

A Dynamic Asset Allocation Model with Downside Risk Control *

Yonggan Zhao and William T. Ziemba

University of British Columbia

Vancouver, B.C. Canada V6T 1Z2

September 26, 1999

Abstract

This paper presents a new stochastic control model for investment. The investors' objective is to maximize the expected growth rate while controlling for downside risk. Assuming lognormally distributed prices, the strategy that determines the optimal dynamic portfolio weights by changing *risk neutral excess rate* is determined by a stochastic differential equation. The maximum loss can be limited almost surely. A constrained optimization model is developed given investors' preference on the minimum subsistence reward among all possible scenarios. The relative changes in the expected terminal wealth, minimum subsistence and the risk aversion are studied. Taking VaR as the risk measure, the return/risk tradeoff efficient frontier is constructed. A comparison of the downside risk control model for a typical example to Buy and Hold (BH) and Fixed Mix (FM) strategic asset allocation models shows that the downside risk control model has superior performance in the return/VaR framework

§1 Introduction

Intertemporal asset allocation models in continuous time have generally been modeled as stochastic control problems. One must find implementable optimal policies in each period as a function of the current and past state observations to optimize investment objectives. Merton (1969, 1971) and Karatzas and Shreve (1986), analyze this problem using complete market assumptions and a dynamic programming approach. Expected utility theory can be used by investors to trade off profits in good future “states” against the losses in the bad “states”. Using simple utility function

*This research was supported by the National Science and Engineering Research Council of Canada.

assumptions and asset price processes, closed form solution are available for some of these models. However, for investors with liability streams, explicit risk control approaches, such as minimum floor, VaR, etc., seem more appropriate. MacLean and Ziemba (1992, 1999) discuss the tradeoff between growth and security. Zhao and Ziemba (1999) introduce a reward function on the portfolio worst case payoff which represents investors' risk aversion to downside losses in a discrete time model. The portfolio target is an endogenous choice variable determined by the risk aversion and investment opportunities. This risk measure arises from an option strategy for asset allocation; see Cariño and Turner (1998).

A problem is how to introduce liabilities. Mean-variance models are unable to model them in a natural way. One may buy a put option on an initial portfolio with strike price K . Another way is synthesizing a put option struck at K with expiry at time T . In asset and liability models, the objective of maximizing the expected return is triggered purely by the *ex ante* choice of the target, regardless of whether this choice is consistent with asset evolution or not. The optimization problem maximizes the expected growth less a penalty if the target is not met, as in the Russell-Yasuda Kasai model (Cariño, Ziemba, et al (1994, 1998(ab)) in discrete time. Since different economic situation may influence the choice of minimum subsistence, the allocation strategies may change correspondingly. In this paper we use the certainty reward function that determines the choice of the target K endogenously through a special control policy given the asset price processes.

Value at Risk (VaR) , see Figure 1, is the maximum potential change in value of a portfolio at a preset confidence interval (probability $1 - p$) over a pre-set horizon. Investors maximize the expected return given their VaR at some certain risk level. This strategy is very plausible, however, it has received severe criticism, see e.g. Artzner et al (1999), Jorion (1996), and Koedijk, Huisman and Pownall (1998). Can we do better? Since VaR only deals with the decisions that are within some confidence level, what if the investor cannot afford the loss caused by extreme events, even though the probability of losing is small; and should not the penalty from the loss be

non-decreasing? Therefore a minimum subsistence level is important to institutional investors. The investor may wish to maximize K as well as an overall potential payoff for all scenarios at the end of planning horizon. We develop a strategy that guarantees a target payoff almost surely and potential upward return when assets jointly follow a Geometric Brownian Motion in a complete market. For a typical example, this downside risk control strategy performs better than both Buy and Hold and Fixed Mix strategies in the framework of return/risk tradeoff, where risk is quantified by the VaR and the criterion for comparison is the expected return subject to the risk constraint.

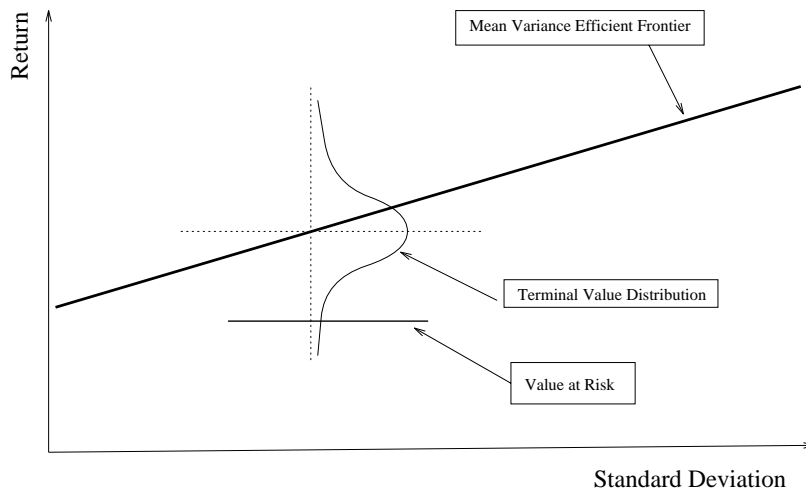


Figure 1: Mean Variance Optimal Portfolio with VaR

§2 Financial Market and Dynamics of Portfolio value

Assume that there are no injections and withdrawals of funds during the planning horizon. Thus via a self-financing strategy so that the opening value of the portfolio at time $t+1$ is the closing value at time t net of transaction costs. Frictions in financial markets change qualitatively the nature of the optimization faced by an investor. It requires one to either act or do nothing, an issue that does not arise in frictionless situations. In continuous time models, transaction costs are extremely difficult to

model because the dynamic control models assume continuous revisions of portfolio weights. For technical reasons, it is assumed that there are no transaction costs. Zhao and Ziemba (1999) focus on this type of computation with transaction costs in a discrete time framework.

Suppose the investment consists of n risky assets and a riskless asset, which can be continuously traded. Assume the financial market is defined by the stochastic differential equations

$$\begin{aligned} \text{riskless asset:} \quad & dX_0(t) = X_0(t)r dt, & X_0(0) = 1 \\ \text{risky assets:} \quad & dX_i(t) = X_i(t)b_i dt + X_i(t) \sum_{j=1}^m \sigma_{ij} dB_j(t) \end{aligned} \quad (1)$$

where $X_0(t)$ and $X_i(t), i = 1, 2, \dots, n$ are the price of riskless and risky assets at time t , respectively, $B(t) = (B_1(t), B_2(t), \dots, B_m(t))'$ is an m -dimensional Brownian motion. The riskless rate r , the instant rates b_i and the volatilities σ_{ij} of assets are assumed to be constants throughout the planning horizon.

An investor with endowment $W(0)$ and an objective function must decide how much wealth $\pi_i(t)$ to be allocated to each of the available asset based on the information observed by time t . The objective function is discussed in the next section. Here we describe the wealth dynamics and asset price evolution. Let $\{\mathcal{F}_t\}$ be the natural filtration generated by $(B(s))_{0 \leq s \leq t}$ which represents the available market information at time t , so, the portfolio $\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_n(t))'$ and the risky asset prices $X(t) = (X_1(t), X_2(t), \dots, X_n(t))'$ are \mathcal{F}_t measurable, which means that decisions are non-anticipative. Denote $b = (b_1, \dots, b_n)'$, $\sigma = (\sigma)_{n \times m}$. There is no arbitrage if and only if there exist some $\theta = (\theta_1, \dots, \theta_m)'$, the market price of risk, such that $\sigma\theta = b - r\mathbf{1}$. It is assumed that this condition is satisfied for model (1).

Let $W(t)$ be the wealth at time t . Then

$$W(t) = \pi_0(t) + \pi_1(t) + \dots + \pi_n(t) = \pi_0(t) + \pi(t)'\mathbf{1}. \quad (2)$$

Let $N_0(t)$ and $N(t) = (N_1(t), \dots, N_n(t))'$ be the number of shares held in the riskless

asset and the n risky assets at time t , respectively. Let

$$I_X = \text{diag}\{X_1(t), \dots, X_n(t)\},$$

then $\pi_0(t) = X_0(t)N_0(t)$ and $\pi(t) = I_X \cdot N(t)$. By Itô's formula

$$d\pi(t) = I_{dX}N(t) + I_X dN(t) + I_{dX} dN(t). \quad (3)$$

With no transaction costs, the self-financing strategy can be characterized by the stochastic differential equation

$$(X_0(t) + dX_0(t))dN_0(t) + (I_X(t) + I_{dX}(t))'dN(t) = 0. \quad (4)$$

Combining (3) and (4) yields

$$\begin{aligned} dW(t) &= d\pi_0(t) + d\pi(t)' \mathbf{1} = N_0(t)dX_0(t) + N(t)'dX(t) \\ &= \pi_0(t)rdt + N(t)'I_X(t)(bdt + \sigma dB(t)) \\ &= W(t)r dt + \pi(t)'[(b - r\mathbf{1})dt + \sigma dB(t)]. \end{aligned} \quad (5)$$

The controlled wealth process can be represented as (cf, Karatzas and Shreve (1998))

$$e^{-rt}W(t) = W(0) + \int_0^t e^{-rs}\pi(s)'[(b - r\mathbf{1})ds + \sigma dB(s)]. \quad (6)$$

Let $\tilde{B}(t) = B(t) + \theta t$. Under the Equivalent Martingale Measure Q defined by

$$Q(A) = \int_A \exp(-\theta' B(T) - \frac{1}{2}\theta'\theta T)dP \quad \forall A \in \mathcal{F}_T,$$

$\tilde{B}(t)$ is a Brownian motion and we have the discounted Q -Martingale

$$e^{-rt}W(t) = W(0) + \int_0^t e^{-rs}\pi(s)'\sigma d\tilde{B}(s), \text{ for } 0 \leq t \leq T \quad (7)$$

which describes a wealth process given a control policy $\pi(s), 0 \leq s \leq t$, at time t .

§3 Model Formulation and Solution

In asset/liability management models, the investment objective is usually chosen to maximize the expected portfolio value less penalties for targets not met; see Carriño,

Ziemba et al (1994, 1998ab) in discrete time. This model is equivalent to a piecewise linear concave utility maximization problem if the penalty is a piecewise linear convex function of the shortfall. The continuous time version of these models are not popular in the literature so far. We look for an alternative approach to achieve risk aversion without using a penalty function. In most asset allocation models, maximizing expected asset value is a primary objective, but the dispersion among scenarios brings large potential losses to the portfolio.

How do we control this risk? One way as in a mean-variance model is to adjust the expected value by a measure of dispersion. Technically, it is easy to handle, especially for the static model, but this lacks control power, because the measure chosen are mostly based on the first two moments and the potential large losses still exist. The VaR approach has been implemented to address this issue. We utilize a new approach to measure risk – *reward on minimum subsistence*.

Definition 1 For an \mathcal{F}_t measurable random variable Y , the certainty payoff of Y is defined to be

$$Y^L = \sup\{k \in \mathfrak{R}; \Pr\{Y \geq k\} = 1\}, \quad (8)$$

that is, Y^L is the essential lower bound of Y . If Y is unbounded below, then Y^L is $-\infty$

As an extension of utility based models, investors are also concerned about the minimum subsistence $W^L(T)$ of investment at the end of horizon. Assigning a reward to $W^L(T)$, characterized by a concave increasing function f , defined on $\mathfrak{R} \cup \{\infty\}$, determines the preference or risk aversion between the expected return and the minimum subsistence in all scenarios. The dynamics of the portfolio value is derived in (7). The stochastic control model is

$$\begin{aligned}
& \max_{\pi(t) \in \mathcal{U}_t} E(W(T)) + f(W^L(T)) \\
& \text{s.t. } e^{-rt}W(T) = W(0) + \int_0^t e^{-rs}\pi(s)' \sigma d\tilde{B}(s), \forall t \in [0, T].
\end{aligned} \tag{9}$$

Before discussing the solvability of the model, we discuss some of its properties.

Definition 2 A function $f(x)$ is called superior to $g(x)$, denoted by $f(x) \succeq g(x)$, if $f(x) \geq g(x)$ and $f'(x) \geq g'(x)$, where primes denote the first order derivatives.

Theorem 1 Let $f(x)$ and $g(x)$ be concave increasing functions, and $W_f(T), W_f^L$ and $W_g(T), W_g^L$ be the corresponding optimal solutions to (9).

(i) If $f(x) \succeq g(x)$, then $W_f^L(T) \geq W_g^L(T)$, $E(W_f(T)) \leq E(W_g(T))$ and $E(W_f(T)) + f(W_f^L(T)) \geq E(W_g(T)) + g(W_g^L(T))$.

(ii) If $h(x) = \eta f(x) + (1 - \eta)g(x)$, $0 < \eta < 1$, then

$$\begin{aligned}
E(W_h(T)) + h(W_h(T)) & \leq \eta(E(W_f(T)) + f(W_f^L(T))) \\
& \quad + (1 - \eta)(E(W_g(T)) + g(W_g^L(T))).
\end{aligned}$$

Proof (i) We suppress the time ‘‘T’’ in the proof. By optimality,

$$\begin{aligned}
E(W_f) + g(W_f^L) & \leq E(W_g) + g(W_g^L) \\
E(W_g) + f(W_g^L) & \leq E(W_f) + f(W_f^L).
\end{aligned} \tag{10}$$

Hence,

$$f(W_g^L) - f(W_f^L) \leq E(W_f) - E(W_g) \leq g(W_g^L) - g(W_f^L). \tag{11}$$

Since $f'(x) \geq g'(x), \forall x \in \mathfrak{R}$, i.e. $f(x) - g(x)$ is an increasing function, so

$$W_f^L \geq W_g^L.$$

Using (10) and the fact that $g(x)$ is an increasing function yields

$$E(W_f) \leq E(W_g).$$

If $f(x) \geq g(x)$ and $f'(x) \geq 0$, then

$$E(W_g) + g(W_g^L) \leq E(W_g) + f(W_g^L) \leq E(W_f) + f(W_f^L). \quad (12)$$

(ii). Let $h(x) = \eta f(x) + (1 - \eta)g(x)$, then

$$h(W_h^L) = \eta f(W_h^L) + (1 - \eta)g(W_h^L).$$

By optimality,

$$\begin{aligned} E(W_f) + f(W_f^L) &\geq E(W_h) + f(W_h^L) \\ E(W_g) + g(W_g^L) &\geq E(W_h) + g(W_h^L). \end{aligned} \quad (13)$$

So,

$$\eta(E(W_f) + f(W_f^L)) + (1 - \eta)(E(W_g) + g(W_g^L)) \geq E(W_h) + h(W_h^L). \quad (14)$$

Theorem 1 indicates how the two objectives, expected growth and the certainty reward, are related when the function f changes. The optimal expected terminal value $E(W(T))$, optimal certainty payoff $W^L(T)$, and the optimal value function $E(W(T)) + f(W^L(T))$ are decreasing, increasing and increasing, respectively, as function f 's superiority increases. The optimal value function $E(W(T)) + f(W^L(T))$ is convex in f in the sense that the optimal value function is defined over the set of concave increasing functions.

Solving (9) is not as easy, in general, as solving a stochastic control problem in which we can choose the utility function form. The problem is the irregularity of the objective function in the state variables (the portfolio values and asset prices). For example, $f(W^L(T))$ is not a smooth function of the state space. $W^L(T)$ is the infimum of the probability support of $W(T)$. If the control space consists of all admissible Markovian policies, then it is hard to write down the HJB equation. The problem is not solvable using stochastic programming theory alone. Therefore, we have to apply a numerical method to approximate the solution. However, if we are primarily concerned that there is a "good" strategy that meets our objective, then we can reduce the difficulty of the problem by restricting our control space.

Solving model (1) yields

$$X_i(t) = X_i(0)e^{\sigma_i \tilde{B}(t) + (r - \frac{1}{2}\sigma_i \sigma_i')t}$$

which implies that

$$\sigma_i \tilde{B}(t) = \ln \left(\frac{X_i(t)}{X_i(0)} \right) - \left(r - \frac{1}{2}\sigma_i \sigma_i' \right) t \quad (15)$$

where σ_i is the i th row of σ .

The quantity $\ln \left(\frac{X_i(t)}{X_i(0)} \right) + \frac{1}{2}\sigma_i \sigma_i' t$ is the risk compensated return of asset i at time t . The expected value of the risk compensated rate of return is equal to the instantaneous rate of return for each asset in the setting of (1). The difference between the risk compensated rate of return and the riskless rate is called the *risk neutral excess rate*, since its expected value under the risk neutral probability is 0. The term $\frac{1}{2}\sigma_i \sigma_i' t$ can be interpreted as the risk premium for transferring a gamble. To control downside loss, we require that the discounted relative changes of the control to the initial wealth be proportional to the changing risk neutral excess rate in the same asset. The stochastic control $\pi(\cdot) = (\pi_1(\cdot), \pi_2(\cdot), \dots, \pi_n(\cdot))'$ satisfies the stochastic differential equations

$$d\pi_i(t) = r\pi_i(t)dt + \alpha_i W(0)e^{rt} \sigma_i d\tilde{B}(t), \quad \alpha_i \geq 0. \quad (16)$$

Definition 3 A stochastic control $\pi(\cdot)$ is called a *risk neutral excess rate control* if it satisfies Equation (16). Let \mathcal{U}_l denote the set of all such controls.

Theorem 2 For the standard complete market (1), there exists an optimal control $\pi(\cdot) \in \mathcal{U}_l$ that solves (9).

Proof Equation (16) implies that there exist a constant vector $\beta = (\beta_1, \dots, \beta_n)'$ and a diagonal matrix $I_\alpha = \text{Diag}(\alpha_1, \dots, \alpha_n), \alpha_i \geq 0$ such that

$$e^{-rt}\pi(t) = W(0)I_\alpha(\sigma \tilde{B}(t) + \beta). \quad (17)$$

The wealth equation becomes

$$\begin{aligned}
e^{-rt}W(t) &= W(0) + W(0) \int_0^t (\tilde{B}(s)' \sigma' + \beta') I_\alpha \sigma d\tilde{B}(s) \\
&= W(0) + W(0) \int_0^t (\tilde{B}(s)' \sigma' I_\alpha \sigma + \beta' I_\alpha \sigma) d\tilde{B}(s) \\
&= W(0) \left(1 + \frac{1}{2} \tilde{B}(t)' \sigma' I_\alpha \sigma \tilde{B}(t) - \frac{1}{2} \text{tr}(\sigma' I_\alpha \sigma) t + \beta' I_\alpha \sigma \tilde{B}(t) \right)
\end{aligned} \tag{18}$$

where $\text{tr}()$ is the trace function. $W^L(T)$ can be determined from

$$\begin{aligned}
e^{-rT}W^L(T) &= \inf_{y \in \mathbb{R}^n} \left\{ W(0) \left(1 + \frac{1}{2} y' \sigma' I_\alpha \sigma y - \frac{1}{2} \text{tr}(\sigma' I_\alpha \sigma) T + \beta' I_\alpha \sigma y \right) \right\} \\
&= \inf_{y \in \mathbb{R}^n} \left\{ W(0) \left(1 - \frac{1}{2} \text{tr}(\sigma' I_\alpha \sigma) T \right) + \frac{1}{2} (\sigma y + \beta)' I_\alpha (\sigma y + \beta) - \frac{1}{2} \beta' I_\alpha \beta \right\} \\
&= W(0) \left(1 - \frac{1}{2} \text{tr}(\sigma' I_\alpha \sigma) T - \frac{1}{2} \beta' I_\alpha \beta \right).
\end{aligned} \tag{19}$$

The expected value $E(W(T))$ can be calculated as

$$\begin{aligned}
e^{-rT}E(W(T)) &= W(0) + W(0)E \left(\int_0^T (\tilde{B}(t)' \sigma' + \beta') I_\alpha \sigma \theta dt \right) \\
&= W(0) + W(0) \int_0^T (\sigma \theta)' t + \beta' I_\alpha \sigma \theta dt \\
&= W(0) \left(1 + \frac{1}{2} (\sigma \theta)' I_\alpha (\sigma \theta) T^2 + \beta' I_\alpha (\sigma \theta) T \right) \\
&= W(0) \left(1 + \frac{1}{2} (b - r\mathbf{1})' I_\alpha (b - r\mathbf{1}) T^2 + \beta' I_\alpha (b - r\mathbf{1}) T \right).
\end{aligned} \tag{20}$$

If f is twice differentiable, solving

$$\begin{aligned}
\sup_{I_\alpha \in \mathbb{R}^n \times \mathbb{R}^n, \beta \in \mathbb{R}^n} \left\{ W - 0e^{rT} \left(1 + \frac{1}{2} (b - r\mathbf{1})' I_\alpha (b - r\mathbf{1}) T^2 + \beta' I_\alpha (b - r\mathbf{1}) T \right) \right. \\
\left. + f \left(W_0 \left(1 - \frac{1}{2} \text{tr}(\sigma' I_\alpha \sigma) T - \frac{1}{2} \beta' I_\alpha \beta \right) \right) \right\}
\end{aligned} \tag{21}$$

yields the optimal α and β

Theorem 2 provides a direct way of implementing this strategy. By assumption, σ is invertible, hence, there is a one to one corresponding relation between the realized asset prices and therefore the realized wealth, and the underlying state of the world. Assuming lognormality, the asset prices can be represented as a function of the underlying Q - Brownian motion.

The strategy given in Theorem 2 can be dynamically implemented by observing current market prices. We can use any available asset pricing model to value and predict future asset returns, namely, appropriate probability distribution for asset future movements. In a lognormal world, this amounts to “calculating” the instantaneous mean rate vector b and the volatility matrix σ for a given riskless interest rate. Then we can implement this model.

Example An investor has an endowment of \$1 and a certainty reward function $f(x) = -2e^{-\frac{1}{2}x}$. There are two assets, one riskless and one risky (the S&P 500). Market parameter estimates are $b = 0.15, \sigma = 0.20$. It is assumed that these parameters are fixed and stationary through the planning horizon $T = 1$ (a year). The riskless rate for this period is $r = 0.05$. The first order condition for (21) becomes

$$(b - r)T - \exp(-\frac{1}{2}x) \cdot \beta = 0$$

$$(b - r)^2T + 2\beta(b - r) - \exp(-\frac{1}{2}x) \cdot (\sigma^2T + \beta^2) = 0.$$

where $x = 1 - \frac{1}{2}\alpha\sigma^2 - \frac{1}{2}\alpha\beta^2$. Substituting in the market parameters yields

$$\alpha = 3.3734 \quad \text{and} \quad \beta = 0.1562. \quad (22)$$

Hence, the investor has $\alpha\beta = 52.8\%$ of the wealth in the risky asset, and the balance in the riskless asset. After the initial investment, the portfolio has to be “continuously” adjusted by observing the stock price. Using (15) computes $\sigma\tilde{B}(t)$, and the portfolio holdings $\pi(t)$ at time t . We simulate a sample of size 100. The maximum loss is about 7.4% while it can achieve a possible maximum profit 95.15% and an expected return of 12.1%. We compare these results to those using two common strategic asset allocation methods, namely, Buy and Hold and Fixed Mix. Buy and Hold allocates wealth at the beginning of the planning horizon and holds the portfolio to the end. There are no transactions except that dividends are reinvested in the same asset. A Fixed Mix strategy allocates wealth according to a preset investment policy such that each asset represents an identical proportion of the wealth at the beginning of

each period. Our model is superior to these two strategies with this data by several statistics. Both of strategic models will lose a maximum 10.5% and 10.2% and the possible upward gains are 48.61% and 44.93%, respectively. Their means are 11.3% and 11.1% with this data.

§4 Comparisons of Strategies under VaR

Using a reward function for risk aversion makes it easy to calculate the optimal policy, but an appropriate function is difficult to find. We focus now on constructing return/risk efficient frontiers under the risk measure, Value at Risk (VaR), for different investment strategies.

Buy and Hold and Fixed Mix strategies are two representative strategic asset allocation models. In these models, risk is defined to be the volatility (standard deviation) of the portfolio at the planning horizon. The optimal asset allocation is determined under a framework of the risk/return tradeoff that is to maximize the expected return given a certain level of risk (volatility). Performance is measured by the Sharpe ratio which is valid assuming that the portfolio return from these two strategies are normally (or lognormally) distributed. Leland (1999) shows that the Sharpe ratio or the mean-variance measure is not suitable when investors have a dynamic model with skewed investment portfolios. The VaR has been popularly implemented as a measure of risk for investment management. A simple version of VaR is *the market loss that is not exceeded at a given confidence interval*. If W is the portfolio payoff at the end of planning horizon, then the VaR of W at confidence level $1 - \alpha$ is

$$VaR(W) = E(W) - \sup\{K \in \mathfrak{R}; \Pr[W \leq K] \leq \alpha\}. \quad (23)$$

This measure provides a new approach to deal with downside risk. The probability that the portfolio will lose more than the VaR is α . Methods of calculating VaR based on the mean-variance approximation are not suitable when the asset prices follow a multivariate geometric Brownian motion, because the returns (arithmetic

or geometric) of dynamic portfolios are generally neither normally nor lognormally distributed. Furthermore, for a positively skewed curve (the density function), the variance is mainly contributed by the overperformance at the right tail which should not be considered as “risk” at all. See Hull and White (1998) and Koedijk et al (1998) for discussions of VaR estimation. We analytically derive the VaRs for the three asset allocation strategies: Buy and Hold, Fixed Mix, and Downside Risk Control. The Downside Risk Control strategy is superior to both Buy and Hold and Fixed Mix strategies in this return/risk framework.

§4.1 Calculation of the Mean and Volatility

Expected return and volatility are directly related to the calculation of VaR, so we need to know how to compute them for a specific policy.

Buy and Hold Strategy

Buy and Hold is static in the sense that assets are allocated at the beginning period and held until the end of the horizon without transactions. Let $W_{BH}(t)$ be the value of the portfolio at time t and u be the proportion of the total wealth allocated to the risky asset at the beginning, then the terminal payoff W_{BH} is

$$\begin{aligned} W_{BH} &= W_{BH}(0)(1-u)e^{rT} + W_{BH}(0)u \cdot \frac{X(T)}{X(0)} \\ &= W_{BH}(0)\left((1-u)e^{rT} + u e^{\sigma B(T) + (b - \frac{1}{2}\sigma^2)T}\right). \end{aligned} \tag{24}$$

The expected final value $E(W_{BH})$ and the volatility $V(W_{BH})$ are

$$\begin{aligned} E(W_{BH}) &= W_{BH}(0)\left((1-u)e^{rT} + u \cdot \frac{E(X(T))}{X(0)}\right) \\ &= W_{BH}(0)e^{rT}(1 + u(e^{(b-r)T} - 1)) \\ V(W_{BH}) &= W_{BH}(0)u V(e^{\sigma B(T) + (b - \frac{1}{2}\sigma^2)T}) \\ &= W_{BH}(0)u e^{bT} \sqrt{(e^{\sigma^2 T} - 1)}. \end{aligned} \tag{25}$$

Fixed Mix Strategy

The Fixed Mix strategy requires one to rebalance the portfolio “continuously”. Investors preset an appropriate portfolio mix u among the asset categories. Then, portfolio weights are balanced by selling assets with high returns to buy assets with low returns; the opposite of portfolio insurance. Let $W_{FS}(t)$ be the portfolio value at time t . The wealth Equation (6) becomes

$$dW_{FM}(t) = r W_{FM}(t)dt + W_{FM}(t) u \sigma d\tilde{B}(t).$$

The terminal payoff W_{FM} is

$$\begin{aligned} W_{FM} &= W_{FM}(0)e^{u\sigma\tilde{B}(T)+(r-\frac{1}{2}u^2\sigma^2)T} \\ &= W_{FM}(0)e^{u\sigma B(T)+u(b-r)T+(r-\frac{1}{2}u^2\sigma^2)T}. \end{aligned} \quad (26)$$

The expected value $E(W_{FM})$ and the volatility $V(W_{FM})$ of the terminal portfolio are

$$\begin{aligned} E(W_{FM}) &= W_{FM}(0)e^{(ub+(1-u)r)T} \\ V(W_{FM}) &= W_{FM}(0) V(e^{v\sigma B(t)+v(b-r)T+(r-\frac{1}{2}v^2\sigma^2)T}) \\ &= W_{FM}(0)e^{vbT+(1-v)rT} \sqrt{(e^{v^2\sigma^2T} - 1)}. \end{aligned} \quad (27)$$

Downside Control

From Equation (18), we can deduce the terminal wealth W_{DC} for the two asset case,

$$W_{DC}(T) = e^{rT} W_{DC}(0) (1 + \alpha(\frac{1}{2}(\sigma\tilde{B}(T))^2 - \frac{1}{2}\sigma^2T + \beta\sigma\tilde{B}(T))). \quad (28)$$

Using stochastic isometry for Itô's integral yields the expected terminal value $E(W_{DC})$ and its volatility $V(W_{DC})$

$$\begin{aligned} E(W_{DC}(T)) &= e^{rT} W_{DC}(0) (1 + \frac{1}{2}\alpha(b-r)^2T^2 + \alpha\beta(b-r)T). \\ V(W_{DC}(T)) &= e^{rT} W_{DC}(0) \alpha \sqrt{\left(\frac{1}{2}\sigma^4T^2 + \sigma^2((b-r)T + \beta)^2T\right)}. \end{aligned} \quad (29)$$

All terminal portfolio values of the three strategies are expressed in terms of normal random variables. Using (15) and (28) yields the portfolio return in terms of the gross asset returns

$$\begin{aligned}
R_{BH} &= (1 - u)e^{rT} + uR_X \\
R_{FM} &= e^{(1-u)(r+\frac{1}{2}u\sigma^2)T} (R_X)^u \\
R_{DC} &= e^{rT} \left(1 + \frac{1}{2}\alpha((\ln R_X)^2 + (2\beta - 2r + \sigma^2) \ln R_X \right. \\
&\quad \left. + \frac{1}{4}\sigma^4 T^2 - 2r\sigma^2 T^2 - \frac{1}{2}\sigma^2 T + r^2 T^2 - rT) \right).
\end{aligned} \tag{30}$$

With the parameters from the example, the relation between portfolio returns and the asset returns are given in Figure 2, where the Value at Risk tolerance is 20%.

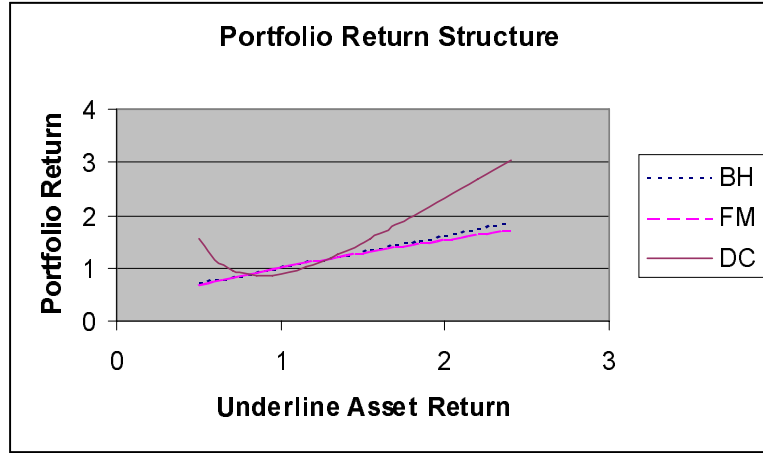


Figure 2: The structure of the portfolio returns

The following proposition is implied from (30).

Proposition 1 *For BH, FM and DC strategies, the returns of the terminal portfolios are linear, concave and convex in terms of the value of the risky asset return, respectively.*

§4.2 Calculation of the VaR

While VaR is accepted by many practitioners, the calculation of an exact VaR has posed a formidable task for a given investment policy; see e.g. Hull and White (1998).

Unlike mean-variance, there does not exist a uniform and analytic way of calculating the VaR, even for typical distributions, such as, the normal and lognormal. Monte Carlo simulation provides a statistical approximation. With the assumed distribution of the “uncertainty” (the simplest being normality), a large sample of the portfolio value is generated which then gives the VaR by finding the left tail cutoff value for the $1 - \alpha$ confidence interval. The following proposition reduces the complexity of calculation

Proposition 2 *Let $VaR(X)$ denote the value at risk of a random variable X . If $f(x)$ is monotonic on \mathfrak{R} , then*

$$VaR(f(X)) = E(f(X)) - f(E(X) - VaR(X)). \quad (31)$$

Proof By definition

$$\begin{aligned} VaR(f(X)) &= E(f(X)) - \sup\{K \in \mathfrak{R}; \Pr[f(X) \leq K] \leq \alpha\} \\ &= E(f(X)) - \sup\{f(f^{-1}(K)); \Pr[X \leq f^{-1}(K)] \leq \alpha, K \in \mathfrak{R}\} \\ &= E(f(X)) - f(\sup\{f^{-1}(K); \Pr[X \leq f^{-1}(K)] \leq \alpha, K \in \mathfrak{R}\}) \\ &= E(f(X)) - f(E(X) - VaR(X)). \end{aligned} \quad (32)$$

Proposition 2 reduces the computation of VaR of $f(X)$ given the distribution of X . This is equivalent to the so called portfolio mapping which maps a complicated portfolio to a simplest form such as a normal distribution.

If we have to resort to a numerical solution of $VaR(X)$, what would be the impact on the calculation of $VaR(f(X))$ caused by the approximation error? This introduces an interesting topic for numerical analysis. The ratio of the estimation errors between the portfolio value $f(X)$ and its building element X can be approximated using numerical analysis techniques. However, Proposition 2 does not always apply. For example it does not, for the Downside Risk Control model. This strategy results in a random payoff which is a quadratic function of a normal distribution. To calculate the “exact” VaR, we start from the definition to calculate the density function. Proposition 3 yields the exact formula for the density function of W_{DC} .

Proposition 3 Let $f(x) = ax^2 + bx + c$, $a > 0$, and X a random variable with density function $\phi(x)$. Then, the density function $\phi_f(x)$ of $f(X)$ is given by

$$\phi_f(x) = \frac{\phi\left(\frac{-b+q(x)}{2a}\right) + \phi\left(\frac{-b-q(x)}{2a}\right)}{q(x)} \quad (33)$$

where $q(x) = \sqrt{4a(x-c) + b^2}$, $x > \frac{4ac-b^2}{4a}$.

Proof By definition, the accumulative function of $f(X)$ is

$$\begin{aligned} \Phi_f(x) &= \Pr[f(X) \leq x] = \Pr[aX^2 + bX + c \leq x] \\ &= \Pr\left[\left(X + \frac{b}{2a}\right)^2 \leq \frac{4a(x-c) + b^2}{4a^2}\right] \\ &= \begin{cases} \Pr\left[\left(X + \frac{b}{2a}\right)^2 \leq \frac{q(x)^2}{4a^2}\right] & \text{if } x > \frac{4ac-b^2}{4a}, \\ 0 & \text{if } x \leq \frac{4ac-b^2}{4a} \end{cases} \\ &= \begin{cases} \Pr\left[\frac{-b-q(x)}{2a} \leq X \leq \frac{-b+q(x)}{2a}\right] & \text{if } x > \frac{4ac-b^2}{4a}, \\ 0 & \text{if } x \leq \frac{4ac-b^2}{4a}. \end{cases} \end{aligned} \quad (34)$$

Hence, the density function of $f(X)$ can be found by taking the derivative $\Phi_f(x)$ with respect to x which verifies (33). Figure 3 shows the relations of the density functions of the portfolio returns determined by the three strategies. The shape of the density function for the Downside Risk Control model shows that portfolio is better off when the market ends much oscillatory. Compared with traditional portfolio insurance, the Downside Risk Control will not only guarantee a floor but also an upward return if market is going down even further. This is equivalent to a straddle option strategy.

Calculation of VaR “exactly” involves finding the left tail cutoff at the confidence level $1 - \alpha$. This amounts to solving Equation (35) for K if X is a continuous random variable

$$\int_{-\infty}^K \phi_f(x) dx = \alpha. \quad (35)$$

With the support of a computer package, such as Maple or Matlab, which can manipulate the symbolic operations and provides efficient numerical solutions, the “exact”

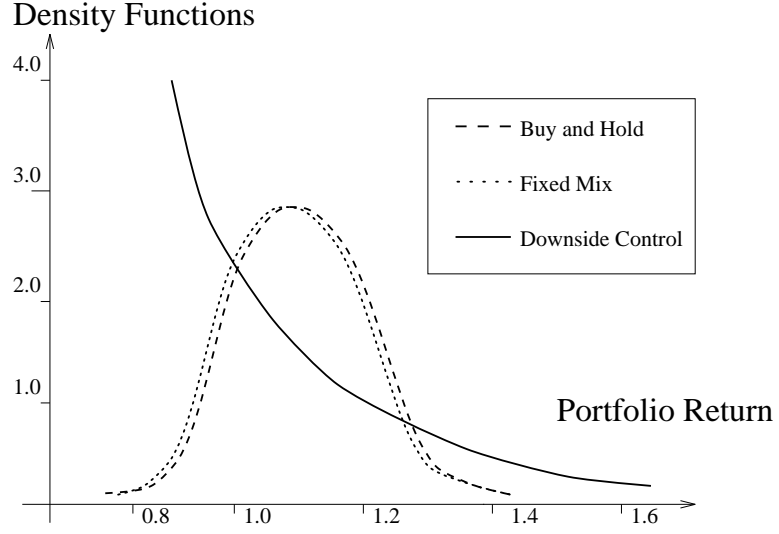


Figure 3: The density functions

VaR can be obtained. The value at risk of X is

$$VaR(X) = E(X) - K. \quad (36)$$

§4.3 The Risk/Return Efficient Frontier

Since the calculation of a given portfolio VaR proposes a major challenge for investors, the inverse problem of developing an optimal strategy given a VaR , v , is even more difficult. This problem requires the solution of the following constrained stochastic control model

$$\begin{aligned} & \max_{\pi(\cdot) \in \mathcal{U}_t} E(W_T) \\ & s.t. \quad e^{-rT} W_T = W_0 + \int_0^T e^{-rt} \pi(t) \sigma d\tilde{B}(t) \\ & \quad \quad VaR = v \end{aligned} \quad (37)$$

The constraint $VaR = v$ is equivalent to the following probability constraint by the definition of VaR ,

$$\Pr[E(W_T) - W_T \leq v] = p.$$

Using equations (16), (35) and (37), we can obtain the portfolio VaR as a function of α and β for the downside control strategy. Then, solving the optimization model (37) yields the optimal policy. This approach requires considerable computation. Here we provide a heuristic method:

- Simulate a large sample path (size 1000) of the Brownian motion $\tilde{B}(t)$;
- Calculate the VaR as the p -quantile of the portfolio for each strategy;
- Solve the optimization model to obtain the optimal α and β .

Figure 4 provides the return/risk efficient frontier for market parameters ($b = 0.15, \sigma = 0.2, r = 0.15$) and the risk level chosen to be the VaR at the 95% confidence interval. From Figure 4, we observe the DC strategy outperforms the BH and FM

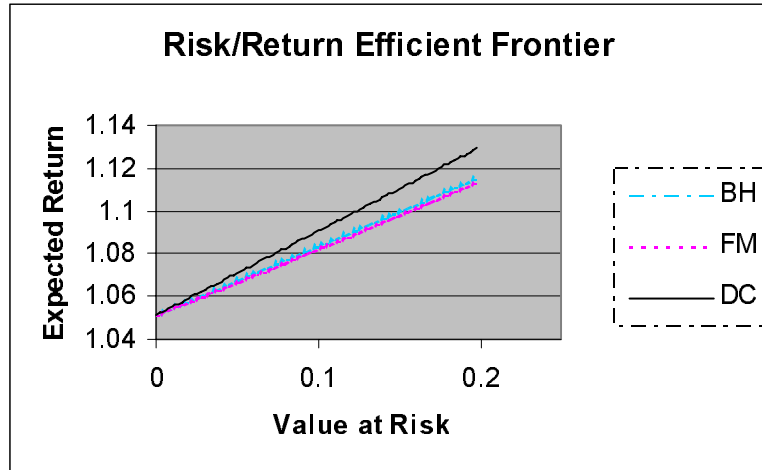


Figure 4: The Risk/Return Efficient Frontier

strategies for all risk levels if the measure of performance is defined similar to the Sharpe Ratio as

$$\frac{E(W_T) - W_0 e^{rT}}{\text{VaR}(W_T)}. \quad (38)$$

While the BH strategy could lose all investment in the risky assets and the FM strategy could lose all the fortune, The Downside Risk Control strategy can guarantee 85% of the initial wealth at the end of horizon with this data.

§5 Conclusions

We have investigated the control of downside risk using a simple asset allocation strategy which requires portfolio dynamic shifts proportional to the change of the risk neutral excess rate. Within a continuous time framework and the assumption of lognormality for stock prices, the downside control model can be used very efficiently. Investors can successfully achieve a return above some floor that meets their liability requirements and, at the same time, upward potentials can be achieved.

Under VaR, comparisons among Buy and Hold (BH), Fixed Mix (FM) and Downside Risk Control (DC) strategies showed that the DC strategy is superior to both BH and FM strategies for a typical example when the asset prices follow a multi-dimensional Geometric Brownian Motion.

The Downside Control is related to Portfolio Insurance in the sense that both strategies “guarantee” a floor of return. However, the Downside Risk Control strategy requires a more intensive portfolio dynamic change which is reflected by the choice of large α , not like the Δ -strategy in traditional portfolio insurance. By examining the portfolio payoff function in terms of the stock prices, we found that the Downside Risk Control strategy is similar to a straddle option strategy.

References

- [1] P. ARTZNER, F. DELBAEN, J.-M. EBER, AND D. HEATH, *Coherent measures of risk*, *Mathematical Finance*, 9 (1999), pp. 203–228.
- [2] D. CARIÑO, D. KENT, D. MYERS, C. STACY, M. SYLVANUS, A. TUNER, K. WATANABLE, AND W. ZIEMBA, *The Russell-Yasuda Kasai model: an asset liability model for a japanese insurance company using multistage stochastic programming*, *Interfaces*, 24 (1994), pp. 29–49.
- [3] D. CARIÑO, D. MYERS, AND W. ZIEMBA, *Concepts, technical issues, and uses of the Russell-Yasuda Kasai financial planning model*, *Operations Research*, 46

- (1998), pp. 449–462.
- [4] D. CARIÑO AND A. TURNER, *Multiperiod asset allocation with derivative assets*, in *World Wide Asset and Liability Modeling*, W. Ziemba and J. Mulvey, eds., Cambridge University Press, 1998, pp. 182–204.
 - [5] D. CARIÑO AND W. ZIEMBA, *Formulation of the Russell-Yasuda Kasai financial planning model*, *Operations Research*, 46 (1998), pp. 433–449.
 - [6] J. COX AND C. HUANG, *Optimal consumption and portfolio policies when asset prices follow a diffusion process*, *Journal of Economic Theory*, 49 (1989), pp. 33–83.
 - [7] J. EVNINE AND R. HENRIKSSON, *Asset allocation and options*, *Journal of Portfolio Management*, 14 (1987), pp. 56–61.
 - [8] W. HARLOW, *Asset allocation in a downside-risk framework*, *Financial Analysts Journal*, 47 (1991), pp. 28–40.
 - [9] J. HULL AND A. WHITE, *Incorporating volatility updating into the historical simulation method for value-at-risk*, *Journal of Risk*, 1 (1998), pp. 5–19.
 - [10] P. JORION, *Risk2: Measuring the risk in value at risk*, *Financial Analysts Journal*, 52 (1996), pp. 47–56.
 - [11] I. KARATZAS, J. LEHOCZKY, AND S. SHREVE, *Optimal portfolio and consumption decisions for a “small investor” on a finite horizon*, *SIAM J. Control & Optimization*, 25 (1987), pp. 1557–1586.
 - [12] I. KARATZAS AND S. E. SHREVE, *Methods of Mathematical Finance*, Springer, 1998.
 - [13] I. KARATZAS, *Optimization problems in the theory of continuous trading*, *SIAM J. of Control and Optimization*, 27 (1989), pp. 1221–1259.
 - [14] K. KOEDIJK, R. HUISMAN, AND R. POWNALL, *Var-x: Fat tails in financial risk management*, *Journal of Risk*, 1 (1998), pp. 47–61.

- [15] H. LELAND, *Beyond mean-variance: performance in a nonsymmetrical world*, Financial Analysts Journal, 55 (1999), pp. 27–36.
- [16] L. MACLEAN AND W. ZIEMBA, *Growth versus security tradeoffs in dynamic investment analysis*, Annals of Operations Research, 85 (1999), pp. 193–225.
- [17] L. MACLEAN, W. ZIEMBA, AND G. BLANZENKO, *Growth versus security in dynamic investment analysis*, Management Science, 38 (1992), pp. 1562–1585.
- [18] Y. ZHAO AND W. ZIEMBA, *Creating synthetic option strategies for asset allocation with transaction costs using multi-period stochastic programming*, forthcoming in Mathematical Programming.
- [19] W. T. ZIEMBA AND J. MULVEY, eds., *World Wide Asset and Liability Modeling*, Princeton University Press, 1998.