Afshari and Karapınar *Advances in Difference Equations* (2020) 2020:616
https://doi.org/10.1186/s13662-020-03076-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces

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Abstract

In this paper, we investigate the existence of positive solutions for the new class of boundary value problems via ψ-Hilfer fractional differential equations. For our purpose, we use the $\alpha - \psi$ Geraghty-type contraction in the framework of the b-metric space. We give an example illustrating the validity of the proved results.

Keywords: ψ-Hilfer fractional derivative; Fixed point; Positive solution; Geraghty-type contraction

1 Introduction

One of the critical techniques of the solving differential equations is using the method of successive approximations, which is the basic of the metric fixed point theory. More precisely, Banach's contraction mapping principle, the first metric fixed point theorem, is obtained by the abstraction of the method of successive approximations. Roughly speaking, starting from the arbitrary initial point, we construct a sequence by recursively applying the given operator. Then, if the obtained sequence converges to a limit, this limit forms a fixed point and solution of the differential equation.

The pioneer result of metric fixed point theory was given by Banach in the framework of complete norm spaces. After then, the praiseworthy fixed point theorem of Banach has been characterized in different structures, such as standard metric spaces, partial metric spaces, quasimetric spaces, fuzzy-metric spaces, modular metric spaces, and b-metric spaces. In this paper, we consider our results in a b-metric space, which is a natural and novel extension of the standard metric spaces. Roughly speaking, the difference of b-metric from the standard metric is the triangle inequality. In the b-metric notion, instead of the triangle inequality, the following inequality is used:

$$d(v,z) \leq c\big[d(v,t) + d(t,z)\big] \quad \text{for all } v,t,z \text{ and some } c \geq 1.$$

In the last few decades, the natural extension of differential equations, fractional differential equations, have been investigated densely in the setting of the standard metric spaces. As it is well known, there are several distinct fractional derivative types, such as Caputo, Hadamard, Grunwald–Letnikov, Hilfer, Riemann–Liouville, Riesz, Atangana–Baleanu, and so on. Among these different types of fractional derivatives, we focus on the Hilfer fractional derivative; see, for example, [1–28]. By using this definition we will investigate the existence of positive solutions for certain boundary value problems in the context of b-metric spaces.

2 Preliminaries

In this section, we recall some notations and definitions of the fractional differential equation. Throughout this paper, we assume that all considered sets are nonempty and denote $\mathbb{R}^+ = [0,\infty)$.

Let $[a,T] \subset \mathbb{R}^+$ with $(0 < a < T < \infty)$, and let $C[a,T]$ be the Banach space of continuous functions $y : [a,T] \to \mathbb{R}$ with the norm

$$\|y\|_{C[a,T]} = \max\{|y(t)| : a \leq t \leq T\}.$$

The weighted space $C_{1-\xi;\delta}[a,T]$ of continuous functions is defined as [22]

$$C_{1-\xi;\delta}[a,T] = \{y : (a,T] \to \mathbb{R}; [\delta(t) - \delta(a)]^{1-\xi} y(t) \in C[a,T]\}, \quad 0 \leq \xi < 1.$$

Obviously, $C_{1-\xi;\delta}[a,T]$ is a Banach space endowed with the norm

$$\|y\|_{c_{1-\xi;\delta}} = \max_{t\in[a,T]} \left|[\delta(t) - \delta(a)]^{1-\xi} y(t)\right|.$$

Definition 2.1 ([22]) Let $\iota > 0$, $y \in L_1[a,b]$, and let $\delta \in C^1[a,b]$ be an increasing function with $\delta'(t) \neq 0$ for all $t \in [a,b]$. Then the left-sided δ-Riemann–Liouville fractional integral of a function y is defined by

$$I_{a^+}^{\iota,\delta} y(t) = \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} y(s)\, ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\iota) = \int_0^\infty s^{\iota-1} e^{-s}\, ds$, $\iota > 0$.

Definition 2.2 ([11]) Let $n - 1 < \iota < n$ $(n = [\iota] + 1)$, and let $y, \delta \in C^n[a,b]$ be two functions with an increasing δ and $\delta'(t) \neq 0$ for all $t \in [a,b]$. Then the left-sided δ-Riemann–Liouville fractional (δ-Caputo) derivative of a function y of order ι is defined by

$$D_{a^+}^{\iota,\delta} y(t) = \left(\frac{1}{\delta'(t)} \frac{d}{dt}\right)^n I_{a^+}^{n-\iota,\delta} y(t)$$

and

$${}^C D_{a^+}^{\iota,\delta} y(t) = I_{a^+}^{n-\iota,\delta} \left(\frac{1}{\delta'(t)} \frac{d}{dt}\right)^n y(t),$$

respectively.

Definition 2.3 ([22]) Let $n-1<\iota<n$ ($n\in\mathbb{N}$), and let $y,\delta\in C^n[a,T]$ be two functions such that δ is increasing and $\delta'(t)\neq 0$ for all $t\in[a,T]$. Then the left-sided δ-Hilfer fractional derivative of a function y of order ι and type $0\le\beta\le 1$ is defined by

$$\begin{aligned} D_{a^+}^{\iota,\beta,\delta}y(t) &= I_{a^+}^{\beta(n-\iota);\delta}\left(\frac{1}{\delta'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\iota);\delta}y(t) \\ &= I_{a^+}^{\beta(n-\iota);\delta}D_{a^+}^{\xi;\delta}y(t) \quad (\xi=\iota+n\beta-\iota\beta). \end{aligned} \tag{1}$$

In this paper, we consider the case $n=1$, because $0<\iota<1$.

Lemma 2.4 ([17]) *Let $\iota>0$ and $0\le\xi<1$. Then $I_{a^+}^{\iota,\delta}$ is bounded from $C_{1-\xi;\delta}[a,b]$ into $C_{1-\xi;\delta}[a,b]$.*

Now we introduce the spaces

$$C_{1-\xi;\delta}^{\iota,\beta}[a,T]=\left\{y\in C_{1-\xi;\delta}[a,T], D_{a^+}^{\iota,\beta;\delta}y\in C_{1-\xi;\delta}[a,T]\right\},\quad 0\le\xi<1,$$

and

$$C_{1-\xi;\delta}^{\xi}[a,T]=\left\{y\in C_{1-\xi;\delta}[a,T], D_{a^+}^{\xi;\delta}y\in C_{1-\xi;\delta}[a,T]\right\},\quad 0\le\xi<1. \tag{2}$$

Lemma 2.5 ([22]) *Let $\xi=\iota+\beta-\iota\beta$, where $\iota\in(0,1)$, $\beta\in[0,1]$, and let $y\in C_{1-\xi;\delta}^{\xi}[a,T]$. Then*

$$I_{a^+}^{\xi;\delta}D_{a^+}^{\xi;\delta}y=I_{a^+}^{\iota;\delta}D_{a^+}^{\iota,\beta;\delta}y$$

and

$$D_{a^+}^{\xi;\delta}I_{a^+}^{\iota;\delta}y=D_{a^+}^{\beta(1-\iota);\delta}y.$$

Lemma 2.6 ([22]) *Let $\iota>0$, $0\le\xi<1$, and $y\in C_{1-\xi}[a,T]$, $\beta\in[0,1]$. Then*

$$D_{a^+}^{\iota,\beta,\delta}I_{a^+}^{\iota,\delta}y(t)=y(t).$$

Lemma 2.7 ([17]) *Let $t>a$. Then for $\iota\ge 0$ and $\xi>0$, we have*

$$I_{a^+}^{\iota,\delta}\left[\delta(t)-\delta(a)\right]^{\xi-1}=\frac{\Gamma(\xi)}{\Gamma(\iota+\xi)}\left(\delta(t)-\delta(a)\right)^{\iota+\xi-1},\quad t>a$$

and

$$D_{a^+}^{\iota,\delta}\left[\delta(t)-\delta(a)\right]^{\iota-1}=0\quad \text{for } \iota\in(0,1).$$

Lemma 2.8 ([22]) *Let $\xi=\iota+\beta-\iota\beta$, where $\iota\in(0,1)$, $\beta\in[0,1]$, let $y\in C_{1-\xi;\delta}^{\xi}[a,T]$, and let $I_{a^+}^{1-\xi;\delta}y\in C_{1-\xi,\delta}^{1}[a,T]$. Then we have*

$$I_{a^+}^{\xi;\delta}D_{a^+}^{\xi,\delta}y(t)=y(t)-\frac{I_{a^+}^{1-\xi;\delta}y(a)}{\Gamma(\xi)}\left(\delta(t)-\delta(a)\right)^{\xi-1}.$$

Lemma 2.9 ([22]) *Let* $\iota > 0$, $0 \leq \xi < \iota$, *and* $y \in C_{1-\xi,\delta}[a,T]$ $(0 < a < T < \infty)$. *If* $\xi < \iota$, *then* $I_{a^+}^{\iota;\delta} : C_{1-\xi,\delta}[a,T] \to C_{1-\xi,\delta}[a,T]$ *is continuous on* $[a,T]$ *and satisfies*

$$I_{a^+}^{\iota;\delta} y(a) = \lim_{t \to a^+} I_{a^+}^{\iota;\delta} y(t) = 0.$$

Definition 2.10 ([18]) Let $\iota > 0$, and let κ be an increasing function having a continuous derivative κ' on (a,b). The left-sided κ-Riemann–Liouville fractional integral of a function h with respect to κ on $[a,b]$ is defined by

$$I_{a^+}^{\iota,\kappa} h(\varrho) = \frac{1}{\Gamma(\iota)} \int_a^{\varrho} \kappa'(\varsigma)\big[\kappa(\varrho) - \kappa(\varsigma)\big]^{\iota-1} h(\varsigma)\, d\varsigma, \quad \varrho > a, \iota > 0,$$

provided that $I_{a^+}^{\iota,\kappa}$ exists. Note that when $\kappa(\varrho) = \varrho$, we obtain the well-known classical Riemann–Liouville fractional integral.

Definition 2.11 ([18, 21]) Let $\iota > 0$, let n be the smallest integer greater than or equal to ι, and let $h \in L^p[a,b]$, $p \geq 1$. Let $\kappa \in C^n[a,b]$ be an increasing function such that $\kappa'(\varrho) \neq 0$ for all $\varrho \in [a,b]$. The left-sided κ-Riemann–Liouville fractional differential of h of order ι is given by

$$D_{a^+}^{\iota;\kappa} h(\varrho) = \left(\frac{1}{\kappa'(\varrho)} \frac{d}{d\varrho}\right)^n I_{a^+}^{n-\iota,\kappa} h(\varrho), \quad n-1 < \iota < n, n \in \mathbb{N}.$$

Definition 2.12 ([9, 11]) Let $n-1 < \iota < n$, $h \in C^n[a,b]$, and let $\kappa \in C^n[a,b]$ be an increasing function such that $\kappa'(\varrho) \neq 0$ for all $\varrho \in [a,b]$. The left-sided κ-Caputo fractional differential of h of order ι is given by

$${}^C D_{a^+}^{\iota;\kappa} h(\varrho) = I_{a^+}^{n-\iota,\kappa} D^{n,\kappa} h(\varrho),$$

where $D^{n,\kappa} := (\frac{1}{\kappa'(\varrho)} \frac{d}{d\varrho})^n$, and $n = [\iota] + 1$.

Definition 2.13 ([12]) Let $c \geq 1$, and let M be a set. The distance function $d : M \times M \to \mathbb{R}^+$ is called b-metric if for all $\varrho, \varsigma, \zeta \in M$, the following are fulfilled:

(bM_1) $d(\varrho,\varsigma) = 0$ if and only if $\varsigma = \varrho$;

(bM_2) $d(\varrho,\varsigma) = d(\varsigma,\varrho)$;

(bM_3) $d(\varrho,\zeta) \leq c[d(\varrho,\varsigma) + d(\varsigma,\zeta)]$.

The triple (M,d,c) is called a b-metric space.

Let Φ be the set of all increasing and continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the property $\phi(c\varrho) \leq c\phi(\varrho) \leq c\varrho$ for $c > 1$ and $\phi(0) = 0$. We denote by $\mathcal{F}$ the family of all nondecreasing functions $\lambda : \mathbb{R}^+ \to [0, \frac{1}{r^2})$ for some $r \geq 1$.

Definition 2.14 ([7]) For b-metric space (M,d,r), an operator $T : M \to M$ is called a generalized α–δ-Geraghty mapping whenever there exists $\alpha : M \times M \to \mathbb{R}^+$ such that

$$\alpha(\varrho,\varsigma)\phi\big(r^3 d(T\varrho, T\varsigma)\big) \leq \lambda\big(\phi\big(d(\varrho,\varsigma)\big)\big)\phi\big(d(\varrho,\varsigma)\big)$$

for $\varrho, \varsigma \in M$, where $\lambda \in \mathcal{F}$ and $\phi \in \Phi$.

Definition 2.15 ([13]) For M $(\neq \emptyset)$, let $T : M \to M$ and $\alpha : M \times M \to \mathbb{R}^+$ be given mappings. We say that T is orbital α-admissible if for $\varrho \in M$, we have

$$\alpha(\varrho, T\varrho) \geq 1 \quad \Longrightarrow \quad \alpha\bigl(T\varrho, T^2\varrho\bigr) \geq 1. \tag{3}$$

Theorem 2.16 ([7]) *Let (M,d) be a complete b-metric space, and let $T : M \to M$ be a generalized α–δ-Geraghty mapping such that*

(i) *T is α-admissible;*
(ii) *there exists $\varrho_0 \in M$ such that $\alpha(\varrho_0, T\varrho_0) \geq 1$;*
(iii) *If $\{\varrho_n\} \subseteq M$ with $\varrho_n \to \varrho$ and $\alpha(\varrho_n, \varrho_{n+1}) \geq 1$, then $\alpha(\varrho_n, \varrho) \geq 1$.*

Then T has a fixed point.

Theorem 2.17 ([10]) *Let $\xi = \iota + \beta - \iota\beta$, where $\iota \in (0,1)$ and $\beta \in [0,1]$. If $f : (a,T] \to \mathbb{R}$ is a function such that $f \in C_{1-\xi,\delta}[a,T]$, then $y \in C^{\xi}_{1-\xi,\delta}(a,T]$ satisfies the problem*

$$\begin{aligned} &{}^{H}D^{\iota,\beta;\delta}_{a^+} y(t) = f\bigl(t, y(t)\bigr), \quad t \in (a,T], a > 0, \\ &y(T) = w \in \mathbb{R}, \end{aligned} \tag{4}$$

if and only if y satisfies the integral equation

$$\begin{aligned} Af(t) := y(t) = {}& \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left[w - \frac{1}{\Gamma(\iota)} \int_a^T \delta'(s)\bigl(\delta(T) - \delta(s)\bigr)^{\iota-1} f(s, y(s))\, ds \right] \\ &+ \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\bigl(\delta(t) - \delta(s)\bigr)^{\iota-1} f(s)\, ds. \end{aligned} \tag{5}$$

3 Main results

Let $M = C^{\xi}_{1-\xi,\delta}(a,T] := C(\mathcal{K})$, where $\mathcal{K} = (a,T]$, and $d : M \times M \to \mathbb{R}^+$ is given by

$$d(\zeta, w) = \bigl\|(\zeta - w)^2\bigr\|_\infty = \sup_{\vartheta \in (a,T]} \bigl(\zeta(\vartheta) - w(\vartheta)\bigr)^2.$$

Then (M,d) is a complete b-metric space with $r = 2$.

Theorem 3.1 *Suppose that*

(i) *$f : \mathcal{K} \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following inequality;*

$$\begin{aligned} &\bigl|f\bigl(\vartheta, \zeta(\vartheta)\bigr) - f\bigl(\vartheta, w(\vartheta)\bigr)\bigr| \\ &\quad \leq \frac{\iota\Gamma(\iota)(\delta(\vartheta) - \delta(a))^{1-\xi}}{4\sqrt{2}(\delta(T) - \delta(a))^{\iota+1-\xi}} \sqrt{\phi\bigl(\|(\zeta - w)^2\|_\infty\bigr)\lambda\bigl(\phi\bigl(\|(\zeta - w)^2\|_\infty\bigr)\bigr)}, \end{aligned}$$

where $\phi \in \Phi$ and $\lambda \in \mathcal{F}$;

(ii) *For A defined in relation (5) there exist $\zeta_0 \in C(\mathcal{K})$ and $\tau : \mathbb{R}^2 \to \mathbb{R}$ with*

$$\tau\bigl(\zeta_0(\vartheta), A\zeta_0(\vartheta)\bigr) \geq 0, \quad \vartheta \in \mathcal{K};$$

(iii) *For $\vartheta \in \mathcal{K}$ and $\zeta, w \in C(\mathcal{K})$, $\tau(\zeta(\vartheta), w(\vartheta)) \geq 0$ implies*

$$\tau\bigl(A\zeta(\vartheta), Aw(\vartheta)\bigr) \geq 0;$$

(iv) *If* $\{\zeta_n\} \subseteq C(\mathcal{K})$ *with* $\zeta_n \to \zeta$ *and* $\tau(\zeta_n, \zeta_{n+1}) \geq 0$, *then* $\tau(\zeta_n, \zeta) \geq 0$.
Then problem (4) *has at least one solution.*

Proof By Theorem 2.17, $\zeta \in C(\mathcal{K})$ is a solution of (4) if and only if a solution of the integral equation (5). Define $O : C(\mathcal{K}) \to C(\mathcal{K})$ by

$$\begin{aligned} Oy(t) &= \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left[w - \frac{1}{\Gamma(\iota)} \int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1} f\big(s, y(s)\big)\, ds \right] \\ &\quad + \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} f\big(s, y(s)\big)\, ds. \end{aligned} \tag{6}$$

We find a fixed point of O. Now let $\zeta, w \in C(\mathcal{K})$ be such that $\tau(\zeta(\varkappa), w(\varkappa)) \geq 0$. Using (i), we get

$$\begin{aligned} &\big|O\zeta(\varkappa) - Ow(\varkappa)\big| \\ &\quad = \left| \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left[w - \frac{1}{\Gamma(\iota)} \int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1} f\big(s, \zeta(s)\big)\, ds \right] \right. \\ &\qquad + \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} f\big(s, \zeta(s)\big)\, ds \\ &\qquad - \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left[w - \frac{1}{\Gamma(\iota)} \int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1} f\big(s, w(s)\big)\, ds \right] \\ &\qquad \left. - \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} f\big(s, w(s)\big)\, ds \right| \\ &\quad = \frac{1}{\Gamma(\iota)} \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left| \left[\int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1} \big(f\big(s, w(s)\big) - f\big(s, \zeta(s)\big)\big)\, ds \right] \right. \\ &\qquad \left. + \frac{1}{\Gamma(\iota)} \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} \big(f\big(s, \zeta(s)\big) - f\big(s, w(s)\big)\big)\, ds \right| \\ &\quad \leq \frac{1}{\Gamma(\iota)} \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \left[\int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1} \big|f\big(s, w(s)\big) - f\big(s, \zeta(s)\big)\big|\, ds \right. \\ &\qquad \left. + \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} \big|f\big(s, \zeta(s)\big) - f\big(s, w(s)\big)\big|\, ds \right] \\ &\quad \leq \frac{1}{\Gamma(\iota)} \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \frac{\iota\Gamma(\iota)(\delta(t) - \delta(a))^{1-\xi}}{4\sqrt{2}(\delta(T) - \delta(a))^{\iota+1-\xi}} \\ &\qquad \times \sqrt{\phi\big(\|(\zeta - w)^2\|_\infty\big)\lambda\big(\phi\big(\|(\zeta - w)^2\|_\infty\big)\big)} \\ &\qquad \times \left(\int_a^T \delta'(s)\big(\delta(T) - \delta(s)\big)^{\iota-1}\, ds + \int_a^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1}\, ds \right) \\ &\quad \leq \frac{1}{\Gamma(\iota)} \frac{(\delta(T) - \delta(a))^{1-\xi}}{(\delta(t) - \delta(a))^{1-\xi}} \frac{\iota\Gamma(\iota)(\delta(t) - \delta(a))^{1-\xi}}{4\sqrt{2}(\delta(T) - \delta(a))^{\iota+1-\xi}} \\ &\qquad \times \sqrt{\phi\big(\|(\zeta - w)^2\|_\infty\big)\lambda\big(\phi\big(\|(\zeta - w)^2\|_\infty\big)\big)} \\ &\qquad \times \left(\frac{2}{\iota}\big(\delta(T) - \delta(a)\big)^{\iota} \right), \end{aligned}$$

and hence

$$\begin{aligned}&\left|O\zeta(\varkappa)-Ow(\varkappa)\right|^2\\&\quad\le\frac{1}{8}\phi\left(\left\|(\zeta-w)^2\right\|_\infty\right)\lambda\left(\phi\left(\left\|(\zeta-w)^2\right\|_\infty\right)\right).\end{aligned}$$

Define $\alpha: C(\mathcal{K})\times C(\mathcal{K})\to\mathbb{R}^+$ by

$$\alpha(\zeta,w)=\begin{cases}1, & \tau(\zeta(\vartheta),w(\vartheta))\ge 0,\vartheta\in\mathcal{K},\\ 0, & \text{otherwise.}\end{cases}$$

So for $\zeta,w\in C(\mathcal{K})$ with $\tau(\zeta(\vartheta),w(\vartheta))\ge 0$, we have

$$\alpha(\zeta,w)8d(O\zeta,Ow)\le 8d(O\zeta,Ow)\le\lambda\bigl(\phi\bigl(d(\zeta,w)\bigr)\bigr)\phi\bigl(d(\zeta,w)\bigr),\quad\lambda\in\mathcal{K}.$$

From (iii) we have

$$\begin{aligned}\alpha(\zeta,w)\ge 1\quad&\Rightarrow\quad\tau\bigl(\zeta(\vartheta),w(\vartheta)\bigr)\ge 0\quad\Rightarrow\quad\tau\bigl(O(\zeta),O(w)\bigr)\ge 0\\&\Rightarrow\quad\alpha\bigl(O(\zeta),O(w)\bigr)\ge 1\end{aligned}$$

for $\zeta,w\in C(\mathcal{K})$. Thus O is α-admissible. By (ii) there exists $\zeta_0\in C(\mathcal{K})$ with $\alpha(\zeta_0,O\zeta_0)\ge 1$. By (iv) and Theorem 2.16 we find out ζ^* with $\zeta^*=O\zeta^*$, which is a positive solution of (4). □

Example 3.2 Consider the δ-Caputo fractional integral BVP

$$\begin{cases}D_{1^+}^{\frac{1}{2},0;e^t}y(t)=f(t,y(t)), & t\in(1,2],\\ y(2)=w\in\mathbb{R},\end{cases}\tag{7}$$

$$C_{1-\xi;\delta}^{\beta(1-\iota)}[1,2]=C_{\frac{1}{2};e^t}^{0}[1,2]=\bigl\{f:(1,2]\times\mathbb{R}^2\to\mathbb{R};\bigl(e^t-e\bigr)^{\frac{1}{2}}f\in C[1,2]\bigr\}$$

with $\iota=\frac{1}{2}$, $\beta=0$, $\xi=\frac{1}{2}$, $\delta(t)=e^t$, $(a,T]=(1,2]$. Clearly, $f\in C_{\frac{1}{2};e^t}[1,2]$. Then u and w satisfy the following condition:

$$\bigl|f(x,u)-f(x,w)\bigr|\le\frac{\sqrt{\pi}(e^t-e^a)}{8\sqrt{2}(e^2-e^a)}\sqrt{\left\|(u-w)^2\right\|_\infty\frac{\sin^2\|(u-w)^2\|_\infty}{4}}.$$

Setting $\phi(x)=x$ and $\lambda(t)=\frac{\sin^2 t}{4}$, we obtain

$$\bigl|f(x,u)-f(x,w)\bigr|\le\frac{\iota\Gamma(\iota)(\delta(t)-\delta(a))^{1-\xi}}{4\sqrt{2}(\delta(T)-\delta(a))^{\iota+1-\xi}}\sqrt{\phi\left(\left\|(u-w)^2\right\|_\infty\right)\lambda\left(\phi\left(\left\|(u-w)^2\right\|_\infty\right)\right)}.$$

Hence all assumptions of Theorem 3.1 hold. Therefore problem (7) has a solution on $\mathcal{K}$.

In [23] the authors investigated the existence, uniqueness, and continuous dependence of global solution to the following singular fractional differential equation involving the left generalized Caputo fractional derivative with respect to another function δ:

$$
\begin{aligned}
&{}^{c}D_{0^+}^{\iota;\delta} u(t) = f\big(t, u(t)\big), \quad t \in (0,b], b > 0, \\
&u(0) = u_0 \in \mathbb{R},
\end{aligned}
\tag{8}
$$

where $0 < \iota \le 1$, and ${}^{c}D_{0^+}^{\iota;\delta}$ is the δ-Caputo fractional derivative introduced by Almeida [11], $f : (0,b] \times \mathcal{R} \to \mathcal{R}$ is given function with $\lim_{t\to 0^+} f(t,\cdot) = \infty$, and u_0 is a constant.

Lemma 3.3 ([23]) *Assume that*:

(A1) $f : (0,b] \times \mathcal{R} \to \mathcal{R}$ *is a continuous with* $\lim_{t\to 0^+} f(t,u) = \infty$, *and there exists a constant* $0 < k < \iota$ *such that* $[\delta(t) - \delta(0)]^k f(t,u)$ *is a continuous function on* $[0,b] \times R$.

(A2) *For the k above, there exists constant* $L > 0$ *such that*

$$\big[\delta(t) - \delta(0)\big]^k \big(f(t,u_1) - f(t,u_2)\big) \le l|u_1 - u_2|$$

for all $t \in [0,b]$ *and* $u_1, u_2 \in \mathcal{R}$.

Then the function $u \in C[0,b]$ *is a solution to Cauchy problem* (8) *if and only if u satisfies the Volterra integral equation*

$$Au(t) := u(t) = u_0 + \frac{1}{\Gamma(\iota)} \int_0^t \delta'(s)\big(\delta(t) - \delta(s)\big)^{\iota-1} f\big(s, u(s)\big)\, ds, \quad t \in (0,b]. \tag{9}$$

Theorem 3.4 *Suppose that the conditions* (A1) *and* (A2) *from Lemma* 3.3 *hold, moreover*

(i) $f : \mathcal{K} \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the following condition*:

$$
\begin{aligned}
&\big|f\big(\vartheta, \zeta(\vartheta)\big) - f\big(\vartheta, w(\vartheta)\big)\big| \\
&\quad \le \frac{\iota\Gamma(\iota)}{2\sqrt{2}(\delta(T) - \delta(a))^{\iota}} \sqrt{\phi\big(\big\|(\zeta - w)^2\big\|_\infty\big)\lambda\big(\phi\big(\big\|(\zeta - w)^2\big\|_\infty\big)\big)},
\end{aligned}
$$

where $\phi \in \Phi, \mathcal{K} = (0,b]$ *and* $\lambda \in \mathcal{F}$;

(ii) *For A defined in relation* (9) *there exist* $\zeta_1 \in C(\mathcal{K})$ *and* $\tau : \mathbb{R}^2 \to \mathbb{R}$ *with*

$$\tau\big(\zeta_1(\vartheta), A\zeta_1(\vartheta)\big) \ge 0, \quad \vartheta \in \mathcal{K};$$

(iii) *For* $\vartheta \in \mathcal{K}$ *and* $\zeta, w \in C(\mathcal{K})$, $\tau(\zeta(\vartheta), w(\vartheta)) \ge 0$ *implies*

$$\tau\big(A\zeta(\vartheta), Aw(\vartheta)\big) \ge 0;$$

(iv) *If* $\{\zeta_n\} \subseteq C(\mathcal{K})$ *with* $\zeta_n \to \zeta$ *and* $\tau(\zeta_n, \zeta_{n+1}) \ge 0$, *then* $\tau(\zeta_n, \zeta) \ge 0$.

Then problem (8) *has at least one solution.*

Proof By Lemma 3.3, $\zeta \in C(\mathcal{K})$ is a solution of (8) if and only if it is a solution of the integral equation (9). Define $O : C(\mathcal{K}) \to C(\mathcal{K})$ by

$$O\zeta(\varkappa) = \zeta_0 + \frac{1}{\Gamma(\iota)} \int_0^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1} f\big(s, \zeta(s)\big)\, ds, \quad \varkappa \in (0,b]. \tag{10}$$

We find a fixed point of O. Now let $\zeta, w \in C(\mathcal{K})$ be such that $\tau(\zeta(\varkappa), w(\varkappa)) \geq 0$. Using (i), we get

$$\begin{aligned}
&\left|O\zeta(\varkappa) - Ow(\varkappa)\right| \\
&= \frac{1}{\Gamma(\iota)}\left|\int_0^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1} f\big(s, \zeta(s)\big)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\iota)}\int_0^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1} f\big(s, w(s)\big)\, ds\right| \\
&= \frac{1}{\Gamma(\iota)}\int_0^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1} \left|f\big(s, w(s)\big) - f\big(s, \zeta(s)\big)\right| ds \\
&\leq \frac{1}{\Gamma(\iota)}\int_a^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1} \left|f\big(s, w(s)\big) - f\big(s, \zeta(s)\big)\right| ds \\
&\leq \frac{1}{\Gamma(\iota)} \frac{\iota\Gamma(\iota}{(\delta(t) - \delta(a))^{\iota}} \sqrt{\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\lambda\big(\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\big)} \\
&\quad \times \int_0^{\varkappa} \delta'(s)\big(\delta(\varkappa) - \delta(s)\big)^{\iota-1}\, ds \\
&= \frac{1}{2\sqrt{2}} \sqrt{\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\lambda\big(\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\big)},
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|O\zeta(\varkappa) - Ow(\varkappa)\right|^2 \\
&\quad \leq \frac{1}{8}\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\lambda\big(\phi\big(\|(\zeta - w)^2\|_{\infty}\big)\big).
\end{aligned}$$

Put $\alpha : C(\mathcal{K}) \times C(\mathcal{K}) \to \mathbb{R}^+$ by

$$\alpha(\zeta, w) = \begin{cases} 1, & \tau(\zeta(\vartheta), w(\vartheta)) \geq 0, \vartheta \in \mathcal{K}, \\ 0, & \text{otherwise}. \end{cases}$$

So for $\zeta, w \in C(\mathcal{K})$ with $\tau(\zeta(\vartheta), w(\vartheta)) \geq 0$, we have

$$\alpha(\zeta, w)8d(O\zeta, Ow) \leq 8d(O\zeta, Ow) \leq \lambda\big(\phi\big(d(\zeta, w)\big)\big)\phi\big(d(\zeta, w)\big), \quad \lambda \in \mathcal{F}.$$

From (iii) we have

$$\begin{aligned}
\alpha(\zeta, w) \geq 1 \quad &\Rightarrow \quad \tau\big(\zeta(\vartheta), w(\vartheta)\big) \geq 0 \quad \Rightarrow \quad \tau\big(O(\zeta), O(w)\big) \geq 0 \\
&\Rightarrow \quad \alpha\big(O(\zeta), O(w)\big) \geq 1
\end{aligned}$$

for $\zeta, w \in C(\mathcal{K})$. Thus O is α-admissible. By (ii) there exists $\zeta_0 \in C(\mathcal{K})$ with $\alpha(\zeta_0, O\zeta_0) \geq 1$. By (iv) and Theorem 2.16 we find out ζ^* with $\zeta^* = O\zeta^*$, which is a positive solution of (8). □

Example 3.5 We fix a kernel $\delta : [0,1] \to \mathcal{R}$ and consider the following equation:

$$\begin{cases} D_{1^+}^{\frac{1}{2};\delta} y(t) = \frac{1}{4\sqrt{2}}[\delta(t) - \delta(0)]^{-\frac{1}{2}}(1 + \frac{1}{3}y)e^{-\|y^2\|_\infty}, & t \in (0,1], \\ y(0) = 2, \end{cases} \tag{11}$$

where $\alpha = \frac{1}{2}$, $f(t,y(t)) = \frac{1}{4\sqrt{2}}[\delta(t) - \delta(0)]^{-\frac{1}{2}}(1 + \frac{1}{3}y)e^{-\|y^2\|_\infty}$ for $(t,y) \in (0,1] \times \mathcal{R}$, and $\lim_{t\to 0^+} f(t,\cdot) = \infty$. Setting $k = \frac{1}{2}$, the function

$$\bigl[\delta(t) - \delta(0)\bigr]^{\frac{1}{2}} f\bigl(t,y(t)\bigr) = \frac{1}{4\sqrt{2}}\left(1 + \frac{1}{3}y\right)e^{-\|y^2\|_\infty}$$

is continuous on $[0,1]$. So hypothesis (A1) from Lemma 3.3 is satisfied.

For $y_1(t), y_2(t) \in \mathcal{R}$ $(t \in (0,1])$ we have

$$\bigl|f\bigl(t,y_1(t)\bigr) - f\bigl(t,y_2(t)\bigr)\bigr| = \frac{1}{12\sqrt{2}}\bigl[\delta(t) - \delta(0)\bigr]^{-\frac{1}{2}}\bigl|y_1(t) - y_2(t)\bigr|e^{-\|(y_1-y_2)^2\|_\infty}.$$

Considering $\delta(t) = \sqrt{t+1}$ for $t \in (0,1]$ we get

$$\bigl|f\bigl(t,y_1(t)\bigr) - f\bigl(t,y_2(t)\bigr)\bigr| = \frac{1}{12\sqrt{2}}[\sqrt{t+1} - 1]^{-\frac{1}{2}}\bigl|y_1(t) - y_2(t)\bigr|e^{-\|(y_1-y_2)^2\|_\infty}.$$

So, hypothesis (A2) from Lemma 3.3 is also satisfied with $L = \frac{1}{12\sqrt{2}}e^{-\|(y_1-y_2)^2\|_\infty}$ and $k = \frac{1}{2}$. Therefore we can apply Lemma 3.3.

For all $y_1(t)$, $y_2(t)$ satisfying in the condition

$$\bigl|e^{-\|y_1{}^2\|_\infty} - e^{-\|y_2{}^2\|_\infty}\bigr| \le e^{-\|(y_1-y_2)^2\|_\infty},$$

we have

$$\bigl|f\bigl(x,y_1(t)\bigr) - f\bigl(x,y_2(t)\bigr)\bigr| \le \frac{\sqrt{\pi}}{8\sqrt{2}(\delta(T) - \delta(a))^{\iota}}\sqrt{\bigl\|(y_1 - y_2)^2\bigr\|_\infty \frac{e^{-\|(y_1-y_2)^2\|_\infty}}{4}}.$$

Setting $\phi(x) = x$ and $\lambda(t) = \frac{e^{-t}}{4}$, we obtain

$$\bigl|f\bigl(x,y_1(t)\bigr) - f\bigl(x,y_2(t)\bigr)\bigr| \le \frac{\iota\Gamma(\iota)}{2\sqrt{2}(\delta(T) - \delta(a))^{\iota}}\sqrt{\phi\bigl(\bigl\|(y_1 - y_2)^2\bigr\|_\infty\bigr)\lambda\bigl(\phi\bigl(\bigl\|(y_1 - y_2)^2\bigr\|_\infty\bigr)\bigr)}.$$

Hence all assumptions of Theorem 3.4 hold. Therefore problem (11) has a solution on $\mathcal{K}$.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

Both authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 August 2020 Accepted: 21 October 2020 Published online: 31 October 2020

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