

**A DISTRIBUTION-FREE UPPER CONFIDENCE BOUND FOR
Pr $\{Y < X\}$, BASED ON INDEPENDENT SAMPLES OF
 X AND Y**

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1. Summary. A solution for the problem of obtaining a distribution-free one-sided confidence interval for $p = \text{Pr} \{Y < X\}$ has been proposed in [1]. At present a numerical procedure is given for computing the sample sizes needed for such a confidence interval with given width and confidence level.

2. Introduction and formulation of the problem. The problem discussed in this paper arises in practical situations such as the following: structural components of a mechanism are mass-produced and each component has a strength at failure Y which, in view of unavoidable variability of the product, is a random variable. A component is then installed in a mechanism and exposed to a stress which reaches its maximum value X , again a random variable. If, due to chance, the values of Y and X are so paired off that $Y < X$, then the component fails in use. It is therefore of considerable importance to have an upper bound for

$$(2.1) \quad p = \text{Pr} \{Y < X\}.$$

Our problem is: Can p be estimated from samples of X and Y alone and, in particular, is there an upper confidence bound for p , i.e., a statistic ψ based on a sample of X and a sample of Y , such that for any $\epsilon > 0$, $\alpha > 0$ there exists a pair of numbers $M_{\epsilon, \alpha}$, $N_{\epsilon, \alpha}$, so that

$$(2.2) \quad \text{Pr} \{p \leq \psi + \epsilon\} \geq 1 - \alpha$$

when the sample of X is of size $m \geq M_{\epsilon, \alpha}$ and the sample of Y of size $n \geq N_{\epsilon, \alpha}$?

The following answer to this question was proposed in [1].

We assume that X and Y are independent random variables with continuous cumulative distribution functions $F(s) = \text{Pr} \{X < s\}$, $G(s) = \text{Pr} \{Y < s\}$. Let $X_1 \leq X_2 \leq \dots \leq X_m$ be an ordered sample of X and $Y_1 \leq Y_2 \leq \dots \leq Y_n$ an ordered sample of Y , and let $F_m(s)$, $G_n(s)$ be the empirical distribution functions corresponding to these samples.

Using the Wilcoxon-Mann-Whitney 'statistic,

$$U = \text{number of pairs } (X_i, Y_k) \text{ such that } Y_k < X_i,$$

we write

$$(2.3) \quad \hat{p} = U/mn.$$

Received June 7, 1957. Research sponsored by the Office of Naval Research and the Boeing Airplane Company.

It is easily verified that \hat{p} is an unbiased estimate of p and

$$p = \int_{-\infty}^{+\infty} G(s) dF(s),$$

$$\hat{p} = \int_{-\infty}^{+\infty} G_n(s) dF_m(s),$$

hence

$$(2.4) \quad p - \hat{p} = \int_{-\infty}^{+\infty} G d(F - F_m) + \int_{-\infty}^{+\infty} (G - G_n) dF_m$$

$$= \int_{-\infty}^{+\infty} (F_m - F) dG + \int_{-\infty}^{+\infty} (G - G_n) dF_m$$

and

$$(2.5) \quad p - \hat{p} \leq D_m^- + D_n^+,$$

where

$$D_m^- = \sup_{-\infty < s < +\infty} \{F_m(s) - F(s)\},$$

$$D_n^+ = \sup_{-\infty < s < +\infty} \{G(s) - G_n(s)\}.$$

It is well known [2] that

$$\Pr \{D_m^- < v\} = \Pr \{D_m^+ < v\} = P_m(v)$$

and

$$\Pr \{D_n^+ < v\} = P_n(v)$$

are cumulative distribution functions which depend on the sample sizes m, n , but not on the c.d.f.'s F and G . It follows from (2.5) that

$$(2.6) \quad \Pr \{p \leq \hat{p} + \epsilon\} \geq \Pr \{D_m^+ + D_n^+ \leq \epsilon\} = P_{m,n}(\epsilon),$$

where $P_{m,n}(\epsilon)$ is the convolution of P_m and P_n , hence does not depend on F and G . The statistic \hat{p} has, therefore, the property required of ψ in (2.2) provided one can, for given ϵ, α , determine numbers $M_{\epsilon,\alpha}, N_{\epsilon,\alpha}$ so that

$$(2.7) \quad P_{m,n}(\epsilon) \geq 1 - \alpha \quad \text{for } m \geq M_{\epsilon,\alpha}, n \geq N_{\epsilon,\alpha}.$$

Some further properties of \hat{p} are discussed in [1].

A numerical procedure for computing $M_{\epsilon,\alpha}, N_{\epsilon,\alpha}$ is presented in the next sections.

3. An approximate expression for $P_{m,n}(\epsilon)$. It was shown by N. Smirnov [3] that

$$(3.1) \quad \lim_{n \rightarrow \infty} \Pr \{D_n^+ \leq z/\sqrt{n}\} = \lim_{n \rightarrow \infty} P_n(z/\sqrt{n}) = 1 - e^{-2z^2} = L(z).$$

Since, for fixed n , $P_n(z/\sqrt{n}) = H_n(z)$ is a cumulative distribution function, and $L(z) = 1 - e^{-2z^2}$ is a continuous c.d.f., it follows by a well-known argument (see, e.g., [4], p. 276) that $H_n(z) \rightarrow L(z)$ uniformly. We may, therefore, conclude that

$$(3.2) \quad \lim_{n \rightarrow \infty} [\Pr \{D_n^+ \leq v\} - L(v\sqrt{n})] = \lim_{n \rightarrow \infty} [H_n(v\sqrt{n}) - L(v\sqrt{n})] = 0$$

uniformly for $0 \leq v \leq 1$. Writing

$$(3.3) \quad \begin{aligned} P_{m,n}(\epsilon) &= \Pr \{D_m^+ + D_n^+ \leq \epsilon\} = \int_0^\epsilon P_n(\epsilon - u) dP_m(u), \\ Q_{m,n}(\epsilon) &= \int_0^\epsilon L[(\epsilon - u)\sqrt{n}] dL(u\sqrt{m}), \end{aligned}$$

we have

$$(3.4) \quad \begin{aligned} |P_{m,n}(\epsilon) - Q_{m,n}(\epsilon)| &\leq \left| \int_0^\epsilon \{P_n(\epsilon - u) - L[(\epsilon - u)\sqrt{n}]\} dP_m(u) \right| \\ &\quad + \left| \int_0^\epsilon \{P_m(\epsilon - v) - L[(\epsilon - v)\sqrt{m}]\} dL(v\sqrt{n}) \right| \\ &\leq \text{Max}_{0 \leq u \leq \epsilon} |P_n(\epsilon - u) - L[(\epsilon - u)\sqrt{n}]| \\ &\quad + \text{Max}_{0 \leq v \leq \epsilon} |P_m(\epsilon - v) - L[(\epsilon - v)\sqrt{m}]|, \end{aligned}$$

which in view of (3.2) shows that

$$(3.5) \quad \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} |P_{m,n}(\epsilon) - Q_{m,n}(\epsilon)| = 0$$

uniformly for $0 \leq \epsilon \leq 1$. This justifies the use of $Q_{m,n}(\epsilon)$ as an approximation to $P_{m,n}(\epsilon)$ for m, n sufficiently large. Some observations on the goodness of this approximation are presented in Section 5.

By straightforward integration one obtains for $Q_{m,n}(\epsilon)$ the expression

$$(3.6) \quad \begin{aligned} Q_{m,n}(\epsilon) &= 1 - \frac{n}{m+n} e^{-2m\epsilon^2} - \frac{m}{m+n} e^{-2n\epsilon^2} \\ &\quad - \frac{2\sqrt{2\pi} mn\epsilon^2}{(m+n)^{3/2}} e^{-2mn\epsilon^2/(m+n)} \frac{1}{\sqrt{2\pi}} \int_{-2n\epsilon/\sqrt{m+n}}^{2m\epsilon/\sqrt{m+n}} e^{-t^2/2} dt. \end{aligned}$$

4. Sample sizes m, n which satisfy $Q_{m,n}(\epsilon) = 1 - \alpha$. With the notations

$$(4.1) \quad \begin{aligned} m + n &= N \\ m / (m + n) &= \lambda, \quad n / (m + n) = 1 - \lambda \\ \epsilon\sqrt{m + n} &= \delta \end{aligned}$$

TABLE I
 Values $\delta_{\lambda,\alpha}$ such that $Q(\delta_{\lambda,\alpha}; \lambda) = 1 - \alpha$

α	λ				
	.1	.2	.3	.4	.5
.10	4.1185	3.2027	2.8501	2.6928	2.6468
.05	4.6115	3.5667	3.1641	2.9844	2.9317
.01	5.5700	4.2745	3.7770	3.5524	3.4870
.005	5.9300	4.5405	4.0050	3.7665	3.6960
.001	6.6800	5.0980	4.4880	4.2150	4.1360

the expression (3.6) for $Q_{m,n}(\epsilon)$ may be written in the form

$$\begin{aligned}
 (4.2) \quad Q(\delta; \lambda) &= 1 - \lambda e^{-2(1-\lambda)\delta^2} - (1 - \lambda)e^{-2\lambda\delta^2} \\
 &\quad - 2\sqrt{2\pi} \lambda(1 - \lambda) \delta e^{-2\lambda(1-\lambda)\delta^2} \frac{1}{\sqrt{2\pi}} \int_{-2(1-\lambda)\delta}^{2\lambda\delta} e^{-t^2/2} dt.
 \end{aligned}$$

Table I contains solutions $\delta_{n,\alpha}$ of the equation.

$$(4.3) \quad Q(\delta; \lambda) = 1 - \alpha$$

for $\alpha = .001, .005, .01, .05, .10$, and $\lambda = .1$ (.1) .5. These solutions were obtained on a desk calculator, using the National Bureau Bureau of Standards Tables of the Exponential Function [5], Descending Exponential [6], and the Normal Distribution Function [7].

The use of the quantities N, λ, δ instead of the original m, n, ϵ has not only the advantage of reducing the computations to a table with double entry, but also makes it possible to design an experiment with a given ratio $\lambda = m/N$. This ratio is often dictated by considerations of cost or time.

Example. We wish to use four times as many Y 's as X 's, i.e., $\lambda = .2$, and require $\epsilon = .10, \alpha = .05$. From Table I we have $\delta_{.2,.05} = 3.5667$; hence, by (4.1), $(.10)\sqrt{N} = 3.5667$, and $N = 1272.13, m = 254.43, n = 1017.70$. The rounded-up sample sizes are therefore 255 for X , 1018 for Y .

5. Concluding remarks. The sample sizes computed for given $\lambda, \epsilon, \alpha$ by the use of Table I are conservative, i.e., too large, for two reasons. The first is that, instead of finding sample sizes m, n such that $P\{p \leq \hat{p} + \epsilon\} = 1 - \alpha$, we used inequality (2.6) and looked for m, n satisfying $P_{m,n}(\epsilon) = 1 - \alpha$, a step which certainly yields larger values. The second reason is that in equation $P_{m,n}(\epsilon) = 1 - \alpha$ the exact expression $P_{m,n}(\epsilon)$ was replaced by the approximate expression $Q_{m,n}(\epsilon)$ and then only m, n were computed. This step was justified by (3.5) which, however, does not indicate which way the sample sizes are affected. The following arguments are offered in favor of the contention that the solutions m, n of $P_{m,n}(\epsilon) = 1 - \alpha$ differ little from those of $Q_{m,n}(\epsilon) = 1 - \epsilon$ and that the solu-

tions of the second equation are more conservative (greater) than those of the first.

The exact form of $P\{D_n^+ \leq v\}$ for finite n is known and numerical computations, some of which are reproduced in [8], show that already for $n \geq 50$ the approximation of $P\{D_n^+ \leq v\}$ by $L(v\sqrt{n})$ is uniformly very good. Since the sample sizes computed from Table I are in all practical situations much larger than 50, (3.4) assures very close agreement between $P_{m,n}(\epsilon)$ and $Q_{m,n}(\epsilon)$.

Furthermore, the following conjecture appears to be substantiated by considerable numerical computations and some analytical considerations, although no proof for it is available: for every integer $n \geq 1$ and for $0 \leq v \leq 1$,

$$(5.1) \quad L(v\sqrt{n}) = 1 - e^{-2nv^2} \leq P\{D_n^+ \leq v\}.$$

From (5.1) would follow that $P_{m,n}(\epsilon) \geq Q_{m,n}(\epsilon)$ for $0 \leq \epsilon \leq 1$, hence $Q_{m,n}(\epsilon) = 1 - \alpha$ would yield sample sizes larger than $P_{m,n}(\epsilon) = 1 - \alpha$.

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