ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 1, Winter 1976

A DOCQUIER-GRAUERT LEMMA FOR STRONGLY PSEUDOCONVEX DOMAINS IN COMPLEX MANIFOLDS

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1. **Introduction.** Let M be a closed complex submanifold of C^n. The lemma referred to in the title is the following projection lemma of Docquier and Grauert.

1.1 THEOREM. [1] *Let K be a compact subset of M. There is a neighborhood U of K and a holomorphic map $\pi : U \rightarrow U \cap M$ such that $\pi(p) = p$ for $p \in U \cap M$.*

This lemma allows for the generalization to Stein manifolds of several function-algebraic results on domains in C^n [1, 9]. The purpose of this note is to prove a similar lemma when K is a strongly pseudo-convex domain on M, so that U is also strongly pseudoconvex, and $U \cap M = K$. We conclude with some applications, notably a generalization of the theorem of Sibony and Wermer [10].

I am indebted to Lee Stout for repeatedly encouraging me to pursue this idea, and to J. E. Fornaess for providing one critical step. Thanks also to N. Kerzman for reminding me recently of the potential usefulness of this lemma.

2. **The Proof of the Lemma.** We begin with the result of J. E. Fornaess, which can, for our purposes, be taken as the definition of s. psc. (strongly pseudoconvex).

2.1 THEOREM. [2] *Let D be a s. psc. domain on a Stein manifold M. There is a neighborhood N of $\bar{D}$, a C^∞ function ρ defined in, and s. psh. (strictly plurisubharmonic) on N such that*

(a) $d\rho \neq 0$ near ∂D,

(b) $D = \{p \in N : \rho(p) < 0\}$.

For a proof of this result see [8].

We shall call ρ a *defining function for D.* The main result of this note is the following assertion.

2.2 THEOREM. *Let M be a closed submanifold of a Stein domain U_0 in C^n. Suppose we are given (by Theorem 1.1) a neighborhood U of*

Received by the editors on September 17, 1974, and in revised form on December 12, 1974.

[1]Partially supported by NSF GP 28323 A3

M and a holomorphic projection $\pi : U \to M$ (in particular M is also a closed submanifold of U). Let D be a s. psc. domain in M ($\bar{D} \subset M$). Then, there exists a s. psc. domain E in C^n such that

(a) $\bar{E} \subset U \cap U_0$

(b) $E \cap M = D$

(c) ∂E *intersects M transversally in* ∂D

(d) $\pi : \bar{E} \to \bar{D}$.

PROOF. Let d be the dimension of M. From the argument of Docquier and Grauert, we know that the fibers $\pi^{-1}\pi x$ of π intersect M transversally and are of dimension $n - d$. Let $f_1, \cdots, f_k \in \mathcal{O}(U_0)$ be chosen so as to generate the idealsheaf of M at every point of $\bar{D}$. Let ρ be the defining function for $D : D = \{p \in M : \rho(p) < 0\}$. ρ is s. psh. near $\bar{D}$, we may take M to be this neighborhood.

Let $F = \sum_{i=1}^{k} |f_i|^2$. We want to show that there is an ϵ_0 and a neighborhood $U_1 \subset U \cap U_0$ of $\bar{D}$ such that dF is nonvanishing on the tangent space to the fibers of π in $N_1 = \{x \in U_1 : 0 < F(x) \leqq \epsilon_0\}$.

First of all, cover $\bar{D}$ by coordinate neighborhoods B (in C^n) with the following conditions satisfied: For $z_1, \cdots, z_d, z_{d+1}, \cdots, z_n$ the coordinates of B,

(a) the level sets $\pi^{-1}\pi x$ are given by $\{z_j = z_j^0 : j \leqq d\}$,

(b) $\{z_{d+1}, \cdots, z_n\} \subset \{f_1, \cdots, f_k\}$.

The following lemma shows that $F \mid \pi^{-1}\pi x$, for $x \in \bar{D}$, has differential nonzero near the origin, but for the origin. It is apparent that this concept of "nearness" varies continuously with the parameters $\{z_j^0 : j \leqq d\}$. Thus, by the compactness of $\bar{D}$ the desired result follows.

2.3. LEMMA. *Let B be a ball centered at the origin in C^{n-d} and let $g_1, \cdots, g_t$ be functions holomorphic in B with $g_i(0) = 0$, $1 \leqq j \leqq t$. Let*

$$G(z) = \sum_{i>d} |z_i|^2 + \sum_{j=1}^{t} |g_j(z)|^2.$$

There is an $\epsilon_0 > 0$ such that for $z \in B(0, \epsilon_0)$, $z \neq 0$, we have $dG(z) \neq 0$.

PROOF. Specifically, we show that for small enough balls, the derivative normal to the sphere is nonzero. Let $\theta = (\theta_{d+1}, \cdots, \theta_n)$ be on the unit sphere. Let $g_j^k = (\partial g_j / \partial z_k)(0)$. Since $g_j(0) = 0$, we have

$$\begin{aligned} G(t\theta) &= \Big(\sum_i |\theta_i|^2 + \sum_j \big| \sum_k g_j^k \theta_k \big|^2 \Big) |t|^2 \\ &\quad + |t|^3 F(t, \theta) \qquad (2.4) \\ &= M(\theta)|t|^2 + |t|^3 F(t, \theta). \end{aligned}$$

Thus

$$G'(t\theta) = 2M(\theta)t + t\tilde{F}(t, \theta).$$

Now, M is the term in parentheses in (2.4), so never vanishes. There is thus an ϵ_0 such that $M(\theta) + t\tilde{F}(t, \theta) \neq 0$ for all θ, and all t, $|t| \leqq \epsilon_0$. Then $G'(t\theta) \neq 0$ for θ on the unit sphere and $0 < |t| \leqq \epsilon_0$.

We return to the proof of theorem 2.2. We have found, using the above lemma, a neighborhood $U_1 \subset U \cap U_0$ of $\bar{D}$ with $dF|_{\pi^{-1}\pi x} \neq 0$ for $x \in U$ and $\epsilon_0 \geqq F(x) > 0$ (i.e., in the set N_1), and in which the set $\{f_1, \cdots, f_k\}$ includes a set of coordinates for the fibers of π.

Choose ϵ_0 even smaller, so that $\{x \in U_1 : \pi(x) \in \bar{D},\ F(x) \leqq \epsilon_0\}$ is compact in U_1. Choose a real number $A > 0$ with $A\epsilon_0 > m = \sup\{|\rho(x)| : x \in D\}$. Let $\sigma = \rho \circ \pi + A\sum_{i=1}^{k} |f_i|^2$, $N_2 = \{x \in N_1 \cup M : \rho \circ \pi(x) < \epsilon_0\}$ and $E = \{p \in N_2 : \sigma(p) < 0\}$.

Now, σ is strictly plurisubharmonic. For $\sqrt{-1}\,\partial\bar{\partial}\sigma = \sqrt{-1}\,\partial\bar{\partial}(\rho \circ \pi) + A\sqrt{-1}\,\partial\bar{\partial}\sum_i f_i\bar{f}_i$. Both summands are positive semidefinite, and the nullspace of the first summand is the tangent space to $\pi^{-1}\pi x$. But $f_1, \cdots, f_k$ include coordinates for $\pi^{-1}\pi x$, so the second summand is positive definite on this space. Thus $\sqrt{-1}\,\partial\bar{\partial}\sigma \gg 0$ throughout $N_1 \cup M$.

$\bar{E}$ is compact in N_2. Let $x \in \partial N_2$. If $\rho \circ \pi(x) > 0$, we must have $\sigma(x) > 0$, so $x \notin \bar{E}$. If $\rho \circ \pi(x) < 0$, then $\pi(x) \in D$, so $x \in U_1$ and $F(x) = \epsilon_0$ (since $\partial N_2 \cap U_1 = \partial N_1$). Thus

$$\sigma(x) = \rho \circ \pi(x) + A\epsilon_0 > -m + A\epsilon_0 > 0,$$

so $x \notin \bar{E}$.

Clearly $\bar{E} \cap M = \bar{D}$. In fact $\pi : \bar{E} \to \bar{D}$. For if $x \in \bar{E}$, $\rho \circ \pi(x) \leqq \sigma(x) \leqq 0$, so $\pi(x) \in \bar{D}$.

All the assertions of the theorem follow easily from these observations.

3. **Applications.** We shall now give some indication of the use of this lemma: to generalize to the context of Stein manifolds certain function algebraic results on s. psc. domains in C^n. With this in mind, we introduce the following notation. If K is a compact set on a complex manifold, $O(K)$ is the ring of all functions holomorphic at all points of K. By $A(K)$ we mean the ring of continuous functions on K, holomorphic at interior points of K. $A(K)$ is a Banach algebra in the uniform norm: $\|f\| = \sup\{|f(x)| : x \in K\}$. $\bar{O}(K)$ is the closure of $O(K)$ in this norm. For $x \in K$, m_x is the (maximal) ideal in $O(K)$ of all functions vanishing at x, and $\bar{m}_x$ is its closure in $A(K)$ in the above norm. m_x is the sheaf of ideals (in $\mathcal{O}$) of functions vanishing at x.

The first result is well-known for manifolds [7]; our point is that it need only be proven for domains in C^n, as in [4, 5].

3.1 THEOREM. [7] *Let D be a s. psc. domain in a Stein manifold M. Then $\bar{O}(\bar{D}) = A(\bar{D})$.*

PROOF. Since M is Stein, we may take M to be a closed submanifold of C^n. Let $\pi : \bar{E} \to \bar{D}$ be as constructed in theorem 2.2. Now, if $f \in A(\bar{D}), f \circ \pi \in A(\bar{E})$. By the approximation theorem of Henkin [4], $f \circ \pi = \lim f_n$ with $f_n \in 0(\bar{E})$. Then $f_n \mid M \in \mathcal{O}(\bar{D})$, and $f_n \mid M \to f$ uniformly on $\bar{D}$.

3.2 THEOREM. *Let D be a s. psc. domain on a Stein manifold M. Let $p \in D$, and suppose $f_1, \cdots, f_k \in A(\bar{D})$ satisfy*
(a) *$df_1(p), \cdots df_k(p)$ span m_p/m_p^2,*
(b) *$\{x \in \bar{D} : f_1(x) = \cdots = f_k(x) = 0\} = \{p\}$.*
Then $f_1, \cdots, f_k$ generate $\bar{m}_p$ as $A(\bar{D})$-module.

PROOF. If M is Euclidean space, this is the theorem of Øvrelid [8] (see also [6]). Here, we again take M as a closed submanifold of C^N, and let $\pi : \bar{E} \to \bar{D}$ be as given by theorem 2.2. Let $g_i = f_i \circ \pi$, $1 \leqq i \leqq k$, $h_j = z_j - z \circ \pi$, $1 \leqq j \leqq N$. Then $g_1, \cdots, g_k, h_1, \cdots, h_N$ are in $A(\bar{E})$ and satisfy conditions (a), (b) for p in $\bar{E}$. Thus, Øvrelid's result applies to this set of functions. If $f \in \bar{w}_p$ in $A(\bar{D})$, then $f \circ \pi \in \bar{w}_p$ in $A(\bar{E})$, so $f \circ \pi = \sum_i u_i g_i + \sum_j v_j h_j$, with $u_i, v_i \in A(\bar{E})$. Then, restricting to $\bar{D}$, we have $f = \sum_i (u_i \mid \bar{D}) f_i$, with $u_i \mid \bar{D} \in A(\bar{D})$.

3.3. REMARK. In both the above theorems, the results should remain true if we replace M by either a holomorphically convex manifold or a Stein space. However, in these situations the Grauert-Docquier lemma fails, and with it goes theorem 2.2 and any hope of proceeding via a reduction to C^n. Nevertheless, these results can be proven by means of another more specific result (see [2]).

Finally, we generalize the theorem of Sibony and Wermer [10]. It should be noted that their result itself requires a version of the Grauert-Docquier lemma which is different from theorem 2.2 and apparently requires much deeper results. We shall remark on this more specifically after the proof.

3.4 THEOREM. *Let D be a s. psc. domain in a Stein manifold M. Suppose $f_1, \cdots, f_k \in C^4(\bar{D})$ are holomorphic in D and satisfy*
(a) *for $x \neq y$ in $\bar{D}$, there is a j with $f_j(x) \neq f_j(y)$,*
(b) *for $x \in \bar{D}$, $df_1(x), \cdots, df_k(x)$ span m_x/m_x^2,*
(c) *$F(\bar{D}) = \{(f_1(x), \cdots, f_k(x)) : x \in \bar{D}\}$ is polynomically convex in C^k.*

Then the ring P of polynomials in $f_1, \cdots, f_k$ *is uniformly dense in* $A(\bar{D})$.

PROOF. As before, we take M to be a closed submanifold of C^N and take $\pi : \bar{E} \to \bar{D}$ as given by theorem 2.2. Let $w_1, \cdots, w_N$ be the coordinates of C^N, and define these functions on $\bar{E}$:

$$g_i(p) = f_i(\pi(p)), \quad 1 \leqq i \leqq k$$

$$h_j(p) = w_j(p) - w_j(\pi(p)), \quad 1 \leqq j \leqq N.$$

Clearly the functions $g_1, \cdots, g_k, h_1, \cdots, h_N$ are in $C^4(\bar{E})$ and satisfy the first two conditions (a), (b) relative to $\bar{E}$. Let $z_1, \cdots, z_{k+N}$ be coordinates of C^{k+N} and define $G : \bar{E} \to C^{k+N}$ by

$$z_i(G(p)) = g_i(p), \ 1 \leqq i \leqq k,$$

$$z_i(G(p)) = h_{i-k}(p), \quad k < i \leqq k + N.$$

Notice that $G(\bar{E}) \cap \{z \in C^{k+N} : z_i = 0, i > k\} = F(\bar{E}) \times \{0\}$. We now apply the argument of Sibony and Wermer to the map G defined on $\bar{E}$. It must be noted that the hypothesis of polynomial convexity does not intervene in their argument until *after* the application of the approximation theorem of Henkin. Thus, we can conclude: if $h \in A(\bar{E}), f = H \circ G$ where $H \in \bar{0}(G(\bar{E}))$.

Now, take $f \in A(\bar{D})$, and apply the above approximation result to $f \circ \pi$. There is a sequence $H_n \in O(G(\bar{E}))$ such that

$$\begin{aligned} f(\pi(p)) = \lim H_n(&f_1(\pi(p)), \cdots, f_k(\pi(p)), \\ &w_1(p) - w_1(\pi(p), \cdots, w_N(p) - w_N(\pi(p))) \end{aligned} \tag{3.5}$$

uniformly in p. Let $f_n(z_1, \cdots, z_k) = H_n(z_1, \cdots, z_k, O, \cdots, O)$. Restricting (3.5) to $\bar{D}$, we obtain $f(x) = \lim f_n(f_1(x), \cdots, f_k(x))$, where $f_n \in O(F(\bar{D}))$, uniformly in x. But $F(\bar{D})$ is *polynomially convex*, so by the Oka-Weil approximation theorem, the f_n, and with it, f, are uniformly approximable by polynomials.

3.6. REMARK. In contrast to the preceding results, this application of theorem 2.2 appears to be genuine: the techniques of Sibony and Wermer will not apply generally to manifolds. Their operative hypotheses are that both T and N are trivial bundles. However, J. Taylor and I have recently found an elementary way of accomplishing what is accomplished by the generalized Docquier-Grauert theorem which was only the fact that $\bar{D}$ is the spectrum of $A(D)$. These results together with applications to more general manifolds-with-boundary will appear elsewhere.

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