A dwell time approach to the stability of switched linear systems based on the distance between eigenvector sets

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Abstract

In this paper a sufficient condition on the minimum dwell time that guarantees the stability of switched linear systems is given. The proposed method interprets the stability of switched linear systems through the distance between the eigenvector sets of subsystem matrices. Thus, an explicit relation in view of stability is obtained between the family of the involved subsytems and the set of admissible switching signals.

Keywords: switched systems, dwell time, stability, eigenvector sets.

1 INTRODUCTION

The stability analysis of switched linear systems is one of the active research areas in system theory during the last decades [1, 2, 3, 4]. In general, a switched system is a hybrid dynamical system defined by a family of subsystems and a switching rule orchestrating the switching between subsystems [2, 5, 6]. Except the problem of finding a particular stabilizing switching signal [6, 7, 8] there exist mainly two stability problems: stability under arbitrary switching and stability under constraint switching, such as dwell time, average dwell time constraints, etc. [1, 3, 9]. The arbitrary switching problem has been widely investigated [10, 11, 12, 13, 14], whereas the stability under constraint switching still needs to be treated throughly [15, 16, 17, 18]. Especially specifying the minimum dwell time that ensures the stability of the switched sytems is an attracting problem. In literature there are some conditions to obtain a minimum dwell time [15, 16, 18], which do not reveal any explicit relation between the minimum dwell time and the properties of the involved subsystems. In this paper a novel method to obtain the minimum dwell time is proposed which improved the results in [19]. The value for the minimum dwell time is numerically observed to be smaller than most of the already existing results. Moreover, the method reveals the missing explicit relation mentioned above, as it shows the dependence of the minimum dwell time on the weakest eigenvalues and on the distance between the eigenvector sets of subsystems.

In the next section, definitions and mathematical preliminaries are given. In Section 3, main results are presented and in Section 4, these are illustrated by examples. Finally discussions and comparisons are given in conclusion.

Notation: $\mathbb{R}^n(\mathbb{C}^n)$ denotes n-dimensional real (complex) vector space and $\mathcal{M}^n_{\mathbb{R}}(\mathcal{M}^n_{\mathbb{C}})$ the set of all $n \times n$ real (complex) matrices. $\|x\|$ is the Euclidean norm if x is a vector and spectral norm if x is a matrix. For a set $\mathcal{X}$, $s(\mathcal{X})$ is the number of elements in $\mathcal{X}$ and for a matrix, $A \in \mathcal{M}^n_{\mathbb{C}}$. A^* denotes the Hermitian transpose of A.

2 PRELIMINARIES

In this section, a brief definition of switched linear systems is given. Then, two semimetrics on the set of $n \times n$ invertible matrices and the notion of walk in digraphs are introduced as they will be used for the main results in the next section.

2.1 Linear Switched Systems

Switched linear systems can be defined as follows,

$$\dot{x}(t) = A_\sigma x(t) \ , \quad \sigma \in \mathcal{S} \ , \quad t \geq 0 \tag{1}$$

where $\mathcal{S}$ is a set of admissible switching signals $\sigma\colon [0, \infty) \to \mathcal{P}$ and $\mathcal{P}$ denotes a finite set of indices. $\{A_p \mid \ p \in \mathcal{P}\}$ is a parametrized family of m subsystems each of which is represented by an $n \times n$ matrix. In the sequel, the generic case where each A_p has n distinct eigenvalues will be considered. A switching signal is a piecewise constant function which is continuous from right and has finitely many discontinuities on any finite interval. The set of all possible switching functions is denoted as $\mathcal{S}_{\text{nonchatt}}$. Another set of switching signals can be defined by restricting the dwell time of switching signals, defined as the minimum length of intervals where the switching signal is constant. Let $\mathcal{S}_{\text{dwell}}[\tau]$ denote the set of all elements of $\mathcal{S}_{\text{nonchatt}}$ with dwell time not less than τ. Obviously, $\mathcal{S}_{\text{dwell}}[0] = \mathcal{S}_{\text{nonchatt}}$. In this work, the system (1) with $\mathcal{S} = \mathcal{S}_{\text{dwell}}[\tau]$ will be considered. For a more broad definition of switched systems and sets of switching signals see [5].

A solution of (1) is a function $x\colon [0,\infty) \to \mathcal{P}$ which satisfies the linear time varying system

$$\dot{x}(t) = A_{\sigma(t)}x(t) \ , \quad t \geq 0 \tag{2}$$

for some $\sigma \in \mathcal{S}$. In order to represent $x(t)$ explicitly, we define the following functions from $\mathcal{S}$ to $[0,\infty)$ as

$$\omega_i(\sigma) := \ i\text{'th discontinuity point of } \sigma \ , \quad i = 1, 2, \ldots,$$

$$\omega_0(\sigma) := 0 \text{ (for convenience)} \ ,$$

$$\Delta_i(\sigma) := \omega_i(\sigma) - \omega_{i-1}(\sigma) \ , \quad i = 1, 2, \ldots,$$

and $p_i(\sigma)$ from $\mathcal{S}$ to $\mathcal{P}$ as

$$p_i(\sigma) := \sigma(\omega_{i-1}(\sigma)),$$

which gives the index of the i'th active subsystem when switching signal is σ. In short, ω_i, Δ_i and p_i will be used instead of $\omega_i(\sigma)$, $\Delta_i(\sigma)$ and $p_i(\sigma)$ (see Fig. 1). Using these functions, the solution of (1) can be stated as

$$x(t) = e^{A_{p_i}(t-\omega_{i-1})}\left(\prod_{k=1}^{i-1} e^{A_{p_k}\Delta_k}\right)x(0) \ , \quad t \in [\omega_{i-1}, \omega_i) \tag{3}$$

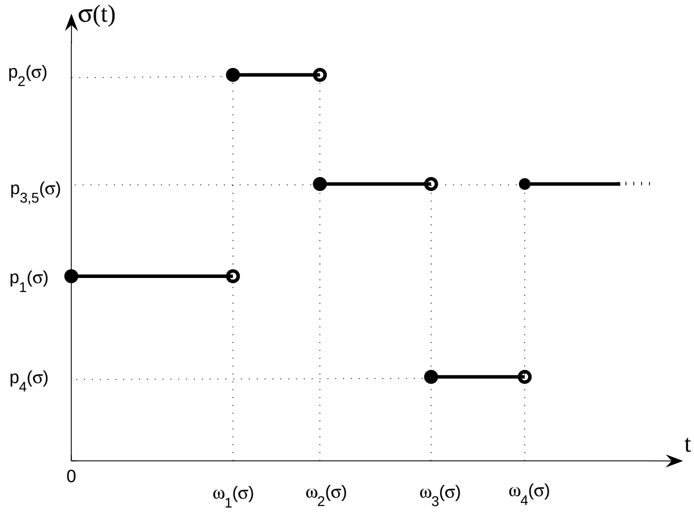

Figure 1: A switching signal

There are various stability definions of switched systems [1, 5, 20]. In this work, some results on stability and asymptotic stability of the switched linear systems will be obtained. Considering the switched linear system (1) and the linear time varying system (2) obtained from (1) choosing a specified switching signal, the system given by (1) is (asymptotically) stable if (2) is (asymptotically) stable for every $\sigma \in \mathcal{S}$. Thus (1) is stable if every solution x of (1) satisfies $\|x(t)\| \leq \gamma \|x(0)\|, \forall t \geq 0$ for some $\gamma \in \mathbb{R}$ and asymptotically stable if it approaches to zero as $t \to 0$ [21]. We are not concerned with exponential stability as it is equivalent to asymptotic stability for the switched linear system (1) (see Lemma 1 in [5]).

2.2 Distance Between Eigenvector Sets

Consider a matrix $A_i \in \mathcal{M}_{\mathbb{R}}^n$ which has n distinct eigenvalues. The *eigenvector set* $\mathcal{V}_i$ is defined as the set of all eigenvectors of A_i. $\mathcal{V}_i$ can also be represented by an invertible matrix $V_i = [v_1^{(i)} v_2^{(i)} \dots v_n^{(i)}] \in \mathcal{M}_{\mathbb{C}}^n$ called as the *eigenvector matrix* of A_i, the columns of which are linearly independent and have norm one. Obviously, an eigenvector set can be represented by different eigenvector matrices. One can change the order and signs of the columns and obtain a different eigenvector matrix representing the same eigenvector set.

Let $\mathcal{V}^{(n)}$ be the set of all eigenvector sets of $n \times n$ real matrices with distinct eigenvalues. The following two semimetrics can be defined on $\mathcal{V}^n$.

$$d_1(\mathcal{V}_i, \mathcal{V}_j) := \ln(\max(\|V_i^{-1}V_j\|, \|V_j^{-1}V_i\|)) \tag{4}$$

$$d_2(\mathcal{V}_i, \mathcal{V}_j) := \ln(\|V_i^{-1}V_j\| \cdot \|V_j^{-1}V_i\|) \tag{5}$$

It is straightforward to show that (4) and (5) satisfy semimetric conditions, which are the three axioms of metric excluding the identity of indiscernibles ($d(x, y) = 0 \Leftrightarrow x = y$). Here V_i and V_j are the eigenvector matrices representing $\mathcal{V}_i$ and $\mathcal{V}_j$, respectively. The semimetrics (4) and (5) are well-defined, for they are independent from the choice of eigenvector matrices. In order to see this let V_i and V_i' be two eigenvector matrices representing the eigenvector set $\mathcal{V}_i$, that is they differ only by the order or sign of their columns. Then there exist a unitary matrix U, such that $V_i' = V_iU$. The spectral norm is unitarily invariant. Moreover $\|UAY\| = \|A\|$ for any pair of unitary matrices (U, Y) [22]. Consequently,

$\|V_i'^{-1}V_j\| = \|U^*V_i^{-1}V_j\| = \|V_i^{-1}V_j\|$ and $\|V_j^{-1}V_i'\| = \|V_j^{-1}V_iU\| = \|V_j^{-1}V_i\|$. These imply that both d_1 and d_2 are independent from the choice of eigenvector matrices.

A matrix $A_i \in \mathcal{M}_{\mathbb{C}}^n$ is said to be normal if $AA^* = A^*A$. A set of matrices $\{A_p \mid p \in \mathcal{P}\}$ is said to be simultaneously normalizable if there exist an invertible matrix X such that $X^{-1}A_pX$ is normal for every $p \in \mathcal{P}$ [23, 24]. One of the important properties of normal matrices is that they are unitarily similar to diagonal matrices. See the spectral theorem for normal matrices in [22]. This gives rise to the following lemma.

Lemma 1 *Let $\mathcal{V}_i$ and $\mathcal{V}_j$ be the eigenvector sets of two simultaneously normalizable matrices. Then $d_1(\mathcal{V}_i, \mathcal{V}_j) = d_2(\mathcal{V}_i, \mathcal{V}_j) = 0$.*

Proof. Let A_i and A_j be two simultaneously normalizable matrices. Then there exist an invertible matrix X such that $X^{-1}A_iX = N_i$ and $X^{-1}A_jX = N_j$, where N_i and N_j are normal matrices. From the spectral theorem of normal matrices $N_i = U_iD_iU_i^*$ and $N_j = U_jD_jU_j^*$ hold, where U_i, U_j are unitary and D_i, D_j are diagonal. Then $A_i = XN_iX^{-1} = XU_iD_iU_i^*X^{-1}$ and $A_j = XU_jD_jU_j^*X^{-1}$, which are the diagonalization form of A_i and A_j. Thus XU_i and XU_j are the eigenvector matrices of A_i and A_j, respectively. This implies

$$\begin{aligned} d_1(\mathcal{V}_i, \mathcal{V}_j) &= \ln(\max(\|U_i^{-1}X^{-1}XU_j\|, \|U_j^{-1}X^{-1}XU_i\|)) \\ &= \ln(\max(\|U_i^{-1}U_j\|, \|U_j^{-1}U_i\|)) \end{aligned}$$

and similarly $d_2(\mathcal{V}_i, \mathcal{V}_j) = \ln(\|U_i^{-1}U_j\| \cdot \|U_j^{-1}U_i\|)$. Since product of unitary matrices is unitary and spectral norm of a unitary matrix is equal to one, we conclude that $d_1(\mathcal{V}_i, \mathcal{V}_j) = d_2(\mathcal{V}_i, \mathcal{V}_j) = 0$. ∎

2.3 Digraphs: walk, path and cycle

A *digraph* (directed graph) $\mathcal{D}$ consists of two sets $\mathcal{D} = \{\mathcal{V}, \mathcal{A}\}$, where $\mathcal{V}$ is a non-empty finite set of *vertices* and $\mathcal{A}$ is a finite set of ordered pairs of vertices called *arcs* [25]. $\mathcal{D} = \{\mathcal{V}, \mathcal{A}\}$ is a *complete digraph* if every ordered pairs of elements of $\mathcal{V}$ is included in $\mathcal{A}$. The number of vertices in a digraph is called as the *order* of $\mathcal{D}$. For an arc $a = (u, v)$ the first vertex u is the *tail* of a and v is the *head* of a. A *walk* in $\mathcal{D}$ is an alternating sequence $W : v_1, a_1, v_2, a_2, \ldots, a_{k-1}, v_k$, where $v_i \in \mathcal{V}$ for $i = 1, 2, \ldots, k$, and $a_i \in \mathcal{A}$ for $i = 1, 2, \ldots, k-1$ such that the tail of a_i is v_i and the head of a_i is v_{i+1} for every $i = 1, 2, \ldots, k-1$. A walk W is called as a *path* if all its vertices are distinct and called as a *cycle* if the first $k-1$ vertices are distinct and $v_k = v_1$. The *length* of a walk is the number of its arcs. A walk is said to be *open* if $v_1 \neq v_k$. The following lemma is a well-known result of graph theory [25], so the proof is skipped.

Lemma 2 (Open walk decomposition) *Every open walk can be decomposed into a path and some cycles.*

A weighted digraph is defined by $D = \{\mathcal{V}, \mathcal{A}, \Omega\}$ where $\Omega\colon \mathcal{A} \to \mathbb{R}$ is a weight function. For a weighted digraph the weight of a walk W with length k is defined as

$$\Omega(W) := \sum_{i=1}^{k} \Omega(a_i).$$

3 MAIN RESULT

In this section, some sufficient conditions will be given for the stability and asymptotic stability of the linear switched system (1) where all A_i's have distinct eigenvalues and

$$\mathcal{S} = \mathcal{S}_{\text{dwell}}[\tau] = \{\sigma \in \mathcal{S}_{\text{nonchatt}} \mid \Delta_i(\sigma) \geq \tau\} \tag{6}$$

or

$$\mathcal{S} = \mathcal{S}'_{\text{dwell}}[\tau] = \{\sigma \in \mathcal{S}_{\text{dwell}} \mid \sum_{i=1}^{\infty}(\Delta_i(\sigma) - \tau) \to \infty\}. \tag{7}$$

It is known that (1) with $\mathcal{S} = \mathcal{S}_{\text{nonchatt}}$ is asymptotically stable if the matrices corresponding to subsystems are Hurwitz and commute pairwise [10]. In the case that all A_i's have distinct eigenvalues the commuting condition is equivalent to the condition of eigenvectors laying in the same directions, namely simultaneously diagonalizability of A_i's. It is also known that the stable subsystems whose trajectories look quite different may result in divergent trajectory by switching, even if they are individually stable (see Fig.2 in [3]). Moreover the forms of trajectories of linear systems are mainly determined by eigenvectors (see page 394 in [26]). These give the idea that the distance between eigenvector sets defined by (4) and (5) may have a role in finding the minimum dwell time which guaranties the stability of (1). For the commuting case, where the eigenvector sets are equal, τ would be zero and the value of τ would increase as the distance between eigenvector sets increases. In order to see this let us consider the solution (3). For stability one needs $\|\prod A_i\| < 1$. Using the eigenvalue decomposition $A_i = V_i D_i V_i^{-1}$ and triangular inequality for norms we can find a sufficient condition for stability as $\|V_i V_j^{-1}\| e^{-\lambda_i^* \tau} < 1$, where λ_i^* is the absolute value of the real part of the weakest eigenvalue. Note that, we need here the equality $\|e^{D_i}\| = e^{-\lambda_i^*}$. On the other hand, the terms $\|V_i V_j^{-1}\|$ are related to the previously defined norms in Section 2.2. Thus it is possible to interpret stability in terms of the distance between eigenvectors and the closeness of the weakest eigenvalue to the imaginary axis. The following theorem reveals this result.

Theorem 1 *Consider the switched linear system*

$$\dot{x}(t) = A_\sigma x(t) \ , \quad \sigma\colon [0,\infty) \to \mathcal{P} \in \mathcal{S}_{dwell}[\tau] \ , \quad t \geq 0 \ , \tag{8}$$

where $\mathcal{P}$ is a set of m indices and for every $p \in \mathcal{P}$, A_p has distinct eigenvalues with nonpositive real parts. Let $\mathcal{V}_i$ be the eigenvector set of A_i and $\lambda_i^* = \min_j |Re(\lambda_j(A_i))|$, where $\lambda_j(A)$ is the j'th eigenvalue of A. Then (8) is stable if

$$\tau \geq \max_{j\in\mathcal{P}}(\max_{i\in\mathcal{P}} \frac{d_1(\mathcal{V}_i, \mathcal{V}_j)}{\lambda_j^*}) \tag{9}$$

Proof. Let V_i be an eigenvector matrix of A_i. Then the eigenvalue decomposition of A_i is $A_i = V_i D_i V_i^{-1}$, where D_i is a diagonal matrix whose diagonal entries are the eigenvalues of A_i. Note that $\|e^{D_i}\| = e^{-\lambda_i^*}$. For $t \in [\omega_{i-1}, \omega_i)$, using (3),

$$
\begin{aligned}
\|x(t)\| &= \left\| e^{A_{p_i}(t-\omega_{i-1})} \left(\prod_{k=1}^{i-1} e^{A_{p_k}\Delta_k} \right) x(0) \right\| \\
&= \left\| V_{p_i} e^{D_{p_i}(t-\omega_{i-1})} \left(\prod_{k=1}^{i-1} V_{p_{k+1}}^{-1} V_{p_k} e^{D_{p_k}\Delta_k} \right) V_1^{-1} x(0) \right\| \\
&\leq \|V_{p_i}\| \|V_1^{-1}\| e^{-\lambda^*_{p_i}(t-\omega_{i-1})} \left(\prod_{k=1}^{i-1} \|V_{p_{k+1}}^{-1} V_{p_k}\| e^{-\lambda^*_{p_k}\Delta_k} \right) \|x(0)\|
\end{aligned}
$$

Since $e^{-\lambda^*_{p_i}(t-\omega_{i-1})} \leq 1$,

$$
\begin{aligned}
\|x(t)\| &\leq \|V_{p_i}\| \|V_1^{-1}\| \left(\prod_{k=1}^{i-1} \|V_{p_{k+1}}^{-1} V_{p_k}\| e^{-\lambda^*_{p_k}\Delta_k} \right) \|x(0)\| \\
&= \|V_{p_i}\| \|V_1^{-1}\| \left(\prod_{k=1}^{i-1} e^{-\lambda^*_{p_k}(\Delta_k - \tau) + \ln \|V_{p_{k+1}}^{-1} V_{p_k}\| - \lambda^*_{p_k}\tau} \right) \|x(0)\|
\end{aligned}
$$

By defining $\alpha := \sum_{k=1}^{i-1} \ln \|V_{p_{k+1}}^{-1} V_{p_k}\| - \lambda^*_{p_k}\tau$, $\beta := \prod_{k=1}^{i-1} e^{-\lambda^*_{p_k}(\Delta_k - \tau)} \leq 1$ since $\Delta_k \geq \tau$, and $\gamma := \max_{i,j\in\mathcal{P}}(\|V_i\|\|V_j^{-1}\|) \geq \|V_{p_i}\|\|V_1^{-1}\|$.

$$
\|x(t)\| \leq \beta\, \gamma\, e^{\alpha} \|x(0)\| \tag{10}
$$

$$
\leq \gamma\, e^{\alpha} \|x(0)\| \tag{11}
$$

Now assume that (9) is satisfied. Then

$$
\begin{aligned}
\ln \|V_{p_{k+1}}^{-1} V_{p_k}\| - \lambda^*_{p_k}\tau &= \lambda^*_{p_k} \left(\frac{\ln \|V_{p_{k+1}}^{-1} V_{p_k}\|}{\lambda^*_{p_k}} - \tau \right) \\
&\leq \lambda^*_{p_k} \left(\max_{j\in\mathcal{P}} \left(\max_{i\in\mathcal{P}} \frac{d_1(\mathcal{V}_i, \mathcal{V}_j)}{\lambda^*_j} \right) - \tau \right) \leq 0
\end{aligned}
$$

which implies $\alpha \leq 0$. Thus from (11), $\|x(t)\| \leq \gamma\, \|x(0)\|$ follows and this result does not depend on the chosen interval of t. So (8) is stable. ■

Remark 1 *The condition given by the inequality (9) shows how the minimum dwell time depends on the weakest eigenvalues and the distance between the* *eigenvector sets of subsystem matrices. In synthesis of switched control systems one should decrease the distance between the eigenvectors of subsystems when one wants to allow faster switching.*

Remark 2 *If all* (A_i, A_j) *pairs commute then choosing same eigenvector matrix for both gives* $d_1(\mathcal{V}_i, \mathcal{V}_j) = 0,\ \forall i, j \in \mathcal{P}$ *and (9) implies that the switched system (1) is stable under arbitrary switching. That is, in case of distinct eigenvectors Theorem 1 is more general than the result in [10].*

Remark 3 *In view of Lemma 1, if the set* $\{A_p \mid p \in \mathcal{P}\}$ *is simultaneously normalizable then the system (1) is stable under arbitrary switching when all individual subsystems are stable. One can also see this considering the fact that* $v = x^T x$ *is a common Lyapunov function for normal matrices.*

In Theorem 1 we have made use of the relation between the eigenvector sets of subsystem matrices. For another condition one can benefit from the innate restriction stemming from the correspondence between switching signals and walks in a digraph. While it is easier to calculate the dwell time by Theorem 1, the following theorem may give a smaller dwell time not only due to the abovementioned approach but also due to the different semimetric (5) used instead of (4).

Theorem 2 *Assume that all assumptions of Theorem 1 are satisfied. Let* $\mathcal{P}_2$ *denote the set of all subsets of* $\mathcal{P}$ *with two or more than two elements. Then*

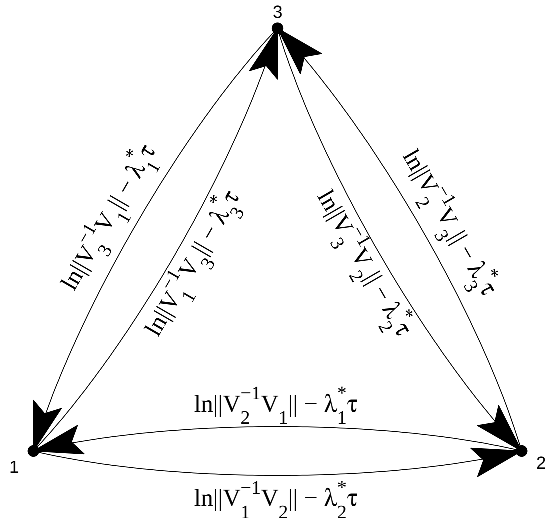

Figure 2: A digraph corresponding to a switched system with $m = 3$

(8) is stable if

$$\tau \geq \mathcal{K} \max_{i,j \in \mathcal{P}} \frac{d_2(\mathcal{V}_i, \mathcal{V}_j)}{\lambda_i^* + \lambda_j^*} \tag{12}$$

or

$$\tau \geq \frac{2\,m - 2}{m} \max_{i,j \in \mathcal{P}} \frac{d_2(\mathcal{V}_i, \mathcal{V}_j)}{\lambda_i^* + \lambda_j^*} \tag{13}$$

where

$$\mathcal{K} = 1 + \max_{\mathcal{I} \in \mathcal{P}_2} \left(\frac{s(\mathcal{I}) - 2}{\sum_{i \in \mathcal{I}} \lambda_i^*} \min_{i \in \mathcal{I}} \lambda_i^* \right) \tag{14}$$

and $m = s(\mathcal{P})$.

Proof. Consider a complete weighted digraph $\mathcal{D} = (\mathcal{V}, \mathcal{A}, \Omega)$ for which there exist a bijective map $p(\cdot)$ from $\mathcal{V}$ to the index set $\mathcal{P}$ (see Fig. 2). By defining the weight of an arc as $\Omega\left((v_i, v_j)\right) = \ln \|V_{p(v_j)}^{-1} V_{p(v_i)}\| - \lambda_{p(v_i)}^* \tau$, the quantity α

in (11) will correspond to the weight of a walk in $\mathcal{D}$. From Lemma 2, this walk can be decomposed into cycles and a path with length of at most $m-1$. Then one can write α defined in the previous proof as $\alpha = \alpha^* + \alpha_2 + \alpha_3 + \cdots + \alpha_m$ where α^* is the weight of the path and α_i is the sum of the weights of cycles with length i for $i = 2, 3, \ldots m$. Since $\mathcal{V}$ is a finite set there exist a number $\bar{\alpha} = \max \alpha^*$, which is the largest weight of all paths in $\mathcal{D}$. Then from (11)

$$\|x(t)\| \leq \gamma\, e^{\bar{\alpha} + \sum_{k=1}^{m} \alpha_k}\, \|x(0)\|.$$

Now we will show that $\alpha_k \leq 0\ \forall k = 2, \ldots, m$.

Consider a cycle $C_k : v_1, a_1, v_2, a_2, \ldots, v_k, a_k, v_1$ in $\mathcal{D}$, where k is the length of the cycle. Let $\mathcal{I} = \{p(v_1), p(v_2), \ldots, p(v_k)\}$, $\delta = \arg\min_{i \in \mathcal{I}} \lambda_i^*$. Then the weight of the cycle is

$$\begin{aligned}
\Omega(C_k) &= \sum_{i=1}^{k-1} \Omega\left((v_i, v_{i+1})\right) + \Omega\left((v_k, v_1)\right) \\
&= \sum_{i=1}^{k-1} \ln \|V_{p(v_i)}^{-1} V_{p(v_{i+1})}\| + \ln \|V_{p(v_k)}^{-1} V_{p(v_1)}\| - \tau \sum_{i \in \mathcal{I}} \lambda_i^* \\
&= \sum_{i=1}^{k-1} \ln \|V_{p(v_i)}^{-1} V_\delta V_\delta^{-1} V_{p(v_{i+1})}\| + \ln \|V_{p(v_k)}^{-1} V_\delta V_\delta^{-1} V_{p(v_1)}\| - \tau \sum_{i \in \mathcal{I}} \lambda_i^* \\
&\leq \sum_{i \in \mathcal{I}} \left(\ln \|V_i^{-1} V_\delta\| + \ln \|V_\delta^{-1} V_i\| \right) - \tau \sum_{i \in \mathcal{I}} \lambda_i^* \\
&\leq \sum_{i \in \mathcal{I}, i \neq \delta} d_2(V_i, V_\delta) - \tau \sum_{i \in \mathcal{I}} \lambda_i^*
\end{aligned}$$

Assume (12) is satisfied, then eliminating the term $d_2(V_i, V_\delta)$

$$\begin{aligned}
\Omega(C_k) &\leq \tau \sum_{i \in \mathcal{I}, i \neq \delta} \frac{\lambda_i^* + \lambda_\delta^*}{\mathcal{K}} - \tau \sum_{i \in \mathcal{I}} \lambda_i^* \\
&= \frac{\tau}{\mathcal{K}} \left(\sum_{i \in \mathcal{I}} \lambda_i^* + \left(s\left(\mathcal{I}\right) - 2\right) \lambda_\delta^* \right) - \tau \sum_{i \in \mathcal{I}} \lambda_i^* \qquad (15)
\end{aligned}$$

From (14)

$$\mathcal{K} \geq \frac{\sum_{i\in\mathcal{I}} \lambda_i^* + (s(\mathcal{I}) - 2)\lambda_\delta^*}{\sum_{i\in\mathcal{I}} \lambda_i^*} \tag{16}$$

Substituting (16) in (15) gives $\Omega(C_k) \leq 0$. Then $\alpha_k \leq 0$, which implies that (8) is stable.

Now assume that (13) is satisfied. Using $\min_{i\in\mathcal{I}} \lambda_i^* \leq \left(\sum_{i\in\mathcal{I}} \lambda_i^*\right)/s(\mathcal{I})$ and the equation (14),

$$\mathcal{K} \leq 1 + \max_{\mathcal{I}\in\mathcal{P}_2} \frac{s(\mathcal{I}) - 1}{s(\mathcal{I})}$$

can be obtained. Since $(x-2)/x$ is an increasing function

$$\mathcal{K} \leq \frac{2m-2}{m}$$

follows. Thus, (12) is satisfied and the switched system (8) is stable. ■

Remark 4 *The condition (12) can give a smaller dwell time than the condition (9). However it is difficult to apply this condition to get a result on dwell time, because one has to do heavy calculations in order to determine $\mathcal{K}$. On the other hand, the condition (13) is much easier to apply, whereas it gives slightly larger value for the minimum dwell time. Both conditions give the same result if $m = 2$ since in this case $\mathcal{K} = \frac{2m-2}{m} = 1$.*

The conditions (9), (12), and (13) imply asymptotic stability when a slightly different set of switching signal is used and all the individual subsystems are assumed to be asymptotically stable. This result is given by the following theorem.

Theorem 3 *Consider the switched linear system*

$$\dot{x}(t) = A_\sigma x(t) \ , \quad \sigma\colon [0,\infty) \to \mathcal{P} \in \mathcal{S}'_{dwell}[\tau] \ , \quad t \geq 0 \ , \tag{17}$$

where $\mathcal{P}$ is a set of m indices and members of the family $\{A_p \mid p \in \mathcal{P}\}$ are Hurwitz matrices with distinct eigenvalues. Let $\mathcal{V}_i$ be the eigenvector set of A_i and $\lambda_i^* = \min_j |Re(\lambda_j(A_i))|$, where $\lambda_j(A)$ is the j'th eigenvalue of A. Then (17) is asymptotically stable if one of the conditions (9), (12), and (13) is satisfied.

Proof. Again (10) is satisfied. Since $\mathcal{S}'_{\text{dwell}}[\tau] \subset \mathcal{S}_{\text{dwell}}[\tau]$ (9), (12), or (13) implies $\alpha \leq 0$ due to the previous theorems. Thus, from (10)

$$\|x(t)\| < \beta\,\gamma\,\|x(0)\| \ , \quad t \in [\omega_{i-1}, \omega_i).$$

From the definition of $\mathcal{S}'_{\text{dwell}}[\tau]$ (see (7)) $\beta \to 0$ as $i \to \infty$. Consequently $\|x(t)\| \to 0$. ■

4 Illustrative Examples

In this section we are going to illustrate our results with examples and compare them with the ones in literature. In the following we will briefly present the already existing results to remind.

In [15], where it is proved that the switched system (8) is stable for sufficiently large τ's, the minimum dwell time is given by

$$\tau_{\min} = \max_{i\in\mathcal{P}} \inf_{\alpha>0,\beta>0} \left\{ \frac{\alpha}{\beta} \mid \|e^{A_i t}\| < e^{(\alpha-\beta t)}, \ \ \forall t \geq 0 \right\}. \tag{18}$$

In [27] this result was extented to the switched systems with average dwell time, which is a generalized concept of the dwell time. In this case $\mathcal{S} = \mathcal{S}_{\text{ave}}[\tau_D, N_0]$, where N_0 is called chatter bound. Although this method is applicable in a much more general setting such as in presence of noise and parameter uncertainties, it overlaps the dwell time approach when $N_0 = 0$. Using this method the minimum dwell time can be found as

$$\tau_{\min} = \max_{Q,\bar{Q}\in\mathcal{Q}} \frac{\ln\left(\sigma_{\max}[Q]\right) - \ln\left(\sigma_{\min}[\bar{Q}]\right)}{2\lambda_0} \tag{19}$$

where λ_0 is a positive number satisfying that all $A_p + \lambda_0 I$, $p \in \mathcal{P}$ are Hurwitz stable, $\mathcal{Q}$ is the set of matrices Q_p satisfying

$$Q_p(A_p + \lambda_0 I) + (A_p + \lambda_0 I)^T Q_p = -I \ , \ p \in \mathcal{P}$$

and $\sigma_{\max}$ ($\sigma_{\min}$) is the largest (smallest) singular value. A recent work [18] proposed a method by which the minimum dwell time can be found much smaller. Using multiple Lyapunov functions technique they have given linear matrix inequality conditions which imply that the value of the Lyapunov functions at switching times gets smaller. Their method based on increasing $\tau_{\min}$ gradually from zero and checking each time whether some linear matrix inequalities are satisfied, which is difficult to perform especially when the number of subsystems or the dimension of each subsystem is increased.

Example 1 *Consider the switched system (8) with* $\mathcal{P} = \{1, 2\}$*, where* $A_1 = \left(\begin{smallmatrix} -0.2 & 5 \\ 1 & -0.3 \end{smallmatrix}\right)$ *and* $A_2 = \left(\begin{smallmatrix} -0.4 & -1 \\ 5 & -0.6 \end{smallmatrix}\right)$*. For these matrices* $\lambda_1^* = 0.25$*,* $\lambda_2^* = 0.5$ *and the distance between eigenvector sets are calculated as* $d_1(\mathcal{V}_1, \mathcal{V}_2) = 0.8056$ *and* $d_2(\mathcal{V}_1, \mathcal{V}_2) = 1.6104$*. Using Theorem 1, the minimum dwell time is estimated as* $\tau_{\min} = 3.2223$*, whereas by Theorem 2,* $\tau_{\min}$ *is obtained as* 2.1472*.*

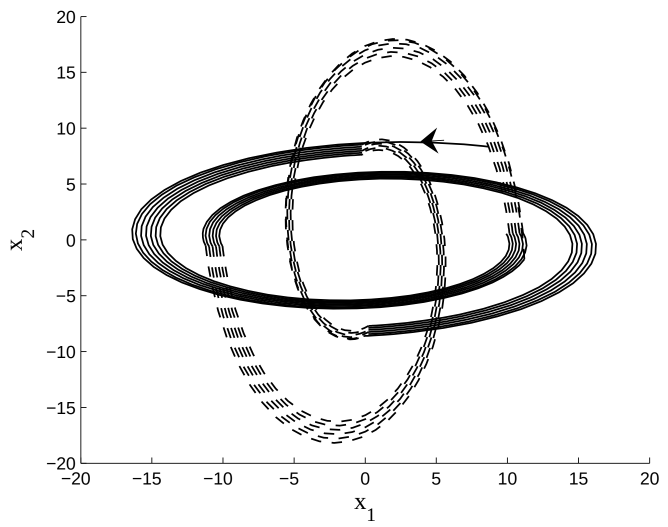

Figure 3: A solution of the switched system in Ex. 1 (solid and dashed lines correspond to the solution for $p = 1$ and $p = 2$, respectively.)

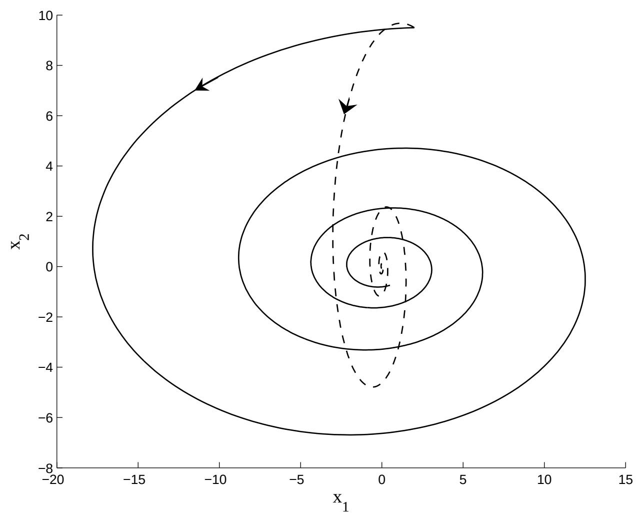

Figure 4: Trajectories of subsystems in Ex. 1 (solid and dashed lines correspond to the solution for $p = 1$ and $p = 2$, respectively.)

The trajectories of the subsystems given in Example 1 is sketched in Figure 4 and a solution obtained by switching these subsystems with a periodic switching signal with period $\tau_{\min} = 2.1472$ is illustrated in Figure 3. For this example the condition (18) taken from [15] gives $\tau_{\min} = 3.24$ and the average dwell time method in [27] results in $\tau_{\min} = 8.51$, which are larger than our results. On the other hand, using the method given in [18] we obtain $\tau_{\min} = 2.1384$, which is slightly better than our results. During our excessive trials carried out by randomly creating stable subsystems we observed that the method in [18] always gives the best result whereas the condition (12) in Theorem 2 is always the second best amongst the other methods. The following example illustrates how the minimum dwell time becomes smaller when the distance between eigenvector sets is decreased.

Example 2 *Consider the switched system (8) with* $\mathcal{P} = \{1, 2\}$*, where* $A_1 = \left(\begin{smallmatrix} -0.2 & -5 \\ 1 & -0.3 \end{smallmatrix}\right)$ *and* $A_2 = \left(\begin{smallmatrix} -0.2 & -5 \\ 2 & -0.3 \end{smallmatrix}\right)$*. For these matrices* λ_1^* *and* λ_2^* *are equal to 0.25 and the distance between eigenvector sets are found as* $d_1(\mathcal{V}_1, \mathcal{V}_2) = 0.2696$ *and* $d_2(\mathcal{V}_1, \mathcal{V}_2) = 0.3467$*. Using Theorem 1 the minimum dwell time is found as* $\tau_{\min} = 1.0785$ *and by Theorem 2 it is found as* 0.6934*.*

For this example the method in [18] gives again a better result $\tau_{\min} = 0.6394$, whereas from (18) and (19) we have estimated the values $\tau_{\min} = 3.24$ and $\tau_{\min} = 3.2333$, respectively, which are obviously larger than our results. In Example 2 the distance between eigenvector sets is smaller than the distance obtained for the system in Example 1. This can also be followed by comparing Figure 4 and Figure 6. Thus the last example also demonstrates the fact that

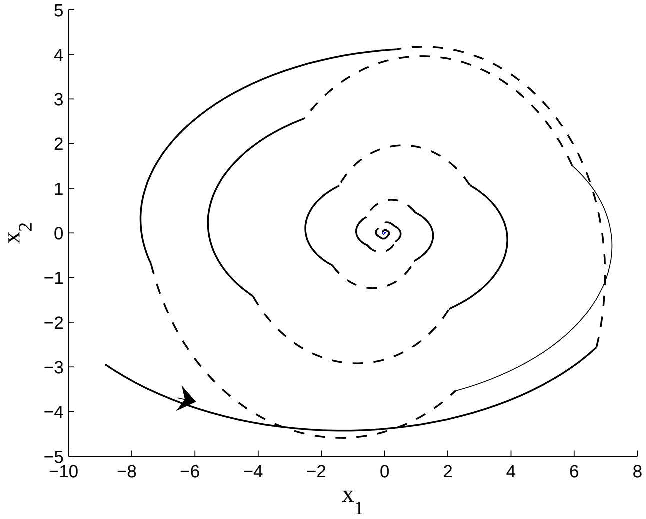

Figure 5: A solution of the switched system in Ex. 2 (solid and dashed lines correspond to the solution for $p = 1$ and $p = 2$, respectively.)

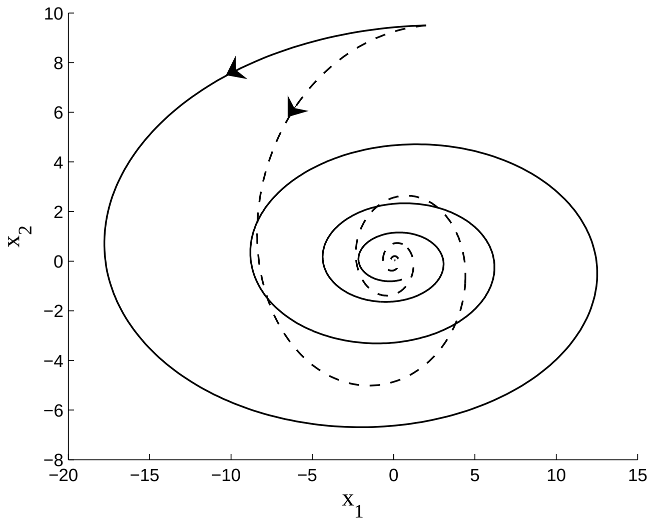

Figure 6: Trajectories of subsystems in Ex. 2 (solid and dashed lines corresponds to the solution for $p = 1$ $p = 2$, respectively.)

if the distance between eigenvector sets is reduced then it is possible to choose $\tau_{\min}$ smaller and obtain a stable switched system. Figure 5 illustrates a solution of the switched system in Example 2, where switching signal is a periodic signal with period $\tau_{\min} = 0.6934$.

5 Conclusion

In this work a method to obtain the minimum dwell time for switched linear systems is given. The method, besides being easy to use, gives a bound on dwell time for the stability of switched linear systems and can give a dwell time smaller than the ones obtained by the already existing methods. The important aspect of the conditions obtained by the proposed method and stated by (9) in Theorem 1 and by (12) and (13) in Theorem 2 is their explicitness. These explicit conditions clearly expose the dependence of the minimum dwell time on the subsystem properties. So, one can easily follow that to reduce the distance between eigenvector sets of subsystems or to shift the weakest eigenvalues to left in complex plane will decrease the minimum dwell time.

The method of the previous studies, such as [15] and [27], considered each subsystem seperately. First they derived a dwell time for each subsystem guaranteeing contraction on the state space and then take the maximum of them. In this work we considered the interaction between the subsystems, which is interpreted by the distance between eigenvector sets. We believe that to benefit from the distance between eigenvectors makes it possible to obtain a smaller dwell time.

Even though in this work stable subsytems are taken into account, the proposed method can also be extended to give conditions for different cases, such as switching between stable and unstable systems. In this case the dwell time should be bounded from above for unstable subsystems, whereas it remaines bounded from below for stable subsystems. Moreover the set of admissible switching signals should exclude the ones that switches only among unstable subsytems.

The proposed method gives the condition in [10] as a special case. Thus, the condition for diagonalizable matrices can be easily obtained considering the distance between eigenvector sets. On the contrary, the proposed method does not comprehend the result given on the triangularizability of subsystem matrices in [11] as a special case. One way to extend the result given here in order to include the triangularizability condition as a special case when τ is equal to zero could be to consider the distance between invariant subspaces of subsystem matrices instead of using the semimetrics defined for eigenvector sets. This approach could further give a better result on the minimum dwell time.

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References

[1] D. Liberzon. *Switching in Systems and Control.* Birkhauser, Boston, 2003.

[2] Z. Sun and S.S. Ge. *Switched Linear Systems: Control and Design.* Springer-Verlag, London, 2005.

[3] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

[4] R.A. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proceedings of the IEEE*, 88(7):1069–1082, 2000.

[5] J.P. Hespanha. Uniform stability of switched linear systems: extensions of LaSalle's invariance principle. *IEEE Trans. on Auto. Cont.*, 49(4):470–482, 2004.

[6] G. Xie and L. Wang. Periodical stabilization of switched linear systems. *Journal of Comp. and Appl. Math*, 181:176–187, 2005.

[7] X. Xu and P.J. Antsaklis. Stabilization of second-order lti switched systems. *Int. J. Control*, 73(14):1261–1279, 2000.

[8] Z. Sun. Stabilizability and insensitivity of switched linear systems. *IEEE Trans. on Auto. Cont.*, 49(7):1133–1137, 2004.

[9] M. Margaliot. Stability analysis of switched systems using variational principles: An introduction. *Automatica*, 42:2059–2077, 2006.

[10] K.S. Narendra and J. Balakrishnan. A common Lyapunov function for stable LTI systems with commuting a-matrices. *IEEE Trans. on Auto. Cont.*, 39(12):2469–2471, 1994.

[11] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common Lyapunov function theorem for continuous-time systems. In *Proc. of the 36. Conf. on Decision and Control*, pages 3530–3531, San Diego, California, 1997.

[12] D. Liberzon, J.P. Hespanha, and A.S. Morse. Stability of switched systems:a lie algebraic condition. *Systems and Control Letters*, 37(3):117–122, 1999.

[13] A.A. Agrachev and D. Liberzon. Lie-algebraic stability criteria for switched systems. *Siam J. on Contr. Optim.*, 40(1):253–269, 2001.

[14] Z. Sun and R. Shorten. On convergence rates of simultaneously triangularizable switched linear systems. *IEEE Trans. on Auto. Cont.*, 50(8):1224–1228, 2005.

[15] S. Morse. Supervisory control of families of linear set-point conrollers-part 1: exact matching. *IEEE Trans. on Auto. Cont.*, 41(10):1413–1431, 1996.

[16] J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. In *Proc. of the 38. Conf. on Decision and Control*, pages 2655–2660, Phoenix, Arizona, 1999.

[17] G. Zhai, B. Hu, K. Yasuda, and A.N. Michel. Stability analsis of switched systems with stable and unstable subsystems:an average dwell time approach. *Int. J. System Science*, 32(8):1055–1061, 2001.

[18] J.C. Geromel and P. Colaneri. Stability and stabilization of continuous-time switched linear systems. *Siam J. on Contr. Optim.*, 45(5):1915–1930, 2006.

[19] Ö. Karabacak. Stability of switched linear systems (in turkish). Master's thesis, Istanbul Technical University, Inst. of Sci. and Tech., 2007.

[20] H. Ye, A.N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. on Auto. Cont.*, 43(4):461–474, 1998.

[21] W.J. Rugh. *Linear System Theory*. Prentice Hall, New Jersey, 1996.

[22] R.A. Horn. *Matrix Analysis*. Cambridge University Press, New York, 1985.

[23] C. Pommer and W. Kliem. Simultaneously normalizable matrices. *Linear Algebra and its Applications*, 94:113–125, 1987.

[24] M. Thathachar and P. Viswanath. On the stability of fuzzy systems. *IEEE Trans. Fuzzy Systems*, 5(1):145–151, 1997.

[25] J.B. Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer-Verlag, London, 2000.

[26] C.A. Desoer, L.O. Chua, and E.S. Kuh. *Linear Nonlinear Circuits*. McGraw-Hill, Singapore, 1987.

[27] J.P. Hespanha and A.S. Morse. Stability of switched systems with average dwell-time. Technical report, Electrical Engineering Dep., University of Southern California, 1999.

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