}''_l : \|g\|_b \leq \ell\}$, then $F_\ell \subseteq \mathcal{B}''_l$ is uniformly bounded, $g \in F_\ell$, and referring to Lemma 2.5,

$$\begin{aligned} \|g_t + \hat{\eta}_t\|_{\mathcal{B}_l} &\leq \|g_t\|_{\mathcal{B}_l} + \|\hat{\eta}_t\|_{\mathcal{B}_l} \\ &\leq j\left(\ell + \tilde{\mathcal{J}}_m \frac{Mt^{\mu-1}}{\Gamma(\alpha(1-\beta) + \beta)} \mathcal{J}_m |\phi|\right) + \|\phi\|_{\mathcal{B}_l} = \ell'. \end{aligned} \tag{3.3}$$

We define the operator $\tilde{\Upsilon} : \mathcal{B}''_l \rightarrow \mathcal{B}''_l$ as follows:

$$\tilde{\Upsilon} g(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) \times f(s, g_t + \hat{\eta}_t, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau) ds, & t \in \mathbb{N}. \end{cases} \tag{3.4}$$

To prove that $\tilde{\Upsilon}$ has a fixed point.

Now we divide the proof into a few steps for our benefit.

Step 1: For $\ell > 0$, $\tilde{\Upsilon}(F_\ell) \subseteq F_\ell$. If it is false, then $g^\ell(\cdot) \in F_\ell$ and $t \in \mathbb{N}$ such that $\|(\tilde{\Upsilon} g^\ell)(t)\| > \ell$. Suppose $\ell > 0$, and consider $\{F_\ell = z \in \mathcal{C}_{1-\nu} : \|z\|_\nu \leq \ell\}$. It is understood that F_ℓ is a closed, bounded, and convex set of C . Furthermore, from Lemma 2.8, (A₁), and Hölder’s inequality, we have

$$\begin{aligned} \ell &< \sup_{t \in \mathbb{N}} t^{1-\nu} \|(\tilde{\Upsilon} g^\ell)(t)\| \\ &\leq b^{1-\nu} \left\| \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, g^\ell_s + \hat{\eta}_s, \int_0^s e(s, \tau, g^\ell_\tau + \hat{\eta}_\tau) d\tau\right) ds \right\| \end{aligned}$$

$$\begin{aligned} &\leq b^{1-\nu} \left\| \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) m_1(s) \Omega(\ell' + bE_0(1 + \ell')) ds \right\| \\ &\leq b^{1-\nu} \frac{MM_1 \tilde{\mathcal{J}}_m}{\Gamma(1 + \beta)} \Omega(\ell' + bE_0(1 + \ell')). \end{aligned} \tag{3.5}$$

We now divide (3.5) by ℓ , and taking $\ell \rightarrow \infty$, it contradicts with (3.1). Therefore, $\tilde{\Upsilon}(F_\ell) \subseteq F_\ell$.

Step 2: $\tilde{\Upsilon}$ is continuous on F_ℓ .

For all $g^m, g \in F_\ell(\mathbb{N})$, $m = 0, 1, 2, \dots$, with $\lim_{m \rightarrow \infty} g^m = g$, then we have $\lim_{m \rightarrow \infty} g^m(t) = g(t)$ and

$$\lim_{m \rightarrow \infty} t^{1-\nu} g^m(t) = t^{1-\nu} g(t).$$

Consider $f(t, t^{1-\nu} z^m(t), \int_0^t e(t, s, s^{1-\nu} z^m(s)) ds)$, and we take

$$\begin{aligned} F_m(s) &= f\left(s, s^{1-\nu}(g_s^m + \hat{\eta}_s), \int_0^s e(s, \varepsilon, \varepsilon^{1-\nu}(g_\varepsilon^m + \hat{\eta}_\varepsilon)) d\varepsilon\right) \quad \text{and} \\ F(s) &= f\left(s, s^{1-\nu}(g_s + \hat{\eta}_s), \int_0^s e(s, \varepsilon, \varepsilon^{1-\nu}(g_\varepsilon + \hat{\eta}_\varepsilon)) d\varepsilon\right). \end{aligned}$$

Lebesgue’s dominated convergence theorem and hypotheses $(A_1), (A_2)$ give

$$\int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) \|F_m(s) - F(s)\| ds \rightarrow 0 \quad \text{as } m \rightarrow \infty, t \in \mathbb{N}. \tag{3.6}$$

Now, by (A_1) , we have

$$\|\tilde{\Upsilon}g^m - \tilde{\Upsilon}g\|_C \leq \frac{b^{1-\nu} M \tilde{\mathcal{J}}_m}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|F_m(s) - F(s)\| ds. \tag{3.7}$$

Using (3.6) in (3.7), we get

$$\|\tilde{\Upsilon}g^m - \tilde{\Upsilon}g\|_C \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

therefore, $\tilde{\Upsilon}$ is continuous on F_ℓ .

Step 3: $\tilde{\Upsilon}(F_\ell)$ is equicontinuous on \mathbb{N} . Let $\gamma \in \tilde{\Upsilon}(F_\ell)$.

For $0 < t_1 < t_2 < b$, we have

$$\begin{aligned} \|\gamma(t_2) - \gamma(t_1)\| &= \left\| \mathcal{J}^{-1} \int_0^{t_2} t_2^{1-\nu}(t_2-s)^{\beta-1} Q_\beta(t_2-s) F(s) ds \right. \\ &\quad \left. - \mathcal{J}^{-1} \int_0^{t_1} t_1^{1-\nu}(t_1-s)^{\beta-1} Q_\beta(t_1-s) F(s) ds \right\| \\ &\leq \left\| \int_{t_1}^{t_2} t_2^{1-\nu}(t_2-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t_1-s) F(s) ds \right\| \\ &\quad + \left\| \int_0^{t_1} [t_2^{1-\nu}(t_2-s)^{\beta-1} - t_1^{1-\nu}(t_1-s)^{\beta-1}] \mathcal{J}^{-1} Q_\beta(t_2-s) F(s) ds \right\| \\ &\quad + \left\| \int_0^{t_1} t_1^{1-\nu}(t_1-s)^{\beta-1} \mathcal{J}^{-1} [Q_\beta(t_2-s) - Q_\beta(t_1-s)] F(s) ds \right\| \end{aligned}$$

$$\begin{aligned} &\leq \frac{M\tilde{\mathcal{J}}_m}{\Gamma(\beta)} \Omega(\ell' + bE_0(1 + \ell')) \left\| \int_{t_1}^{t_2} t_2^{1-\nu}(t_2 - s)^{\beta-1} m_1(s) ds \right\| \\ &\quad + \frac{2M\tilde{\mathcal{J}}_m}{\Gamma(\beta)} \left\| \int_0^{t_1} [t_2^{1-\nu}(t_2 - s)^{\beta-1} - t_1^{1-\nu}(t_1 - s)^{\beta-1}] m_1(s) ds \right\| \\ &\quad + \left\| \int_0^{t_1} t_1^{1-\nu}(t_1 - s)^{\beta-1} \mathcal{J}^{-1}[Q_\beta(t_2 - s) - Q_\beta(t_1 - s)] F(s) ds \right\|. \end{aligned}$$

By Lemma 2.9 and Lebesgue’s integral dominance convergence theorem, we get $\|\gamma(t_2) - \gamma(t_1)\|$ becomes zero as $t_2 - t_1 \rightarrow 0$.

Thus, $\tilde{\Upsilon}(F_\ell)$ is equicontinuous on \mathbb{N} .

Step 4: Mönch’s condition: Assume that $\mathcal{K} \subseteq F_\ell$ is countable and $\mathcal{K} \subseteq \text{conv}(\{0\} \cup \tilde{\Upsilon}(\mathcal{K}))$. To prove $\hat{\mu}(\mathcal{K}) = 0$, where $\hat{\mu}$ is the Hausdorff measure of noncompactness. If $\mathcal{K} = \{g^m\}_{m=1}^\infty$, thus we show that $\tilde{\Upsilon}(\mathcal{K})(t)$ is relatively compact in Z for all $t \in \mathbb{N}$. By using Theorem 2.12,

$$\begin{aligned} &\hat{\mu}(\{(\tilde{\Upsilon}g^m)(t)\}_{m=1}^\infty) \\ &= \hat{\mu}\left(\left\{t^{1-\nu} \int_0^t (t - s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t - s) F_m(s) ds\right\}_{m=1}^\infty\right) \\ &\leq \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} \hat{\mu}(\{F_m(s)\}_{m=1}^\infty) ds \\ &\leq \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} m_2(s) \left[\sup_{-\infty < \tau \leq 0} \mathcal{S}(\{g^m(s + \tau) + \hat{\eta}(s + \tau)\}_{m=1}^\infty) \right. \\ &\quad \left. + \hat{\mu}\left(\left\{\int_0^s e(s, \varepsilon, g_\varepsilon^m + \hat{\eta}_\varepsilon) d\varepsilon\right\}_{m=1}^\infty\right) \right] ds \\ &\leq \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} m_2(s) \sup_{-\infty < \tau \leq s} \hat{\mu}(\{g^m(\tau) + \hat{\eta}(\tau)\}_{m=1}^\infty) ds \\ &\quad + \frac{4M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} m_2(s) \int_0^s \hat{\mu}(\{e(s, \varepsilon, g_\varepsilon^m + \hat{\eta}_\varepsilon) d\varepsilon\}_{m=1}^\infty) ds \\ &\leq \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} m_2(s) \sup_{-\infty < \tau \leq s} \hat{\mu}(\{g^m(\tau) + \hat{\eta}(\tau)\}_{m=1}^\infty) ds \\ &\quad + \frac{4M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} m_2(s) \int_0^s m_3(s, \varepsilon) \left[\sup_{-\infty < \xi \leq 0} \hat{\mu}((g_\varepsilon^m + \hat{\eta}_\varepsilon)(\xi)) d\varepsilon \right] ds \\ &\leq \frac{2MM_2\tilde{\mathcal{J}}_m c^{1-\nu}}{\Gamma(\beta)} \sup_{-\infty < \tau \leq s} \hat{\mu}(\{g^m(\tau) + \hat{\eta}(\tau)\}_{m=1}^\infty) \\ &\quad + \frac{4MM_2\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^s m_3(s, \varepsilon) \left[\sup_{-\infty < \xi \leq 0} \hat{\mu}((g_\varepsilon^m + \hat{\eta}_\varepsilon)(\xi)) d\varepsilon \right] \\ &\leq \frac{2MM_2\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} [1 + 2m_3^*] \sup_{0 \leq \tau \leq s} \hat{\mu}(\{g^m(\tau) + \hat{\eta}(\tau)\}_{m=1}^\infty) \\ &\leq \frac{2MM_2\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} [1 + 2m_3^*] \sup_{0 \leq \tau \leq s} \hat{\mu}(\mathcal{K}(\tau)). \end{aligned}$$

That is,

$$\hat{\mu}(\tilde{\Upsilon}\mathcal{K}(t)) \leq \frac{2MM_2\tilde{\mathcal{J}}_m b^{1-\nu} (1 + 2m_3^*)}{\Gamma(\beta)} \sup_{0 \leq \tau \leq s} \hat{\mu}(\mathcal{K}(\tau)).$$

Therefore, by using Lemma 2.11,

$$\widehat{\mu}(\mathcal{K}) \leq \widehat{\mu}(\text{conv}(\{0\} \cup (\widetilde{\Upsilon}(\mathcal{K}))) = \widehat{\mu}(\widetilde{\Upsilon}(\mathcal{K})) \leq P^* \widehat{\mu}(\mathcal{K}),$$

where P^* is defined in (3.1), hence $\widehat{\mu}(\mathcal{K}) = 0$.

Lemma 2.13 shows that $\widetilde{\Upsilon}$ has a fixed point $\mathcal{K} \in F_\ell$. Consequently, $z = g + \widehat{\eta}$ is a mild solution of (1.1)–(1.2). The proof is now completed. \square

4 Controllability

In this section, we mainly focus on the controllability of (1.3)–(1.4). So, we now introduce the mild solution of (1.3)–(1.4) as follows.

Definition 4.1 A function $z : (-\infty, b] \rightarrow Z$ is said to be a mild solution of (1.1)–(1.2) if and only if $z(0) = \phi \in \mathcal{B}_l$ on $(-\infty, 0]$ and z satisfies

$$\begin{aligned} z(t) = & \mathcal{J}^{-1} \mathcal{S}_{\alpha, \beta}(t) \mathcal{J} \phi + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, z_s, \int_0^s e(s, \tau, z_\tau) d\tau\right) ds \\ & + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_z(s) ds, \quad t \in \mathbb{N}. \end{aligned} \tag{4.1}$$

Lemma 4.2 System (1.3)–(1.4) is said to be controllable on \mathbb{N} if, for every $\phi \in \mathcal{B}_l, z^1 \in Z$, there exists $u \in L^2(\mathbb{N}, U)$ such that $z(t)$ of (1.3)–(1.4) satisfies $z(b) = z^1$.

Controllability results are proved in relation to the following hypotheses:

(A₃) The operator $B : L^2(\mathbb{N}, U) \rightarrow L^1(\mathbb{N}, Z)$ which is bounded, $W : L^2(\mathbb{N}, U) \rightarrow Z$ defined by

$$Wu = \int_0^b (b-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(b-s) B u(s) ds,$$

satisfies:

- (i) W^{-1} takes the value in $L^2((0, b], U) / \text{Ker } W$, there exist $M_b > 0, M_w > 0$ such that $\|B\| \leq M_b$ and $\|W^{-1}\| \leq M_w$.
- (ii) There exists $\beta_4 \in (0, \beta)$, and for every $K \in Z, m_4 \in L^{\frac{1}{\beta_4}}((0, b], \mathbb{R}^+)$ such that $\widehat{\mu}((W^{-1}K)(t)) \leq m_4(t) \widehat{\mu}(K)$. Here, $m_i \in L^{\frac{1}{\beta_i}}([0, b], \mathbb{R}^+), \beta_i \in (0, \beta), i = 1, 2, 3, 4$.

For our convenience, we introduce

$$\begin{aligned} M_4 = & k_4 \|m_4\|_{L^{\frac{1}{\beta_4}}(\mathbb{N}, \mathbb{R}^+)}, \quad C^* = \sqrt{\frac{b^{2\beta-1}}{2\beta-1}}, \\ k_i = & \left[\left(\frac{1-\beta_i}{\beta-\beta_i} \right) b^{\frac{\beta-\beta_i}{1-\beta_i}} \right]^{1-\beta_i}, \quad i = 1, 2, 3, 4. \end{aligned}$$

Theorem 4.3 Assume that (A₀)–(A₃) are satisfied, then (1.3)–(1.4) is controllable on $(0, b]$ if

$$\frac{2M \widetilde{\mathcal{J}}_m M_2 b^{1-\nu} (1 + 2m_3^*)}{\Gamma(\beta)} \left[1 + \frac{2MM_b M_4 \widetilde{\mathcal{J}}_m}{\Gamma(\beta)} \right] \widehat{\mu}(\mathcal{K}(\tau)) < 1 \quad \text{for some } \frac{1}{2} < \beta < 1. \tag{4.2}$$

Proof By using (A₃), we define $u_z(t)$ by

$$u_z(t) = W^{-1} \left[z^1 - \mathcal{J}^{-1} \mathcal{S}_{\alpha,\beta}(t) \mathcal{J} \phi - \int_0^b (b-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(b-s) f \left(s, g_s, \int_0^s e(s, \tau, g_\tau) d\tau \right) ds \right] (t).$$

Let $\Upsilon : \mathcal{B}'_l \rightarrow \mathcal{B}'_l$ be defined by

$$\Upsilon z(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{J}^{-1} \mathcal{S}_{\alpha,\beta}(t) \mathcal{J}(t) \phi + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f(s, z_s, \int_0^s e(s, \tau, z_\tau) d\tau) ds + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_z(s) ds, & t \in \mathbb{N}. \end{cases} \tag{4.3}$$

For $\phi \in \mathcal{B}_l$, we have

$$\hat{\eta}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{J}^{-1} \mathcal{S}_{\alpha,\beta}(t) \mathcal{J} \phi, & t \in \mathbb{N}, \end{cases}$$

then $\hat{\eta} \in \mathcal{B}'_l$. Let $z(t) = g(t) + \hat{\eta}(t)$, $-\infty < t \leq b$. Now, we identified that z satisfies (4.1) if and only if g satisfies $g_0 = 0$ and

$$g(t) = \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f \left(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau \right) ds + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_z(s) ds,$$

where

$$u_z(s) = W^{-1} \left[z^1 - \mathcal{J}^{-1} \mathcal{S}_{\alpha,\beta}(t) \mathcal{J} \phi - \int_0^b (b-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(b-s) \times f \left(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau \right) ds \right] (t).$$

We define the operator $\tilde{\Upsilon} : \mathcal{B}''_h \rightarrow \mathcal{B}''_h$ by

$$\tilde{\Upsilon} g(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau) ds + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_z(s) ds, & t \in \mathbb{N}. \end{cases} \tag{4.4}$$

Now, to show $\tilde{\Upsilon}$ has a fixed point. We divide the proof into the following steps for our convenience.

Step 1: To prove that there exists a constant $\ell > 0$ such that $\tilde{\Upsilon}(F_\ell) \subseteq F_\ell$. If it fails, then $g^\ell(\cdot) \in F_\ell$ and $t \in \mathbb{N}$ such that $\|\tilde{\Upsilon}(g^\ell)(t)\| > \ell$.

Take $\ell > 0$ and consider $\{F_\ell = z \in \mathcal{C}_{1-\nu} : \|z\|_\nu \leq \ell\}$. Apparently, F_ℓ is a closed, bounded, and convex set of \mathcal{C} .

Using Lemma 2.8, (A₁), (A₃), and Holder’s inequality, we have

$$\begin{aligned}
 \ell &< \sup_{x \in \mathbb{N}} t^{1-\nu} \left\| (\tilde{\Upsilon} g^\ell)(t) \right\| \\
 &\leq b^{1-\nu} \left\| \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, g_\tau + \hat{\eta}_\tau) d\tau\right) ds \right\| \\
 &\quad + b^{1-\nu} \left\| \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_{g^\ell}(s) ds \right\|, \quad t \in \mathbb{N} \\
 &\leq \frac{M \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \left\| \int_0^t (t-s)^{\beta-1} m_1(s) \Omega(\ell' + bE_0(1 + \ell')) ds \right\| \\
 &\quad + \frac{MM_b \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \sqrt{\frac{b^{2\beta-1}}{2\beta-1}} \|u_{g^\ell}\|_{L^2} \\
 &\leq \frac{MM_1 \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \Omega(\ell' + bE_0(1 + \ell')) \\
 &\quad + \frac{MM_b M_w \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \sqrt{\frac{b^{2\beta-1}}{2\beta-1}} \left[\|z^1\| + \mathcal{J}^{-1} \mathcal{S}_{\alpha, \beta}(t)(b) \mathcal{J} \phi \right. \\
 &\quad \left. + \int_0^b (b-\gamma)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-\gamma) f\left(\gamma, g_\gamma + \hat{\eta}_\gamma, \int_0^\gamma e(\gamma, \tau, g_\tau + \hat{\eta}_\tau) d\tau\right) d\gamma \right] \\
 &\leq \frac{MM_1 \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \Omega(\ell' + bE_0(1 + \ell')) + \frac{MM_b M_w \tilde{\mathcal{J}}_m}{\Gamma(\beta)} C^* \left[b^{1-\nu} \|z^1\| \right. \\
 &\quad \left. + \frac{\tilde{\mathcal{J}}_m M \mathcal{J}_m}{\Gamma(\gamma(1-\beta) + \beta)} \|\phi\| + \frac{MM_1 \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \Omega(\ell' + bE_0(1 + \ell')) \right] \\
 &\leq \frac{MM_1 \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \Omega(\ell' + bE_0(1 + \ell')) \left[1 + \frac{M \tilde{\mathcal{J}}_m M_b M_w}{\Gamma(\beta)} C^* \right] \\
 &\quad + \frac{M \tilde{\mathcal{J}}_m M_b M_w}{\Gamma(\beta)} C^* \left[b^{1-\nu} \|z^1\| + \frac{\tilde{\mathcal{J}}_m M \mathcal{J}_m}{\Gamma(\gamma(1-\beta) + \beta)} \|\phi\| \right]. \tag{4.5}
 \end{aligned}$$

Take $\rho = \ell' + bE_0(1 + \ell')$, note that $\rho \rightarrow \infty$ as $\ell \rightarrow \infty$.

Dividing (4.5) by ℓ and taking $\ell \rightarrow \infty$, we have

$$1 \leq \frac{MM_1 \tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \liminf_{\rho \rightarrow \infty} \frac{\hat{\mu}(\rho)}{\rho} \frac{\rho}{\ell} \left[1 + \frac{M \tilde{\mathcal{J}}_m M_b M_w}{\Gamma(\beta)} C^* \right], \tag{4.6}$$

then by (A₁)(ii), (4.6) is a contradiction. Hence $\tilde{\Upsilon}(F_\ell) \subseteq F_\ell$.

Step 2: Similar to *Step 2* of Theorem 3.1.

Step 3: For $g \in F_\ell$, assume $g(t) = t^{\nu-1} \Upsilon z(t)$, $\tilde{\Upsilon}$ provides bounded sets into equicontinuous sets of C for all $y \in F_\ell$, there exists $\Upsilon \in \tilde{\Upsilon}(z)$ such that $\|\Upsilon z(t_2) - \Upsilon z(t_1)\| \rightarrow 0$ as $t_2 \rightarrow t_1$.

$$\begin{aligned}
 \Upsilon z(t) &= \mathcal{J}^{-1} \mathcal{S}_{\alpha, \beta}(t) \mathcal{J}(t) \phi + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) \\
 &\quad (\times) f\left(s, g_s + \hat{\eta}_s, \int_0^s e(s, \tau, z_\tau) d\tau\right) ds \\
 &\quad + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) B u_g(s) ds.
 \end{aligned}$$

Hereafter, we continue our proof as per *Step 3* of Theorem 3.1, and hence $\tilde{\Upsilon}(F_\ell)$ is equicontinuous.

Step 4: Mönch’s condition: Consider $\mathcal{K} \subseteq F_\ell$ is countable and $\mathcal{K} \subseteq \text{conv}(\{0\} \cup \tilde{\Upsilon}(\mathcal{K}))$. To prove $\hat{\mu}(\mathcal{K}) = 0$, here $\hat{\mu}$ is the Hausdorff measure of noncompactness. If $\mathcal{K} = \{g^m\}_{m=1}^\infty$, then we show that $\tilde{\Upsilon}(\mathcal{K})(t)$ is relatively compact in Z for all $t \in \mathbb{N}$. By Theorem 2.12, we obtain

$$\begin{aligned} \hat{\mu}(\{(\tilde{\Upsilon}g^m)(t)\}_{m=1}^\infty) &= \hat{\mu}\left(\left\{\int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s)[F_m(s) + Bu_{g^m}(s)] ds\right\}_{m=1}^\infty\right) \\ &\leq \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{\mu}(\{F_m(s)\}_{m=1}^\infty) ds \\ &\quad + \frac{2MM_b\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{\mu}(\{u_{g^m}(s)\}_{m=1}^\infty) ds \\ &\leq \mathbb{I}_1 + \mathbb{I}_2, \end{aligned}$$

where

$$\begin{aligned} \mathbb{I}_1 &= \frac{2M\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{\mu}(\{F_m(s)\}_{m=1}^\infty) ds \\ &\leq \frac{2MM_2\tilde{\mathcal{J}}_m b^{1-\nu}(1+2m_3^*)}{\Gamma(\beta)} \sup_{0 \leq \tau \leq s} \hat{\mu}(\mathcal{K}(\tau)) \quad (\text{from Step 4 of Theorem 3.1}), \\ \mathbb{I}_2 &= \frac{2MM_b\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \hat{\mu}(\{u_{g^m}(s)\}_{m=1}^\infty) ds \\ &\leq \frac{2MM_b\tilde{\mathcal{J}}_m b^{1-\nu}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} m_4(s) \frac{2M\tilde{\mathcal{J}}_m}{\Gamma(\beta)} \int_0^b (b-s)^{\beta-1} \hat{\mu}(F_m(s)) ds \\ &\leq \frac{4M^2\tilde{\mathcal{J}}_m^2 M_b M_2 M_4 b^{1-\nu}}{\Gamma(\beta)^2} (1+2m_3^*) \sup_{0 \leq \tau \leq s} \hat{\mu}(\mathcal{K}(\tau)), \\ \mathbb{I}_1 + \mathbb{I}_2 &\leq \frac{2M\tilde{\mathcal{J}}_m M_2 b^{1-\nu}(1+2m_3^*)}{\Gamma(\beta)} \left[1 + \frac{2MM_b M_4 \tilde{\mathcal{J}}_m}{\Gamma(\beta)}\right] \hat{\mu}(\mathcal{K}(\tau)). \end{aligned}$$

That is, $\hat{\mu}(\tilde{\Upsilon}\mathcal{K}(t)) \leq \frac{2M\tilde{\mathcal{J}}_m M_2 b^{1-\nu}(1+2m_3^*)}{\Gamma(\beta)} \left[1 + \frac{2MM_b M_4 \tilde{\mathcal{J}}_m}{\Gamma(\beta)}\right] \hat{\mu}(\mathcal{K}(\tau))$. Therefore, using Mönch’s condition, we get $\hat{\mu}(\mathcal{K}) \leq \hat{\mu}(\text{conv}(\{0\} \cup \tilde{\Upsilon}(\mathcal{K}))) = \hat{\mu}(\tilde{\Upsilon}(\mathcal{K})) \leq P^* \hat{\mu}(\mathcal{K})$, where P^* is defined in (4.2), and hence Lemma 2.13 shows that (1.1)–(1.2) has a fixed point \mathcal{K} in F_ℓ . Hence, $z = g + \hat{\eta}$ is the mild solution of (1.1)–(1.2) satisfying $z(b) = z^1$. Consequently (1.3)–(1.4) is controllable on \mathbb{N} . □

5 Nonlocal conditions

The nonlocal Cauchy problem for differential equation was first studied by Byszewski [69]. Their research is driven by imaginative enthusiasm and the manner in which these types of problems usually occur when proving practical applications. For example, material science and life sciences can be depicted by techniques for the differential framework subject to nonlocal limit conditions, the readers can refer to [48, 60, 62, 69, 70]. We presently expect that the nonlocal Sobolev-type Hilfer fractional integro-differential equations with control

are as follows:

$$D_{0^+}^{\alpha, \beta} [\mathcal{J}z(t)] = Az(t) + f\left(t, z_t, \int_0^t e(t, s, z_s) ds\right) + Bu(t), \quad t \in \mathbb{N} = (0, b], \tag{5.1}$$

$$\mathcal{I}_{0^+}^{(1-\alpha)(1-\beta)} z(0) = \phi + J(t_{i_1}, t_{i_2}, t_{i_3}, \dots, t_{i_k}) \in \mathcal{B}_l, \quad 0 < i_k \leq b, k = 1, 2, \dots, n. \tag{5.2}$$

The result is proved in relation to the following hypothesis:

(A₄) Function $J : \mathcal{B}^n \rightarrow \mathcal{B}$ is continuous, there exists $L_i(h) > 0$ such that

$$\|J(v_1, v_2, \dots, v_k) - J(w_1, w_2, \dots, w_k)\| \leq \sum_{k=1}^n L_k(J) \|v_k - w_k\|_{\mathcal{B}_l}$$

for all $v_k, w_k \in \mathcal{B}_l$, and consider $L_k = \sup\{\|J(v_1, v_2, \dots, v_k)\| : v_k \in \mathcal{B}_l\}$.

Definition 5.1 A function $z : (-\infty, b] \rightarrow Z$ is a mild solution of (5.1)–(5.2) only if $z_0 = \phi + J(t_{x_1}, t_{x_2}, \dots, t_{x_n}) \in \mathcal{B}_l$ on $(-\infty, 0]$ and

$$\begin{aligned} z(t) &= \mathcal{J}^{-1} S_{\alpha, \beta}(t) \mathcal{J}[\phi + J(t_{x_1}, t_{x_2}, t_{x_3}, \dots, t_{x_n})(0)] \\ &\quad + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) f\left(s, z_s, \int_0^s e(s, \tau, z_\tau) d\tau\right) ds \\ &\quad + \int_0^t (t-s)^{\beta-1} \mathcal{J}^{-1} Q_\beta(t-s) Bu_z(s) ds, \quad t \in \mathbb{N}, \end{aligned} \tag{5.3}$$

is satisfied.

Theorem 5.2 Assume that (A₀)–(A₄) are satisfied, then (5.1)–(5.2) is controllable on $(0, b]$ if

$$\frac{2M\tilde{\mathcal{J}}_m M_2 b^{1-\nu} (1 + 2m_3^*)}{\Gamma(\beta)} \left\{ 1 + \frac{2MM_b M_4 \tilde{\mathcal{J}}_m}{\Gamma(\beta)} \right\} \hat{\mu}(\mathcal{K}(\tau)) < 1 \quad \text{for some } \frac{1}{2} < \beta < 1$$

is satisfied.

6 Example

Assume that the fractional evolution system with control is as follows:

$$\begin{cases} D_{0^+}^{\alpha, \frac{2}{3}} [z(\phi, \varsigma) - \frac{\partial^2}{\partial \varsigma^2} z(\phi, \varsigma)] \\ \quad = \frac{\partial^2}{\partial \varsigma^2} z(\phi, \varsigma) + \mathcal{U}(\phi, \varsigma) \\ \quad \quad + g(\phi, \int_{-\infty}^\phi t_1(\sigma - \phi)z(\sigma, \varsigma) d\sigma, \int_0^x \int_{-\infty}^0 \xi_2(r, \varsigma, \varepsilon - r)z(\varepsilon, \varsigma) d\varepsilon dr), \\ \quad r \in [0, \pi], z \in (0, b], \\ \mathcal{I}^{(1-\alpha)\frac{1}{3}} [z(\phi, \varsigma)]|_{\varsigma=0} = z_0(\varsigma), \quad \varsigma \in [0, \pi], \\ z(\phi, 0) = z(\phi, \pi) = 0, \quad \phi \geq 0, \\ z(0, \beta) = \phi(\beta), \quad 0 \leq \beta \leq \pi, \end{cases} \tag{6.1}$$

where $D_{0^+}^{\alpha, \frac{2}{3}}$ denotes the Hilfer fractional derivative of order $\frac{2}{3}$, type $\alpha, g : \mathbb{N} \times [0, 1] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

To convert (6.1) into an abstract form, consider $Z = L^2(0, \pi)$ and $A : D(A) \subset Z \rightarrow Z$, let $\mathcal{J} : D(\mathcal{J}) \subset Z \rightarrow Z$ be defined as $Av = v''$, and $\mathcal{J}v = v - A$, $D(A)$, $D(\mathcal{J})$ are given by

$$\{v \in Z : v, v' \text{ are absolutely continuous, } v(0) = v(\pi) = 0\}.$$

Additionally, A and P are presented as $Av = \sum_{m=1}^{\infty} m^2 \langle v, z_m \rangle z_m$, $v \in D(A)$ and

$$\mathcal{J}v = \sum_{m=1}^{\infty} (1 + m^2) \langle v, z_m \rangle z_m, \quad v \in D(\mathcal{J}),$$

where $z_m(t) = \sqrt{\frac{2}{\pi}} \sin(mt)$, $m = 1, 2, \dots$. Also, for $z \in Z$, we have

$$P^{-1}z = \sum_{m=1}^{\infty} \frac{1}{(1 + m^2)} \langle z, z_m \rangle z_m, \quad A\mathcal{J}^{-1}z = \sum_{m=1}^{\infty} \frac{m^2}{(1 + m^2)} \langle z, z_m \rangle z_m$$

and $Q_\beta(x)z = \sum_{m=1}^{\infty} \exp(\frac{m^2 t}{1+m^2}) \langle z, z_m \rangle z_m$.

Now, from [62] $z(t)z(s) = z(t + s)$ for $z \in Z$, $\widehat{\mu}(T(t)D) \leq \widehat{\mu}(D)$, where $T(t)$ is not compact and $\widehat{\mu}$ is the Hausdorff measure. Moreover, $\phi \rightarrow \nu(\phi^{\frac{2}{3}} + \sigma)$ is equicontinuous for $\phi \geq 0$. Define $f : [0, \pi] \times Z \rightarrow Z$ by

$$e(z, \phi) = \int_{-\infty}^z \xi_2(z, r, \varepsilon) \phi(\varepsilon) d\varepsilon,$$

$$f\left(z, \phi, \int_0^z e(\varepsilon, \phi) d\varepsilon\right) = \zeta\left(\phi, \int_{-\infty}^\phi \zeta_1(\sigma - \phi) u(\sigma, \gamma) d\sigma, \int_0^z e(\varepsilon, \phi)(r) d\varepsilon\right),$$

and $D_{0+}^{\alpha, \frac{2}{3}}(u)(\phi)(\beta) = \frac{\partial^{\frac{2}{3}}}{\partial \beta^{\frac{2}{3}}} u(\phi, \beta)$, $u(\phi)(\zeta) = u(\phi, \zeta)$.

Let $B : U \rightarrow Z$ be defined by $(Bw)(\phi)(\zeta) = \mathcal{U}w(\phi, \zeta)$, $0 < \zeta < 1$. For $\zeta \in (0, \pi)$, W is given by

$$Ww(\zeta) = \int_0^1 (1 - \phi)^{-\frac{1}{3}} P^{-1} Q_\gamma (1 - \phi) \mathcal{U}w(\phi, \zeta) d\phi,$$

where

$$Q_{\frac{2}{3}} = \frac{2}{3} \int_0^\infty \zeta \Upsilon_{\frac{2}{3}}(\zeta) w(x^{\frac{2}{3}} \zeta) d\zeta,$$

and for $\Upsilon \in (0, \infty)$,

$$\Upsilon_{\frac{2}{3}}(\zeta) = \frac{3}{2} \zeta^{-1 - \frac{5}{2}} \overline{w}_{\frac{2}{3}}(\zeta^{\frac{-3}{2}}),$$

$$\overline{w}_{\frac{2}{3}}(\zeta) = \left(\frac{1}{\pi}\right) \sum_{r=1}^{\infty} (-1)^{r-1} x^{-\frac{2}{3}r-1} \left[\frac{\Gamma(\frac{2}{3}r + 1)}{r!}\right] \sin\left(\frac{2r\pi}{3}\right).$$

Here, $\Upsilon_{\frac{2}{3}}$ is defined on $(0, \infty)$, that is,

$$\Upsilon_{\frac{2}{3}}(\zeta) \geq 0, \quad \zeta \in (0, \infty) \quad \text{as well as} \quad \int_0^\infty \Upsilon_{\frac{2}{3}}(\zeta) d\zeta = 1,$$

f and \mathcal{U} fulfills (A₁)–(A₃). We conclude that (6.1) is controllable on \mathbb{N} .

7 Conclusion

In this article, we have fundamentally focused on a class of Sobolev-type Hilfer fractional integro-differential framework with infinite delay, which generalized the Riemann–Liouville fractional derivative. At first, we dealt with the new existence result of a mild solution with the assumptions that the framework satisfies the initial condition and non-compactness measure condition. Later, we have presented the controllability results of the thought about the fractional framework. In the end, we introduced an example to show the procured hypothetical results. We will try to investigate the neutral differential equation and controllability of a similar problem in our future research work.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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