Geophys. J. Int. (1995) **121,** 579–584

A discussion of the uniqueness of a Laplacian potential when given only partial field information on a sphere

F. J. Lowes,[1] A. De Santis[2] and B. Duka[3]

[1] *Physics Department, University of Newcastle, Newcastle upon Tyne* NE1 7RU, *UK*
[2] *Istituto Nazionale di Geofisica, Via di Vigna Murata* 605, 00143 *Roma, Italy*
[3] *Faculty of Natural Sciences, Department of Physics, University of Tirana, Tirana, Albania*

Accepted 1994 November 14. Received 1994 November 3; in original form 1994 August 12

SUMMARY

For a vector field defined by a scalar potential outside a surface enclosing all the sources, it is well known that the potential is defined uniquely if either the potential itself, or its derivative normal to the surface, is known everywhere on the surface. For a spherical surface, the normal derivative is the radial component of the field: the horizontal (vector) component of the field also gives uniqueness (except for any monopole contribution).

This paper discusses the way other partial information of the field on the spherical surface can give a unique, or almost unique, knowledge of the external potential/field, bringing together and correcting previous work. For convenience the results are given in the context of the geomagnetic field $\boldsymbol{B}$. This is often expressed in terms of its local Cartesian components (X, Y, Z), equivalent to $(-B_\theta, B_\phi, -B_r)$; it can also be expressed in terms of Z and the vector horizontal component $\boldsymbol{H} = (X, Y)$. Alternatively, local 'spherical polar' components (F, I, D) are used, where $F = |\boldsymbol{B}|$, the inclination I is the angle in the vertical plane downward from $\boldsymbol{H}$ to $\boldsymbol{B}$, and the declination D is the angle in the horizontal plane eastward from north to $\boldsymbol{H}$.

Knowledge of X over the sphere gives a complete knowledge of the potential, apart from that of any monopole (which is zero in geomagnetism), and Y gives the potential except for any axially symmetric part (which can be provided by a knowledge of X along a meridian, or of $\boldsymbol{H}$ along any path from pole to pole). In terms of (F, I, D) the situation is more complicated; either F or the total angle (I, D) needs to be known throughout a finite volume; for the latter, this paper shows how, in principle, the actual potential can be determined (except for an unknown scaling factor). Similarly D on the sphere also needs a knowledge of $|\boldsymbol{H}|$ on a line from (magnetic) pole to pole.

We also discuss how these various properties affect the determination, by surface integration, of the Gauss coefficients of the field representation in terms of spherical harmonics.

Key words: geomagnetic field, potential theory, spherical harmonic analysis, uniqueness.

1 INTRODUCTION

In a region where a vector field $\boldsymbol{B}$ is curl-free (always true for a gravitational field, true in the absence of local current density for a magnetic field), the field can be specified as the gradient of a scalar potential U. If there is no local mass/pole density, we also have $\operatorname{div} \boldsymbol{B} = 0$, and the potential satisfies the Laplace equation

$$\nabla^2 U = 0. \tag{1}$$

It is well known that eq. (1) has a unique solution outside a closed surface (containing all the sources) on which we everywhere know either (i) the potential U (Dirichlet problem), or (ii) its derivative normal to the surface (Neumann problem); hence in such cases the field $\boldsymbol{B}$ is therefore also determined everywhere outside (and on) the surface. Once the potential, or its radial derivative, is known on the surface, the actual potential at any exterior point can be determined numerically by using the appropriate Green's function to integrate over the surface.

In geomagnetism, what is measured is the vector field $\boldsymbol{B} = -\text{grad}\, U$. It is clear from the Neumann situation that a full knowledge of the field vector over the surface will enable the external potential, and hence field, to be determined uniquely; in fact we only need the field component that is locally normal to the surface. This paper considers other sorts of partial knowledge of $\boldsymbol{B}$ that also give (almost) unique solutions when the surface is spherical, or when the knowledge is extended over a finite shell. In the next section we consider the problem of uniqueness, bringing together previous work, and adding new results of our own. In Section 3 we discuss how the solutions can be found in several cases, when working in terms of spherical harmonics. Our discussion is given in the notation of geomagnetism, but we point out where the situation for a gravitational field is different.

2 UNIQUENESS WHEN ONLY PARTIAL INFORMATION IS AVAILABLE

2.1 Using normal and tangential components of the field

We saw above that a full knowledge of the normal component of the field over the surface gives a unique solution. In fact, for a *spherical* surface, the tangential component $\boldsymbol{H} = B_\theta \hat{\boldsymbol{\theta}} + B_\phi \hat{\boldsymbol{\phi}}$ of the field is also sufficient to give (almost) uniqueness (Schmidt 1889, though in a spheroidal context; Langel 1987); assuming the potential at one point to be U_0, the potential at any other point on the sphere can be determined by adding to U_0 the line integral from that point $U_H = \int \boldsymbol{H} \cdot d\boldsymbol{s}$. We can then put

$$U = U_H + U_0 = (U' + U_m) + U_0 = U' + U_1, \qquad (2)$$

where U_m is the mean over the sphere of U_H, and hence $U_1 = U_m + U_0$ is an arbitrary potential, constant over the sphere. As the problem is linear, we can then apply Dirichlet to specify the external potentials corresponding to U' and U_1 separately. But we know that, for the spherical surface, the constant potential U_1 can only be that of a central monopole (corresponding to total mass, or net magnetic pole strength). As in geomagnetism the monopole moment is zero, we have $U_1 = 0$, so that the U' given by $\boldsymbol{H}$ completely defines the external potential. (For gravitational and other fields the monopole moment, which gives a purely radial field, is not determined.)

The same approach can be used also for a *spheroidal* surface, giving an unknown constant potential over that surface; this constant potential is now that of the zero-order, *spheroidal* harmonic, which is zero in our situation.

2.2 Using geomagnetic (X, Y, Z) components of the field

It is conventional in geomagnetism to express the field $\boldsymbol{B}$ not in terms of spherical polar coordinates (B_r, B_θ, B_ϕ), but in terms of an equivalent local Cartesian coordinate system which puts $\boldsymbol{B} = (X, Y, Z)$, where X, Y, Z are the local northward $(-B_\theta)$, eastward (B_ϕ), and downward radial $(-B_r)$, components. (Throughout this paper we assume that this (X, Y, Z) system is defined for the *spherical* surface.) Note that the (separate) values of the X and Y components depend on the (otherwise essentially arbitrary) choice of the $\theta = 0$ 'polar' axis, and of the $\phi = 0$ meridian.

We have (in effect) already discussed above the case of the Z component, which is normal to the sphere. We also saw that, on the sphere $r = a$, and given zero monopole field, a knowledge of the total horizontal vector component $\boldsymbol{H} = (X, Y)$ is sufficient to give uniqueness; in fact (Schmidt 1889), we only need to know the north–south component $X(a, \theta, \phi)$. Starting from $\theta = 0$, the relative potential anywhere else on the surface can be found by integrating X down the appropriate line of longitude, and then we have the same situation as above.

It is clear that a knowledge only of the east–west component, $Y(a, \theta, \phi)$, of $\boldsymbol{H}$ can tell us nothing about any part of the potential that has axial symmetry about the $\theta = 0$ axis. But, starting with the (unknown) potential distribution along, say, the $\phi = 0$ meridian on a sphere, the relative potential elsewhere can be obtained by integrating along the appropriate circle of latitude (Schmidt 1889). An argument similar to that used above then shows that this is sufficient to define uniquely any non-axisymmetrical part of the potential. As stated by Vestine (1941), if we also know the value of X along this $\phi = 0$ meridian, the potential is then completely defined (except for any monopole); equivalently, a knowledge of $\boldsymbol{H}$ along any path from geographic pole to pole would suffice.

As in Section 2.1 these results also apply to the corresponding components on a spheroidal surface.

That the Y component of the horizontal (vector) $\boldsymbol{H}$ contains less information than the X component follows from the geometry of its definition; while knowing X allows the equivalent of $\int \boldsymbol{H} \cdot d\boldsymbol{s}$ to be calculated between one point (a pole) and any other point, this is not possible with Y.

2.3 Using geomagnetic (F, I, D) components of the field

Instead of expressing $\boldsymbol{B}$ in terms of components in a local Cartesian coordinate system, we can use a local spherical polar system, with its polar axis vertically downward, to give $\boldsymbol{B} = (F, I, D)$, where F is the (radial) length of the vector $\boldsymbol{B}$, and the inclination I and declination D correspond to the 'latitude' and 'longitude' of the direction of the vector $\boldsymbol{B}$ in the local coordinate system. (I is the angle in the vertical plane downward from $\boldsymbol{H}$ to $\boldsymbol{B}$, and D is the angle in the horizontal plane eastward from north to $\boldsymbol{H}$.) We can then ask similar questions as to whether partial specification of $\boldsymbol{B}$, in terms of one or more of F, I, D, on a closed surface, is sufficient to give uniqueness.

For a complete knowledge of F on an arbitrary closed surface, Backus (1968, 1970) showed that, provided there is a finite monopole moment, the potential is uniquely defined (except as to sign if the sign of the monopole is unknown) on and outside the surface. If, as in geomagnetism, there is *no* monopole moment, then, provided U is known to have only a *finite* number of spherical harmonics, a knowledge of F on a *spherical* surface again determines U uniquely, except as to sign, on and outside the sphere. If U consists of an *infinite* number of harmonics, however, the position is unclear; although Backus (1970) produced a counter-example that there are an infinite number of pairs of solutions having the same F on a spherical surface, this does *not* rule out the possibility that there are *some* magnetic fields that *are* determined uniquely by knowing F on a sphere. (In practice, if only F values are used in a numerical approximation to the geomagnetic field, then large 'perpendicular errors' (Lowes 1975) are produced, analogous to the Backus counter-example. It is not known, however, if this would happen with perfect data over the sphere.)

If knowledge of F is expanded to cover a finite volume, however, Backus (1968 for the volume outside a sphere, 1970 for spherical shell (though without a formal proof), and 1974 for arbitrary finite volume) showed that the solution *is* unique (except as to sign) everywhere outside the source region.

Even if there is a unique solution, there appears to be no formal method of producing an exact solution (except for the analytical continuation involved in Backus' (1974) proof).

Kono (1976) attempted to prove that a knowledge of the full direction (I, D) on the sphere would give uniqueness. If a field $\boldsymbol{B}(r, \theta, \phi)$ were to exist which fitted the observed (I, D) on $r = a$, then so would the field $k(a, \theta, \phi)\,\boldsymbol{B}(a, \theta, \phi)$, where k is a scalar. Kono claimed to have shown that k must in fact be constant both on and outside the sphere, and that therefore $\boldsymbol{B}$ was unique. However his 'proof' assumed that on the sphere the contours of k were *not* perpendicular to $\boldsymbol{B}$, but Gubbins (1986) showed this was *not* true (although Gubbins did not relate this to Kono's work). Later, Proctor & Gubbins (1990) showed that there was in fact no restriction on how k behaved *away* from the surface, and also produced a counter-example.

Kono (1976) also gave counter-examples to show that knowledge of only X/Z (analogous to I) or X/Y (analogous to D) on the sphere is not necessarily sufficient to give uniqueness. For I itself he showed that if two independent potentials give the same values of I on the sphere, then at least one of them must contain an infinite number of spherical harmonics; this is analogous to the result of Backus (1968) for F.

Gubbins (1986) showed that if D is known everywhere on a sphere, and a field $\boldsymbol{B}$ has been found that matches it there, then (for an Earth-like field) a knowledge of the horizontal intensity along a line joining the dip-poles defines the field uniquely; this is analogous to the way a geographic pole-to-pole knowledge of $\boldsymbol{H}$ removes the ambiguity if only the rectangular component Y is known.

If the full direction (I, D), or its equivalent, is known throughout a finite volume, Bloxham (1985), reported by Proctor & Gubbins (1990), proved that the field is then uniquely defined everywhere, up to a multiplicative constant. (They gave their proof for a spherical annulus, but the restriction is not necessary.)

We now show how the field can be determined analytically, at least in principle, in this situation. Let $\boldsymbol{\tau}(\boldsymbol{r})$ be the unit vector in the direction of the field, and $F(\boldsymbol{r})$ the (*unknown*) corresponding field magnitude. We then have

$$\operatorname{div}\boldsymbol{B} = \operatorname{div}(F\boldsymbol{\tau}) = (\operatorname{grad} F)\cdot\boldsymbol{\tau} + F\operatorname{div}\boldsymbol{\tau} = 0, \tag{3}$$

and

$$\operatorname{curl}\boldsymbol{B} = \operatorname{curl}(F\boldsymbol{\tau}) = (\operatorname{grad} F)\times\boldsymbol{\tau} + F\operatorname{curl}\boldsymbol{\tau} = 0. \tag{4}$$

Putting

$$\boldsymbol{A} = (\operatorname{grad} F)/F, \tag{5}$$

we then have

$$\boldsymbol{A}\cdot\boldsymbol{\tau} = -\operatorname{div}\boldsymbol{\tau} \quad \text{and} \quad \boldsymbol{A}\times\boldsymbol{\tau} = -\operatorname{curl}\boldsymbol{\tau}. \tag{6}$$

As $\boldsymbol{\tau}$ is specified everywhere in the annulus, then from (6) so also is the vector $\boldsymbol{A} = (\operatorname{grad} F)/F$. (Expressions for $\boldsymbol{A}$ in terms of its components are given in the Appendix.) Now,

$$(\operatorname{grad} F)/F = \operatorname{grad}(\ln F), \tag{7}$$

so, if $\ln F$ is assumed to have the value $\ln F_0$ at some point $\boldsymbol{r}_0$ in the annulus, the value of $\ln F$ at any other point $\boldsymbol{r}_p$ in the annulus can be found by the line integral

$$\ln F_p = \ln F_0 + \int_{r_0}^{r_p} \boldsymbol{A}\cdot d\boldsymbol{s}, \tag{8}$$

giving

$$F_p = F_0 \exp\left[\int \boldsymbol{A}\cdot d\boldsymbol{s}\right]. \tag{9}$$

Combining this field magnitude F with the known direction $\boldsymbol{\tau}$, we now know the vector field $\boldsymbol{B}$ throughout the annulus. We therefore know it, and hence its normal derivative, on some closed surface, so we have the Neuman problem again (except for an arbitrary scale factor).

In this derivation the finite volume throughout which the direction is known does not have to be a spherical shell, and, although the derivation has been expressed in terms of $\boldsymbol{B}$, there is no assumption about the monopole moment being zero.

3 SPHERICAL HARMONIC APPROACH

3.1 Potential and vector field

Once the potential, or its radial derivative, is known on the sphere, then the potential at any exterior point can be determined by using the appropriate Green's function to integrate over the sphere. A much simpler approach is possible, however, if we restrict the surface to be a sphere of radius a, and express the potential U as the sum of spherical harmonic terms of the form

$$U = \sum_n \sum_m a(a/r)^{(n+1)}(g_n^m \cos m\phi + h_n^m \sin m\phi) \times \mathrm{P}_n^m(\cos\theta), \tag{10}$$

where in geomagnetism the g_n^m and h_n^m are the numerical Gauss coefficients. We can write this more compactly as

$$U = \sum\sum g_n^{mb} U_n^{mb}. \tag{11}$$

where the superscript b denotes either cos or sin (for example g_n^{mc} stands for g_n^m, and g_n^{ms} stands for h_n^m).

In geomagnetism what is measured is the vector field $\boldsymbol{B} = -\text{grad}\, U$, and we can write

$$\boldsymbol{B} = \sum \sum g_n^{mb} \boldsymbol{B}_n^{mb}, \tag{12}$$

where

$$\boldsymbol{B}_n^{mb} = -\text{grad}\, U_n^{mb}.$$

On the sphere the U_n^{mb} are orthogonal, so the g_n^{mb} can be determined directly by integration over the sphere. For the Dirichlet problem, if U is the observed potential, we have

$$g_n^{mb} = \int U U_n^{mb}\, dS \bigg/ \int (U_n^{mb})^2\, dS$$

$$= (2n+1) \int U U_n^{mb}\, dS / 4\pi a^4 \tag{13}$$

for the Schmidt semi-normalized associated Legendre polynomials.

The $\boldsymbol{B}_n^{mb}$ are also orthogonal on the sphere (Lowes 1966), so if $\boldsymbol{B}$ is the observed field we have

$$g_n^{mb} = \int \boldsymbol{B} \cdot \boldsymbol{B}_n^{mb}\, dS \bigg/ \int (\boldsymbol{B}_n^{mb})^2\, dS$$

$$= \int \boldsymbol{B} \cdot \boldsymbol{B}_n^{mb}\, dS / (n+1) 4\pi a^2. \tag{14}$$

3.2 Using (X, Y, Z) components

If we put $\boldsymbol{B} = (X, Y, Z)$, where X, Y, Z are the conventional geomagnetic field components defined in Section 2.2, then we can write

$$\boldsymbol{B} = \sum \sum g_n^{mb} \boldsymbol{B}_n^{mb} = \sum \sum g_n^{mb} (X_n^{mb}, Y_n^{mb}, Z_n^{mb}), \tag{15}$$

where each term is derived from the appropriate potential:

$$X_n^{mc} = (a/r)^{n+2} \cos m\phi\, d\mathrm{P}_n^m(\cos\theta)/d\theta,$$

$$Y_n^{mc} = m(a/r)^{n+2} \sin m\phi \mathrm{P}_n^m(\cos\theta)/\sin\theta, \tag{16}$$

$$Z_n^{mc} = -(n+1)(a/r)^{n+2} \cos m\phi \mathrm{P}_n^m(\cos\theta),$$

and similarly for the sine terms (except for the change of sign in Y_n^{ms}).

We have already seen that a complete knowledge of the radial component $Z(a, \theta, \phi)$ is sufficient; using the results of Lowes (1966) we have, for the Neumann problem,

$$g_n^{mb} = \int Z Z_n^{mb}\, dS \bigg/ \int (Z_n^{mb})^2\, dS$$

$$= (2n+1) \int Z Z_n^{mb}\, dS / 4\pi a^2 (n+1)^2. \tag{17}$$

Just as the $\boldsymbol{B}_n^{mb}$ and the Z_n^{mb} $(= \boldsymbol{B}_n^{mb} \cdot \boldsymbol{n})$ are each orthogonal, so also is their vector difference, the corresponding horizontal component

$$\boldsymbol{H}_n^{mb} = (X_n^{mb}, Y_n^{mb}).$$

We therefore have (Lowes 1966)

$$g_n^{mb} = \int \boldsymbol{H} \cdot \boldsymbol{H}_n^{mb}\, dS \bigg/ \int (\boldsymbol{H}_n^{mb})^2\, dS$$

$$= (2n+1) \int \boldsymbol{H} \cdot \boldsymbol{H}_n^{mb}\, dS / 4\pi a^2 n(n+1), \tag{18}$$

so, as expected from Section 2.1 above, we can obtain all the Gauss coefficients given *either* the vertical (scalar) component, *or* the horizontal (vector) component on a sphere.

But the case of X_n^{mb} and Y_n^{mb} separately is more complicated. As was shown by Lucke (1957) these are separately *not* orthogonal, so the sort of surface integration used above cannot be used directly to determine individual coefficients; Langel (1987, p. 347) is wrong in implying that this can be done. (Of course the line-integral approach of Section 2.2 can be used to give the distribution of the potential on the surface, and then (13) can be used.)

There is also the complication that, because of their definitions, X and Y are discontinuous at the two geographic poles (although of course the field they represent is continuous). To avoid this latter difficulty, Schmidt (1889, expanded in 1895) introduced the use of $(X \sin\theta)$ and $(Y \sin\theta)$; in fact the $(Y_n^m \sin\theta)$ are orthogonal over the sphere, although the $(X_n^m \sin\theta)$ are not completely orthogonal.

Because of the orthogonality of the $(Y_n^m \sin\theta)$ we have

$$g_n^m = \int (y \sin\theta)(Y_n^{ms} \sin\theta)\, dS \bigg/ \int (Y_n^{ms} \sin\theta)^2\, dS$$

$$= (2n+1) \int (Y \sin\theta)(Y_n^{ms} \sin\theta)\, dS / 4\pi a^2 m^2, \tag{19}$$

and similarly, except for change of sign, for h_n^m. Of course no information can be obtained about the g_n^0. Presumably, using $Y \sin\theta$ is equivalent to performing the line integral $\int Y\, ds = \int Y \sin\theta\, a\, d\phi$ of Section 2.2.

Because of the lack of orthogonality for the $(X_n^m \sin\theta)$ the algebra is more complicated. It is a standard result that

$$\sin\theta\, d\mathrm{P}_n^m/d\theta = a_n^m \mathrm{P}_{n-1}^m - b_n^m \mathrm{P}_{n+1}^m, \tag{20}$$

where the a_n^m and b_n^m are known factors (with $b_n^m = 0$ for $n < m$). Therefore when $(X \sin\theta)$ is analysed, using the equivalent of (10), to give

$$(X \sin\theta) = \sum \sum (a/r)^{(n+2)} (e_n^m \cos m\phi + f_n^m \sin m\phi) \times \mathrm{P}_n^m(\cos\theta), \tag{21}$$

then we find that in general each e_n^m has contributions from both g_{n-1}^m and g_{n+1}^m. Each g_{m+1}^m starts a sequence of equations in which e_m^m occurs only in the first equation, so this e_m^m gives the corresponding g_{m+1}^m directly, and then g_{m+3}^m, g_{m+5}^m, etc. can be solved for recursively. However, each g_m^m starts a sequence of equations in which the first equation has only e_m^m, which also occurs in the next equation, so this set of equations *cannot* be solved; this was explained by Kawasaki, Matsushita & Cain (1989). Schmidt (1889) attempted to get round this problem by *assuming* that the harmonic expansion was truncated at a known point, so that the *last* equation of the sequence had an e_n^m which occurred only once, so that the equations could be solved, at least formally, but this seems a *very* dubious procedure (although perhaps analogous to the uniqueness given by I or by F when the field consists only of a finite number of harmonics). Certainly, the phrase 'By a comparison of corresponding terms the coefficients g_n^m can be expressed by a combination of coefficients $e_m^m \cdots$' used by Chapman & Bartels (1940, p. 637) does not seem justified. It is somewhat surprising that while X, unlike some other components of $\boldsymbol{B}$, imposes a unique solution for the full potential, and hence in principle for all the harmonic coefficients g_n^{mb}, there does not seem to be a way to determine more than about half of them.

4 DISCUSSION

This paper has considered how (parts of) the external field/potential are determined uniquely, and how they can be determined analytically, when given various sorts of partial, but exact, information about the field over a spherical surface enclosing the sources.

The potential U itself, and the vector field $\boldsymbol{B}$, or its radial component $Z = -B_r$, all give complete uniqueness, as does the vector horizontal component $\boldsymbol{H}$ (except for a possible monopole field, which is zero in geomagnetism); because of orthogonality, the corresponding spherical harmonic coefficients can be determined analytically.

Knowing the north–south horizontal component, $X = -B_\theta$, over the sphere determines the field uniquely (for zero monopole field). The east–west, $Y = B_\phi$, component determines the field except for its axially symmetric part, and this can be added either by a knowledge of X along a meridian, or of $\boldsymbol{H}$ along any line joining the poles. While the corresponding spherical harmonic coefficients can be determined analytically for the case of Y, in the case of X only about half can be determined, unless the potential is known to consist only of a finite number of harmonics.

Alternatively, $\boldsymbol{B}$ can be expressed in terms of its intensity and direction in the form of (F, I, D). A knowledge of F on the sphere probably does not give uniqueness unless there *is* a monopole (as for gravitational fields), or it is known that the potential consists only of a finite number of spherical harmonics. At best, the total angular information (I, D) by itself could determine the field only to a scale factor in magnitude, but uniqueness has not been proved; surface I by itself gives uniqueness only if there are only a finite number of harmonics, and surface D needs also a knowledge of $\boldsymbol{H}$ along a line from dip-pole to dip-pole.

The intensity F, or total angle (I, D), gives uniqueness if known throughout a finite volume, and this paper shows how the solution can be found, at least in principle, for the latter case.

The above situations all involve exact knowledge over the whole sphere, and in principle will give *exact* results (except for any truncation of a series solution when using a spherical harmonic approach). Of course in practice we only know (components of) the field at a finite number of discrete points on the sphere, and the Gauss coefficients of a series solution for the potential are only *estimated,* usually by some sort of least squares' processs. The way the (lack of) orthogonality affects this process in the various situations is discussed by DeSantis, Falcone & Lowes (1995).

ACKNOWLEDGMENTS

We would like to thank C. Falcone for her stimulating discussion, and D. R. Barraclough and an anonymous referee for helpful comments.

REFERENCES

Backus, G.E., 1968. Applications of a non-linear boundary value problem for Laplace's equation to gravity and geomagnetic intensity surveys, *Q. J. Mech. appl. Math.*, **21,** 195–221.

Backus, G.E., 1970. Non-uniqueness of the external geomagnetic field determined by surface intensity measurements, *J. geophys. Res.*, **75,** 6339–6341.

Backus, G.E., 1974. Determination of the external geomagnetic field from intensity measurements, *Geophys. Res. Lett.*, **1,** 21.

Bloxham, J., 1985. Geomagnetic secular variation, *PhD Thesis*, Cambridge University, UK.

Chapman, S. & Bartels, J., 1940. *Geomagnetism*, Vol II, Clarendon Press, Oxford.

De Santis, A., Falcone, C. & Lowes, F.J., 1995. Remarks on the mean-square values of the geomagnetic field and its components, *Annali di Geofisica*, submitted.

Gubbins, D., 1986. Global models of the magnetic field in historical times: augmenting declination observations with archeo- and paleo-magnetic data, *J. Geomagn. Geoelect.*, **38,** 715–720.

Kawasaki, K., Matsushita, S. & Cain, J.C., 1989. Least squares and integral methods for the spherical harmonic analysis of the Sq-field, *Pure appl. Geophys.*, **131,** 357–370.

Kono, M., 1976. Uniqueness problems in the spherical harmonic analysis of the geomagnetic field direction data, *J. Geomagn. Geoelect.*, **28,** 11–29.

Langel, R.A., 1987. The main field, in *Geomagnetism*, Vol. 1, pp 249–512, ed. Jacobs, J.A., Academic Press, London.

Lowes, F.J., 1966. Mean-square values on sphere of spherical harmonic vector fields, *J. geophys. Res.*, **71,** 2179.

Lowes, F.J., 1975. Vector errors in spherical harmonic analysis of scalar data, *Geophys. J. R. astr. Soc.*, **42,** 637–651.

Lucke, O., 1957. Über Mittelwerte von Energiedichten der Kraftfelder, in *Wiss. Z. der Päd. Hochschule Potsdam*, Jahrgang 3, Heft 1, 39–467.

Proctor, M.R.E. & Gubbins, D., 1990. Analysis of geomagnetic directional data, *Geophys. J. Int.*, **100,** 69–77.

Schmidt, A., 1889. Mathematische Entwicklungen zur allgemeinen Theorie des Erdmagnetismus, in *Archiv det Deutschen Seewarte*, XII annual edition, no. 3, 1–29, Hamburg.

Schmidt, A., 1895. Mitteilungen über eine neue Berechnung des Erdmagnetischen Potentials, *Abhandl. Bayer. Akad. Wiss.*, **19,** 1–66.

Vestine, E.H., 1941. On the analysis of surface magnetic fields by integrals, Part I, *Terr. Magn. Atm. Electr.*, **46,** 27–41.

APPENDIX A: THE DETERMINATION OF (grad F)/F

For $\boldsymbol{A} = (\text{grad } F)/F$, we found above that

$$\boldsymbol{A} \cdot \boldsymbol{\tau} = -\text{div } \boldsymbol{\tau} \quad \text{and} \quad \boldsymbol{A} \times \boldsymbol{\tau} = -\text{curl } \boldsymbol{\tau}. \tag{6}$$

Writing these vector equations in terms of scalar component equations we have

$$A_r\tau_r + A_\theta\tau_\theta + A_\phi\tau_\phi + \frac{1}{r^2}\frac{\partial(r^2\tau_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\tau_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_\phi}{\partial_\phi} = 0,$$

$$A_\theta\tau_\phi - A_\phi\tau_\theta = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\,\tau_\phi)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial\tau_\theta}{\partial\phi}, \tag{A1}$$

$$A_\phi\tau_r - A_r\tau_\phi = \frac{1}{r\sin\theta}\frac{\partial\tau_r}{\partial\phi} - \frac{1}{r}\frac{\partial(r\tau_\phi)}{\partial r},$$

$$A_r\tau_\theta - A_\theta\tau_r = \frac{1}{r}\frac{\partial(r\tau_\theta)}{\partial r} - \frac{1}{r}\frac{\partial\tau_r}{\partial\theta}.$$

Although there appear to be four equations for the three components A_r, A_θ, A_ϕ, only three are independent, as the determinant of the last three equations is zero. (Physically, $\boldsymbol{A}\times\boldsymbol{\tau}$ gives no information about the projection of $\boldsymbol{A}$ along the direction of $\boldsymbol{\tau}$.)

Solving algebraically, we find

$$A_r = -\tau_r\,\mathrm{div}\,\boldsymbol{\tau} - \frac{\tau_\theta}{r}\frac{\partial\tau_r}{\partial\theta} + \frac{\tau_\theta}{r}\frac{\partial(r\tau_\theta)}{\partial r} - \frac{\tau_\phi}{r\sin\theta}\frac{\partial\tau_r}{\partial\phi} + \frac{\tau_\phi}{r}\frac{\partial(r\tau_\phi)}{\partial r} = f_r(r,\theta,\phi), \tag{A2}$$

and similarly for $A_\theta = f_\theta(r,\theta,\phi)$ and $A_\phi = f_\phi(r,\theta,\phi)$. We then have

$$[(\mathrm{grad}\,F)/F]_r = f_r(r,\theta,\phi),$$
$$[(\mathrm{grad}\,F)/F]_\theta = f_\theta(r,\theta,\phi), \tag{A3}$$
$$[(\mathrm{grad}\,F)/F]_\phi = f_\phi(r,\theta,\phi),$$

and the line integral $\int[(\mathrm{grad}\,F)/F]\cdot d\boldsymbol{s}$ can then be done.