A DUALITY THEORY FOR CONVEX PROGRAMS WITH CONVEX CONSTRAINTS

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The existence of a solution to the problem of minimizing a convex function subject to restriction of the variables to a closed convex set in n-space ("convex programming") has been characterized (for suitable differentiability conditions) by the Kuhn-Tucker theorem [5]. In general, no dual programming problem (not involving the variables of the direct problem) has been associated with this situation except in the linear programming case, and very recently by E. Eisenberg in [3], for homogeneity of order one in the function and linear inequality constraints, and by R. J. Duffin [2] in an inverse manner for a highly specialized problem.

Starting with a little known paper of A. Haar [4] in the light of current linear programming constructs (e.g., "regularization" [1]), we effect a generalization of these ideas (with maximal finite algebra and minimal topology) so that a dual theory practically as straightforward as linear programming theory is obtained, and which includes a dual theorem covering the most general convex programming situation (e.g. no differentiability conditions qualifying the convex function or constraints, or homogeneity, etc.).

This general theorem is made possible by associating a suitably restricted, usually infinite-dimensional space problem with the minimization problem in n-space instead of the usual association of another finite m-space problem. The space we use is a "generalized finite sequence space" (g.f.s.s.), defined with respect to an index set I of arbitrary cardinality as the vector space, S, of all vectors $\lambda = [\lambda_i : i \in I]$ over an ordered field F with only finitely many nonzero entries.

Such spaces possess the following key characteristics for linear programming of ordinary n-spaces. Let V be a vector space over F and consider a collection of vectors: $P_0, P_i : i \in I$ in V. Let R be the subspace spanned by these vectors, and let

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$$\Lambda = \left\{ \lambda \in S : \sum_{i \in I} \lambda_i P_i = P_0, \lambda \geqq 0 \right\}.$$

Clearly Λ is convex in S and we have (assuming V finite-dimensional):

THEOREM 1. *$\lambda \neq 0$ is an extreme point of Λ in S if and only if the nonzero coordinates of λ correspond to coefficients of linearly independent vectors in R.*

THEOREM 2. *Λ is generated by its extreme points if and only if for any $\alpha \in S$, $\alpha \neq 0$, $\sum_{i \in I} \alpha_i P_i = 0$ implies some α_r and some α_s are of opposite signs.*

REMARK. Λ need not be bounded as in n-space. ("Bounded" means there exists $M \in F$ such that $\sum_i |\lambda_i| \leqq M$ for all λ in the set.)

These theorems can be proved in similar fashion to their finite space forms due respectively to Charnes and to Charnes-Cooper (see [1]).

By "dual semi-infinite programs" we mean the following pair of problems formed from the same data:

I	II
$\min u^T P_0$	$\max \sum_{i \in I} c_i \lambda_i$
subject to $u^T P_i \geqq c_i \ i \in I$	subject to $\sum_{i \in I} P_i \lambda_i = P_0$
	$\lambda \in S, \ \lambda \geqq 0.$

We restrict ourselves now to the real field and to semi-infinite programs whose $\{P_i, c_i\}$ are "canonically closed" in the sense that in an equivalent inequality system in which the new $\{P_i, c_i\}$ form a bounded set, e.g. by dividing each inequality by some $d_i > 0$, the set is also closed. We call such programs "dual Haar programs."

We require next the inhomogeneous (inequality system) theorem of Haar [4].

THEOREM 3. *Let $u^T P_i \geqq c_i$, $i \in I$ be a canonically closed system. If $u^T P \geqq c$ holds whenever $u^T P_i \geqq c_i$ for all $i \in I$, then there exist $\lambda_k \geqq 0$, $\lambda_0 \geqq 0$, with at most $n+1$ nonzero such that*

$$u^T P - c = \sum_k \lambda_k (u^T P_k - c_k) + \lambda_0.$$

Haar does not specifically use the notion of canonical closure, but as counter-examples show he must have intended something of this sort. By use of Theorem 3 we obtain the following lemma.

LEMMA 1. *For Haar programs if both* I *and* II *are consistent, then*

$$\inf u^T P_0 = \sup \sum_{i \in I} u^T P_i \lambda_i = \sum_{i \in I} c_i \lambda_i^* \qquad \textit{for some } \lambda^* \in \Lambda.$$

Hence we conclude

THEOREM 4 (EXTENDED DUAL THEOREM). *For any pair of dual Haar programs precisely one of the following occurs.*

(i) $\sup \sum_{i \in I} c_i \lambda_i = \infty$ *and* I *is inconsistent.*

(ii) $\inf u^T P_0 = -\infty$ *and* II *is inconsistent.*

(iii) I *and* II *are both inconsistent.*

(iv) $\inf u^T P_0 = \sup \sum_{i \in I} c_i \lambda_i^*$ *for some* $\lambda^* \in \Lambda$.

REMARK. Only the Farkas-Minkowski property of Theorem 3 is employed to obtain Theorem 4. Canonical closure is a sufficient but not a necessary condition for this.

To obtain the general convex programming dual theorem, we move the functional into the constraints and replace it with a linear function as follows. Suppose the direct problem is: min $C(u)$ subject to $G(u) \geqq 0$, where $G^T = (\cdots, G_i(u), \cdots)$ is a finite vector of concave functions which defines the convex set W of the u's. Let $u^T P_i \geqq c_i$, $i \in I$ be a system of supports for W, and $z - u^T Q_\alpha \geqq d_\alpha$, $\alpha \in A$ be a system of supports for $z - C(u) \geqq 0$. Then the direct problem may be rewritten as:

$$\min z, \quad \text{subject to } z - u^T Q_\alpha \geqq d_\alpha, \quad u^T P_i \geqq c_i, \quad \alpha \in A, i \in I.$$

Thus we have

THEOREM 5. *Assuming the Farkas-Minkowski property for this system, the extended dual theorem applies to the following dual programs:*

$$\begin{array}{ll} \min z & \max \quad \sum_\alpha d_\alpha \eta_\alpha + \sum_i c_i \lambda_i \\ \textit{subject to } z - u^T Q_\alpha \geqq d_\alpha & \textit{subject to} \sum_\alpha \eta_\alpha \qquad\qquad = 1 \\ \qquad u^T P_i \geqq c_i & \quad - \sum_\alpha Q_\alpha \eta_\alpha + \sum_i P_i \lambda_i = 0 \\ & \qquad\qquad\qquad \eta_\alpha, \lambda_i \geqq 0. \end{array}$$

Complete generality may now be obtained since an arbitrary semi-infinite program may be replaced by a Haar program according to the following observation:

THEOREM 6. *The canonical closure* $u^T P_i \geqq c_i$, $i \in \hat{I}$ *of the system* $u^T P_i \geqq c_i$, $i \in I$, *has precisely the same set of solutions* $\{u\}$, *where* $\hat{I} \supseteq I$

denotes the increased index set to index limit points of the (P_i, c_i) *not indexed by* I.

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