A DISTRIBUTIONAL GENERALISED STIELTJES TRANSFORMATION

by R. S. PATHAK

(Received 3rd July 1974)

1. Introduction

The generalised Stieltjes transform of a function $f(t) \in L(0, \infty)$ is defined by

$$F(s) \triangleq \int_0^\infty \frac{f(t)dt}{(s+t)^\rho}, \quad (\rho > 0). \tag{1}$$

The integral converges for complex s in the region Ω, where Ω is the s-plane cut from the origin along the negative real axis.

Pollard (3) defined the operator

$$L^\rho_{k,t}[F(t)] \triangleq (-1)^{k-1} \frac{2^{\rho-1}(2k-1)!\Gamma(\rho)}{k!(k-2)!\Gamma(2k+\rho+1)} [t^{2k+\rho-2}F^{(k-1)}(t)]^{(k)} \tag{2}$$

and gave the real inversion formula

$$\lim_{k\to\infty} L^\rho_{k,t}[F(t)] = f(t). \tag{3}$$

Sumner (4) defined the complex inversion operator M_{η_t} by

$$M_{\eta t}[F] \triangleq -\frac{1}{2\pi i}\int_{C_{\eta t}} (z+t)^{\rho-1}F'(z)dz,$$

where C_{η_t} is the contour which starts at the point $-t-i\eta$, proceeds along the straight line $\text{Im}(z) = -\eta$ to the point $-i\eta$, then along the semicircle $|z| = \eta$, $\text{Re}\, z \geqq 0$, to the point $i\eta$, and finally along the line $\text{Im}(z) = \eta$ to the point $-t+i\eta$.

Sumner (4) showed that if $f(u) \in L(0 \leqq u \leqq R)$ for all positive R and is such that (1) converges, then

$$\lim_{\eta\to 0+} M_{\eta_t}(F) = \tfrac{1}{2}[f(t+)+f(t-)] \tag{4}$$

for any positive t at which both $f(t+)$ and $f(t-)$ exist.

It is of interest to note that when $\rho = 1$, (1) reduces to the Stieltjes transform

$$F(s) \triangleq \int_0^\infty \frac{f(t)}{s+t}\,dt, \tag{5}$$

(2) reduces to Widder's real inversion formula (6, p. 345)

$$\lim_{k\to\infty} L_{k,t}[F(t)] = f(t)$$

and (4) reduces to the complex inversion formula for the Stieltjes transform (6, p. 340)

$$\lim_{\eta\to 0+}\frac{F(-t-i\eta)-F(-t+i\eta)}{2\pi i}=\tfrac{1}{2}[f(t+)+f(t-)],$$

for any positive t at which $f(t+)$ and $f(t-)$ both exist.

Pandey (**2**) extended the above real and complex inversion formulae for the Stieltjes transform (5) to the space $S'_\alpha(I)$ of generalised functions (defined in Section 2) interpreting the convergence in the weak distributional sense. Our object is to extend the formulae (3) and (5) for generalised Stieltjes transform (1) to Pandey's space $S'_\alpha(I)$.

The notation and terminology of this work follow that of (**2**) in which I stands for the open interval $(0, \infty)$ and $D(I)$ is the space of smooth functions on I having compact supports. The symbol $D'(I)$ stands for the space of distributions defined over the testing function space $D(I)$. The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of Schwartz distributions.

2. The testing function spaces $S_\alpha(I)$ and $\underset{\sim}{S}_\alpha(I)$

The generalised functions appearing in this paper were discussed in (**2**). We briefly review their definitions and paramount properties here.

An infinitely differentiable and complex-valued function $\phi(x)$ defined over I belongs to $S_\alpha(I)$ if

$$\gamma_k(\phi)\triangleq\sup_{0<x<\infty}(1+x)^\alpha\left|\left(x\frac{d}{dx}\right)^k\phi(x)\right|<\infty$$

for any fixed k, where k assumes values 0, 1, 2, … and α is a fixed real number less than or equal to ρ. We assign to $S_\alpha(I)$ the topology generated by the collection of semi-norms $\{\gamma_k\}_{k=0}^\infty$. $S_\alpha(I)$ is sequentially complete Hausdorff locally convex topological linear space. The dual $S'_\alpha(I)$ of $S_\alpha(I)$ is also sequentially complete. The members of $S'_\alpha(I)$ are called generalised functions. The topology of $D(I)$ is stronger than that induced on it by $S_\alpha(I)$. The restriction of any $f\in S'_\alpha(I)$ to $D(I)$ is in $D'(I)$.

An infinitely differentiable and complex valued function $\psi(x)$ defined over I belongs to $\underset{\sim}{S}_\alpha(I)$ if

$$\rho_k(\psi)\triangleq\sup_{0<x<\infty}(1+x)^\alpha\left|x^k\left(\frac{d}{dx}\right)^k\psi(x)\right|<\infty$$

for all $k = 0, 1, 2, \ldots$, where α is a fixed real number less than or equal to ρ. The concepts of convergence and completeness in $\underset{\sim}{S}_\alpha(I)$ are defined in a way similar to those defined in $S_\alpha(I)$. The space $\underset{\sim}{S}_\alpha(I)$ is also a locally convex Hausdorff topological vector space. The restriction of any member of $\underset{\sim}{S}'_\alpha$ to $D(I)$ is in $D'(I)$.

Moreover, $S_\alpha(I) = \underset{\sim}{S}_\alpha(I)$ in store of elements. The topology T_1 generated in $S_\alpha(I)$ by the sequence of seminorms $\{\gamma_k\}_{k=0}^\infty$ is the same as the topology T_2 generated on $\underset{\sim}{S}_\alpha(I)$ by the sequence of seminorms $\{\rho_k\}_{k=0}^\infty$.

Remark. In the definitions of the testing function spaces $S_\alpha(I)$, $\underline{S}_\alpha(I)$ and also in their duals Pandey restricted α according to $\alpha \leqq 1$ whereas in the present paper we need $\alpha \leqq \rho$.

3. Some preliminary lemmas

In this section we state six lemmas. These are generalisations of Pandey's lemmas 2-6. Their proofs are similar to those given by Pandey (**2**, pp. 86-89) and hence are omitted.

Lemma 1. *For complex* $s \in \Omega$, $\dfrac{1}{(s+x)^{\rho+k}} \in S_\alpha(I)$ *for each* $k = 0, 1, 2, 3, \ldots$ *and* $\alpha \leqq \rho$.

Lemma 2. *Let* $f(x) \in S'_\alpha(I)$ *where* $\alpha \leqq \rho$ *and let*

$$F(s) \triangleq \left\langle f(x), \frac{1}{(s+x)^\rho} \right\rangle \tag{6}$$

for $s \in \Omega$. *Then*

$$F^{(k)}(s) = \left\langle f(x), \frac{(-1)^k(\rho)_k}{(s+x)^{\rho+k}} \right\rangle \tag{7}$$

for $k = 1, 2, \ldots$.

Lemma 3. *For an arbitrary* $\varepsilon > 0$ *there exists a positive* $\eta < \frac{1}{2}$ *such that for*

$$1-\eta < x < 1+\eta \quad \text{and} \quad t > 0,$$

$$(1+t)^\alpha |\phi(tx) - \phi(t)| < \varepsilon$$

where $\phi(x) \in D(I)$ *and* $\alpha \leqq \rho$ (**2**, Lemma 3).

Lemma 4. *If* η *is a fixed positive number less than* $\frac{1}{2}$, *then*

$$\lim_{k\to\infty} \sqrt{k}\, 2^{2k} \int_0^{1-\eta} \frac{x^{k+\rho-2}}{(1+x)^{2k+\rho-1}}\, dx = 0,$$

where k *assumes only integral values and* ρ *is positive.*

Lemma 5. *If* η *is a fixed positive number less than* $\frac{1}{2}$ *then*

$$\lim_{k\to\infty} 2^{2k}\sqrt{k} \int_{1+\eta}^{\infty} \frac{x^{k+\rho-2}}{(1+x)^{2k+\rho-1}}\, dx = 0,$$

where k *assumes only integral values and* ρ *is positive.*

Lemma 6. *For fixed* $\alpha \leqq \rho$ *and* $\phi(x)$ *belonging to* $D(I)$,

$$\frac{2^{\rho-1}(2k-1)!}{k!(k-2)!}(1+t)^\alpha \int_0^\infty (\phi(x)-\phi(t))t^k \frac{x^{k+\rho-2}}{(t+x)^{2k+\rho-1}}\, dx \to 0$$

as $k \to \infty$, *uniformly for all* $t > 0$.

4. The real inversion formula

Theorem 1. *For a fixed* $\alpha \leqq \rho$, $\rho > 0$ *and* $x>0$ *let*

$$F(x) \triangleq \left\langle f(t), \frac{1}{(x+t)^{\rho}} \right\rangle,$$

where $f(t)$ *is an element of* $S'_{\alpha}(I)$. *Then for each* $\phi(x) \in D(I)$,

$$\langle L^{\rho}_{k,x} F(x), \phi(x) \rangle \to \langle f, \phi \rangle \quad \text{as } k \to \infty,$$

where

$$L^{\rho}_{k,x}\psi(x) \triangleq \frac{(-1)^{k-1}2^{\rho-1}(2k-1)!\Gamma(\rho)}{k!(k-2)!\Gamma(2k+\rho+1)} [x^{2k+\rho-2}\psi^{(k-1)}(x)]^{(k)}, \tag{8}$$

where $\psi(x) \in S'_{\alpha}(I)$ *and the differentiation in* (8) *is supposed to be in the distributional sense.*

Proof. By direct computation it follows that

$$L^{\rho}_{k,x}\psi(x) = x^{\rho-1} P\left(x\frac{d}{dx}\right)\psi(x)$$

where $P(x)$ is a polynomial in x of finite degree depending upon k. For $\phi(x) \in D(I)$, we have

$$\langle L^{\rho}_{k,x} F(x), \phi(x) \rangle = \left\langle x^{\rho-1} P\left(x\frac{d}{dx}\right) F(x), \phi(x) \right\rangle$$

$$= \left\langle P\left(x\frac{d}{dx}\right) F(x), x^{\rho-1}\phi(x) \right\rangle \tag{9}$$

$$= \left\langle F(x), P\left(-x\frac{d}{dx}-1\right) x^{\rho-1}\phi(x) \right\rangle \tag{10}$$

$$= \left\langle \left\langle f(t), \frac{1}{(x+t)^{\rho}} \right\rangle, \zeta(x) \right\rangle \tag{11}$$

$$= \left\langle f(t), \left\langle \frac{1}{(x+t)^{\rho}}, \zeta(x) \right\rangle \right\rangle, \tag{12}$$

where

$$\zeta(x) = P\left(-x\frac{d}{dx}-1\right) x^{\rho-1}\phi(x).$$

The step (10) follows from (9) on integration by parts. The equality of (11) and (12) can be proved by the technique of Riemann sums (**1**, pp. 151-152).

Now, we shall show that

$$\left\langle f(t), \left\langle \frac{1}{(x+t)^{\rho}}, \zeta(x) \right\rangle \right\rangle \to \langle f(t), \phi(t) \rangle \quad \text{as } k \to \infty.$$

For this we have to prove that

$$\left\langle \zeta(x), \frac{1}{(x+t)^{\rho}} \right\rangle \to \phi(t) \quad \text{in } S_{\alpha}(I) \text{ as } k \to \infty.$$

Now,

$$\left(t\frac{d}{dt}\right)^{k} \left\langle \zeta(x), \frac{1}{(x+t)^{\rho}} \right\rangle = \left\langle \zeta(x), \left(t\frac{d}{dt}\right)^{k} \frac{1}{(x+t)^{\rho}} \right\rangle$$

$$= \left\langle \zeta(x), \left(-x\frac{d}{dx}-\rho\right)^{k} \frac{1}{(x+t)^{\rho}} \right\rangle$$

$$= \left\langle \left(x\frac{d}{dx}+1-\rho\right)^{k} \zeta(x), \frac{1}{(x+t)^{\rho}} \right\rangle \quad \text{(integration by parts)}$$

$$= \left\langle \left(x\frac{d}{dx}+1-\rho\right)^{k} P\left(-x\frac{d}{dx}-1\right) x^{\rho-1}\phi(x), \frac{1}{(x+t)^{\rho}} \right\rangle$$

$$= \left\langle P\left(-x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}+1-\rho\right)^{k} x^{\rho-1}\phi(x), \frac{1}{(x+t)^{\rho}} \right\rangle$$

$$= \left\langle \left(x\frac{d}{dx}+1-\rho\right)^{k} x^{\rho-1}\phi(x), P\left(x\frac{d}{dx}\right) \frac{1}{(x+t)^{\rho}} \right\rangle$$

$$= \left\langle x^{-\rho+1}\left(x\frac{d}{dx}+1-\rho\right)^{k} x^{\rho-1}\phi(x), L^{\rho}_{k,x}\left(\frac{1}{(x+t)^{\rho}}\right) \right\rangle$$

$$= \left\langle \left(x\frac{d}{dx}\right)^{k} \phi(x), L^{\rho}_{k,x}\left(\frac{1}{(x+t)^{\rho}}\right) \right\rangle$$

$$= \frac{(-1)^{k-1}2^{\rho-1}(2k-1)!\Gamma(\rho)}{k!(k-2)!\Gamma(2k+\rho-1)} \int_0^{\infty} \left(x\frac{d}{dx}\right)^{k} \phi(x) \left[x^{2k+\rho-2}D^{(k-1)}\frac{1}{(x+t)^{\rho}}\right]^{(k)} dx$$

$$= \frac{2^{\rho-1}(2k-1)!t^{k}}{k!(k-2)!} \int_0^{\infty} \phi_k(x) \frac{x^{k+\rho-2}}{(x+t)^{2k+\rho-1}}\, dx,$$

where

$$\phi_k(x) \equiv \left(x\frac{d}{dx}\right)^{k} \phi(x).$$

Hence, as $k \to \infty$, we have

$$(1+t)^{\alpha}\left(t\frac{d}{dt}\right)^{k}\left[\left\langle \zeta(x), \frac{1}{(x+t)^{\rho}} \right\rangle - \phi(x)\right]$$

$$= \frac{2^{\rho-1}(2k-1)!}{k!(k-2)!}(1+t)^{\alpha} \int_0^{\infty} [\phi_k(x)-\phi_k(t)] \frac{t^k x^{k+\rho-2}}{(x+t)^{2k+\rho-1}}\, dx. \tag{13}$$

In view of Lemma 6 the right-hand side of (13) tends to zero uniformly for all $t>0$ as $k\to\infty$. This completes the proof of the theorem.

A special case. Setting $\rho = 1$ in Theorem 1 we obtain the real inversion formula for the Stieltjes transform of generalised functions given by Pandey (**2**, Theorem 1, p. 90).

5. The complex inversion formula

Theorem 2. *Let* $f(t) \in S'_\alpha(I)$ *where* $\alpha \leqq \rho$ *and let* $F(s)$ *be the generalised Stieltjes transform of* $f(t)$ *as defined by* (6). *Then for each* $\phi(x) \in D(I)$, *we have*

$$\left\langle -\frac{1}{2\pi i}\int_{C_{\eta t}} (z+t)^{\rho-1}F'(z)dz,\ \phi(t)\right\rangle \rightarrow \langle f, \phi\rangle \quad \text{as } \eta \rightarrow 0+ \tag{14}$$

where C_{η_t} *is the contour defined in Section* 1.

Proof. The theorem will be proved by justifying the steps in the following manipulations

$$\left\langle -\frac{1}{2\pi i}\int_{C_{\eta t}} (z+t)^{\rho-1}F'(z)dz,\ \phi(t)\right\rangle \tag{15}$$

$$= \left\langle \phi(t), \frac{1}{2\pi i}\int_{C_{\eta t}} (z+t)^{\rho-1}\left\langle f(x), \frac{\rho}{(z+x)^{\rho+1}}\right\rangle dz\right\rangle \tag{16}$$

$$= \left\langle \phi(t), \left\langle f(x), \frac{1}{2\pi i}\int_{C_{\eta t}} \frac{\rho}{(z+x)^{\rho+1}}(z+t)^{\rho-1}dz\right\rangle\right\rangle \tag{17}$$

$$= \left\langle f(x), \left\langle \phi(t), \frac{1}{2\pi i}\int_{C_{\eta t}} \frac{\rho(z+t)^{\rho-1}}{(z+x)^{\rho+1}}\, dz\right\rangle\right\rangle \tag{18}$$

$$\rightarrow \langle f(x), \phi(x)\rangle \quad \text{as } \eta \rightarrow 0+. \tag{19}$$

Since the integrand in (15) is analytic and one-valued on Ω (**4**, p. 178), the integral on C_{η_t} is an analytic function of t (**5**, p. 99). Consequently, (15) has a meaning for $\phi \in D(I)$ and is, in fact, an ordinary integration on t. That (15) equals (16) is obvious in view of (7). The equality of (16) to (17) and also that of (17) to (18) can be proved by the technique of Riemann sums (**1**, pp. 151-153).

To show that (18) goes into (19) we need to prove the following two lemmas.

Lemma 7. *For* $\rho > 0$, *let*

$$G(\eta;\ t, x) = \frac{\eta^\rho}{2\pi i}\,\frac{1}{t-x}\left[\frac{1}{\{\eta - i(t-x)\}^\rho} - \frac{1}{\{\eta + i(t-x)\}^\rho}\right].$$

Then

$$\lim_{\eta\rightarrow 0+}\int_0^\infty G(\eta;\ t, x)dx = 1.$$

Proof. By splitting the integral into two parts corresponding to the interval $(0, t)$ and (t, ∞) and applying the transformations $t-x = v$ in the first part and $x-t = v$ in the second respectively we can write

$$\int_0^\infty G(\eta;\ t, x)dx = \frac{\eta^\rho}{2\pi i}\int_0^t \frac{1}{v}\left[\frac{1}{(\eta-iv)^\rho}-\frac{1}{(\eta+iv)^\rho}\right]dv + \frac{\eta^\rho}{2\pi i}\int_0^\infty \frac{1}{v}\left[\frac{1}{(\eta-iv)^\rho}-\frac{1}{(\eta+iv)^\rho}\right]dv.$$

Now, using the fact

$$\lim_{\eta\to 0+}\frac{\eta^\rho}{2\pi i}\int_0^R\left[\frac{1}{(\eta-iu)^\rho}-\frac{1}{(\eta+iu)^\rho}\right]\frac{du}{u} = \tfrac{1}{2} \quad \text{(4, Lemma 4a, p. 178)}$$

where $0 < R \leqq \infty$, we see that

$$\lim_{\eta\to 0+}\int_0^\infty G(\eta;\ t, x)dx = \tfrac{1}{2}+\tfrac{1}{2} = 1.$$

Lemma 8. *For* $\alpha \leqq \rho$ *and* $m = 0, 1, 2, \ldots,$

$$\sup_{0<x<\infty}\left|(1+x)^\alpha x^m \frac{d^m}{dx^m}\left[\int_0^\infty G(\eta;\ t, x)\phi(t)dt-\phi(x)\right]\right| \to 0 \quad as\ \eta\to 0+$$

where $G(\eta;\ t, x)$ and $\phi(x)$ *are the same as defined in Lemma 7 and Theorem 2.*

Proof. Using the reference (4) one can show that

$$\int_{x-\delta}^{x+\delta}\left|\frac{1}{(t-x)}\left[\frac{1}{\{\eta-i(t-x)\}^\rho}-\frac{1}{\{\eta+i(t-x)\}^\rho}\right]\right| dt < \delta L$$

where L is independent of x. The lemma can be proved using this fact and the technique followed by Pandey in (2; Theorem 2).

A special case. Taking $\rho = 1$ in (14) we obtain Pandey's complex inversion formula (2, Theorem 2, p. 91) for the Stieltjes transform of generalised functions belonging to $S'_\alpha(I)$.

The author expresses his gratitude to the referee for some valuable suggestions.

REFERENCES

(1) J. N. Pandey and A. H. Zemanian, Complex inversion for the generalized convolution transformation, *Pacific J. Math.* **25** (1968), 147-157.

(2) J. N. Pandey, On the Stieltjes transform of generalized functions, *Proc. Cambridge Philos. Soc.* **71** (1972), 85-96.

(3) H. Pollard, *Studies on the Stieltjes transform*, Dissertation, Harvard, *Bull. Amer. Math. Soc.* **48** (1942), 214.

(4) D. B. Sumner, An inversion formula for the generalized Stieltjes transform, *Bull. Amer. Math. Soc.* **55** (1949), 174-183.

(5) E. C. Titchmarsh, *Theory of functions*, 2nd edition (Oxford Univ. Press, 1939).

(6) D. V. Widder, *The Laplace transform* (Princeton, 1946).

(7) A. H. Zemanian, *Generalized integral transformations* (Interscience Publishers, 1968).

Banaras Hindu University
Varanasi 221005
India

https://doi.org/10.1017/S0013091500015716 Published online by Cambridge University Press