A Dynamic Equilibrium $\mathbb{Q}$ Probability Measure for Poisson Yield Spreads – review of European Corporate Bonds

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Abstract

This paper presents a new model for term risk, yield curve, and credit risk in spreads in a unified approach. The originality lies in the structuring of the Poisson stochastic of risk in a form suitable for finding the differential equation for the yield curve and its spreads as the Poisson Yield Spread Model (PYSM). A new $\mathbb{P}$ to $\mathbb{Q}$ change in measure is found for the purpose of parameterizing the stochastic component of the yield curve, based on a frequency specified version of the single event Poisson process. The PYSM determines the behaviour of discount rates and yield spreads over EU debt risk extremes.

Keywords: yield term structure, arbitrage-free Poisson, forwards-spread model, interest rate AR(1) theory.

JEL Classification: C32, D53, E43, G13

1. Introduction

Recent financial crises have highlighted important challenges both theoretically and empirically in the modelling of prices in interest rate and swap markets, when models have generated much larger discrepancies than previously observed (Beirne, [12]; Acharya et al. [1]). Our current study of the reaction of the EU yield curves over the Lehman's crash and the Greek crisis and beyond is one example of seeking new information on market dynamics, and a re-appraisal of an interest rate theoretical model more appropriate to extreme market movements.

The main research question investigates the fundamental model of the term risk pricing kernel and how the formulation proposed here solves for the failure of the existing Wiener mean-reversion model (Barra et al., [9]). Based on the Wiener-Khintchine theorem for equilibrium correlation spectral dynamics (Khintchine; Champeney [45];[23]), we are able to add more theory on how to define the time dependence of the risk premium, or in scientific terms the phase delay in a term risk transform. This study explores how to specify a new martingale measure Radon Nikodym $d\mathbb{Q}/d\mathbb{P}$ differential for AR(1) driven processes. Moreover, it draws attention on how the multi-pathway probabilities in martingale $\mathbb{Q}$ probability-space exist and how they normalise to probability 1, therefore to fulfil the uniqueness property of a new AR(1) based martingale.

To quote an example of where development is needed, theory has now to interpret why important benchmark rates and systems are no longer simply correlated, as discussed by

Mercurio [51]. Moreover, it is important to establish why one factor versions of the yield curve in discount or forward rates are no longer as reliable for prediction as they were before the crash. Early literature found that the differing credit or liquidity premia are related to differing tenors and maturities if the yield curves are required to be fully quantified. (Crepey et al. [28][27]; Zopounidis, Doumpos & Kosmidou []; Moreni & Pallavicini [54]). On the empirical plane, interest rate markets have expanded rapidly in futures, forwards and interest rate swaps to exceed $400 trillion in open interest (ISDA Survey 2017) to put pressure on the adequacy of models to price instruments reliably and precisely. A mistake of a few basis points may translate to a very large value correction, or in wider perspective a market instability. Since the pricing of an interest rate derivative depends ultimately on the accuracy of the yield (or spread) the instrument underlying, this places increased importance on developing the required interest rate model. Many early literatures show that the discrepancies in pricing models for interest rates and derivatives encourage the greater use of no-arbitrage models, including the (Ho-Lee [42]; Hull-White, [43]; Black-Derman-Toy [14]; Black-Karasinski [15]). Furthermore, the discrepancies remaining seem fundamentally related to tail risk (Cont [24]), requiring asymmetric probability or jump risk types of movement (Wright & Zhou [68]. Most models accommodating these effects assume that the shocks due to jumps are so large that they are naturally non-systematic, as proposed in the classical model for jump-diffusion by Merton [53].

The opposing argument to which we subscribe is that, no matter how disjointed the variations in markets, there should be a systematic risk behaviour behind them especially as they grow to current size. This suggest that existing models are anomalous, but in not having the correct martingale, rather than not fitting the Wiener Ito martingale. This study focuses on a Poisson event distribution, applied to the yield curve and its spreads. This new modelling system proposed here is called the Poisson Yield Spread Model (PYSM). This paper is organized as follows. In section 2, we provide a review of the theoretical literature on the analytical stochastic modelling of the yield and spreads in comparison with the exercise of a new distribution. In section 3, we derive the model and specify the data and methodology used to characterize the newly determined yield curve structure and its factors, to be revealed in the new PYS Model. Section 4 is devoted to econometrically testing the new model and its stochastic distribution meaning. Section 5 concludes and discusses implications.

2. Theoretical Literature- Issues and perspectives

2-1 Limitations of current Yields Structures and Spread models

Before embarking on the derivation of a new equilibrium model, we first review the limitations of existing equilibrium interest rate models to define the premium for term risk, and how existing theory seeks to remedy the lack of systematic term risk. Ideally, the preferred system of conditional pricing should entail an equilibrium model composed of the requisite probability distribution of the process, with factors added which are calibrated to the constraint conditions in a given contract. Existing equilibrium models follow a set of stochastic models which seek to describe the behaviour of the spot rate curves, e.g. the series of models starting with Vasicek [65], Cox, Ingersoll & Ross [26]. These comprise Wiener diffusion conditionality with a mean reversion to account for the pull to par. Forward rate models are then built-up by an expectation assumption from the spot rate dynamic. The adoption of these models for term structure of yields of greater accuracy has focused on multi-factor versions for example the Duffie, Pan & Singleton [37] affine term structure models, in which diffusion, jump and volatility processes have been nested and transform solutions developed. Although the affine dynamic term structure modelling through its high universality and mathematical integrity is considered strong enough to be canonical, this is difficult to confirm. It is nonetheless convenient to use this concept as a primary point of reference for existing models when comparing PYS Model performance with the existing model.

Although extensive research has already been made on the causation of the term structure of yields (Dai and Singleton 2000 [30]; Duffee [34]; Duffee & Kan [36], a number of theoretical limitations are still recognized in existing models built on current Wiener dynamic principles and mean-reversion dynamics. The risk neutral dynamics of short-term interest rates when specified by particular functional forms in the above standard models underestimate the amounts investors expect for bearing interest rate risk. Assuming the state variables are independent and affine, the dynamic term structure models allow price estimates to be made reasonably simply in a multivariate framework, with the bond price yields responding linearly. Existing empirical results indicate a number of departures from the above modelling assumptions, which are too important to ignore. These have been reviewed in earlier literature founded by Dai & Singleton [30] that non-zero correlations may occur between the state variables. These authors seek to resolve this by allowing state variables to correlate to some extent. Furthermore, Duffee [34] finds that the standard market price of risk function underestimates the excess returns possible in bonds. Duffee [34] has allowed for a more general form for the dynamics of the price of risk. Duffie & Kan [36] further consolidate these models in the class of multi-factor affine term structure models. Such models are attractive not least since they are tractable and readily applied. However, though they include many of the features thought necessary of term structure of mean reversion, stochastic behaviour of the short rate, time dependence of the risk premia such models can still largely underestimate the amount and time dependence of risk.

Hence, when the standard conventional interest rate fails, there are two possible routes. Either mechanisms are added which model behavioural theory of investor behaviour. Or the alternative is to use the technique of arbitrage-free pricing by the expedient of assuming the quoted prices already reflect true market prices of market risk (conditionally in the future), and then to use these to represent the market risk dynamic extended to more complex instruments. The continuing existence of gaps in time dependent risk premia behaviour has led to the search for models that should be able to explain structural biases in a more explicit agent behavioural fashion (Duarte [33]), for example for liquidity preference, habit preferences, or conditionality in biases. To these, the influence of exogenous variables and mechanisms may also be added to the actions of the above equilibrium stochastic models. To illustrate these mechanisms, Campbell & Cochrane [21] argue a consumption-based explanation, Wachter [66] develop a habit preference formation for time-varying premia due to intertemporal substitution, Rudebusch & Swanton [61] use the preferred habit formation to explain the bond premium within the DSGE modelling framework Rotemberg & Woodford [62]. Piazzesi & Scheinder [57] define a model of subjective expectations and adaptive learning with time-varying risk premia, which can partially explain the expected excess.

The value of the PYSM is that its Poisson Q distribution should be specified *a priori* in both its time *t* and maturity *T* dimensions as $\mathrm{d}L(t,T)$, over the yield curve as a whole. The shape changes in the yield curve determined in time *t* and over maturities in time *T* can then be distinctively identified. By varying time and keeping *T* effects constant at an instant in time, we can observe the movement of the whole of the yield curve due to the *T* correlations effects over current time *t*, as for example the credit environment changes. Varying maturity at a given instant in time measures the structure the yield curve, which parameterises the nature of maturity risk at a given instant. Assuming the PYSM fixes the maturity time behaviour in dL, T correlations variations should not interact by cross-correlation with any of the constant *t* effects. Once the structure of the yield curve and the change in spreads over time are determined, the market price of term risk $\gamma(t,T)$ should be determined as independent effects in t and T, according to the structures we later examine in section 3 on: PYSM, Empirical Data and Graphical Analysis.

2.1 Market price of term risk in interest rate products

The main purpose of this study is to combine the new term risk model of interest rates and spreads (in the PYSM) with the methodology for structuring the yield curve. Since we seek an equilibrium model not a market replication price solution, the methodology we adopt in this paper is to always to compute in the first instance the zero-discount curve, and then to obtain the forward rate curves for pricing instruments when needed. One reason we use this sequence stems from the greater reliability we believe the bootstrapped zero curve offers compared with bootstrapping the forward rate from swaps or LIBOR driven prices (Brigo, Mercurio & Morini [19]). There are arguments on whether zero or forward rates as starting point on both sides. The recent debate as to whether the basic interest rate should be changed e.g. LIBOR replaced say with overnight swaps rates, favours our approach to keep to the zero curves under fundamental modelling rules.

Unlike in the Black Scholes Merton [16],[52] approach for stochastic pricing, the stochastic theory of interest rates cannot be derived by constructing a risk neutral portfolio of cash instruments. Interest rates are not physical entities. This presents the generic problem in interest rate theory that the market price of risk is not an observable but constructed quantity. The Wiener stochastic treatment in the Cox, Ingersoll & Ross [26] CIR model for interest rates as a square root mean reversion Wiener diffusion readily illustrates this effect, in the form:

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}.\, d\widetilde{W}$$

A risk neutral solution can be obtained but only by introducing an explicit term for the market price of risk $\gamma(t)$ to obtain risk neutral solution in the form:

$$dr = (\kappa(\theta - r) - \gamma\sigma)dt + \sigma\sqrt{r}.\, d\widetilde{W}$$

The purpose of $\gamma(t)$ is then to measure the amount of real rate to neutral rate adjustment per unit of risk in equation 2.1

$$\gamma(t) = (\mu(t, T) - r(t))/\sigma(t) \qquad (2.1)$$

where $\mu(t, T)$ is drift of the bond price, t is current time, T is maturity time, σ is the standard deviation of bond price as a Wiener diffusion dW, and $r(t; T)$ is the spot rate in the discount models, or the forward rate as a total yield curve in Heath, Harrow, Merton [39] HJM model. The Nikon-Radodym derivative $\xi_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$ required to find the arbitrage free solution for the Wiener dynamic then applies to find the market pricing solution.

$$\xi_t = \exp(\int_0^t \left[\gamma_u dW_u - \frac{1}{2}\gamma_u^2\right] \quad (2.2)$$

The market price of term risk $\gamma(t,T)$, equation 2.2, of conventional theory in equation 2.1 arises in all classical interest rate models, for zero or forward rate curves with the actual to risk neutral measures from $\mathbb{P}$ to $\mathbb{Q}$ given by equation 2.2. Because we have no prior knowledge of the term risk process in this model, we are left with the problem of estimating its term risk premium by simulation from market quotes as evidence of arbitrage free prices, in totally ad hoc manner. Attempts have been made to obtain an appropriate equilibrium model by a long-term factorization separating the martingale at the long-end from a short-end discount factor model, Qin & Linetsky [60]. In our case, we shall use the PYSM assuming its martingale as an a priori should cover the full yield curve. A real rate drift term $\mu(t)$ does not arise in the PYSM stochastic equation rate equation for example, so that the PYSM yield equation is not impeded from deriving from its term risk kernel. The market pricing logic is provided within the measure change from the data generating probability density $d\mathbb{P}$ to its risk neutral density $d\mathbb{Q}$, given by the Poisson $\mathbb{Q}$ Nikon-Radodym ratio, equation 3.12, p40. The Wiener diffusion equation 2.2 should play no further part in interest rate pricing unless diffusion dynamic has effect. Because diffusion and term risk martingales are orthogonal by definition, they have no stochastic link.

2.2 Nonlinearity in Data Generating Function

The lack of an *a priori* for the market price of risk for the term premia interest rates is just one of the main failings of current interest rate models. Another gap in existing models lies in their difficulty to account for the observed nonlinearities in the term structure curves. For example, observed term risk premia vary disproportionately with credit risk category or with maturity level. Such behaviour should drive further research for an appropriate term risk model, since in arbitrage-free theory a standard martingale-based dynamic must exist. A martingale is a necessary and sufficient condition for linearity. If model with that martingale confers a satisfactory representation of term risk in its $\mathbb{Q}$ distribution, the nonlinearity in observed yield curves and spreads would then no longer be an obstacle in analysis. Once the correct $\mathbb{Q}$ distribution for term risk is identified, $\mathbb{P}$ can always be from its d$\mathbb{Q}$ /d$\mathbb{P}$ Radon Nikodym ratio analytically.

The fact that existing interest rate models clearly do not meet linear superposition conditions is commonly observed. The extensive literature on attempting to resolve this problem with existing models has focused on specifying asymmetric probability generating functions that can produce nonlinear dynamics for both physical drifts and the prices of risk, to compensate for the failings in affine equilibrium models. The models by early literatures follow in this vein (Longstaff [48]; Bernanke & Blinder [13]; Ahn, Dittmar & Gallant [3]; Leippold & Wu [46]; Wachter [66]; Rudebusch & Swanson [60]). However, none of these latter models necessarily explain the full extent of risk in spread analysis suggesting something fundamental is missing either in the dynamic as specified or in the choice of explanatory variables. A key difficulty is to envisage models which are asymmetric enough in their $\mathbb{P}$ distribution and yet linearly tractable in their $\mathbb{Q}$ solution. Without any *a priori* to its structure, the choice of an asymmetric distribution which is extreme enough to provide the biased risk in $\mathbb{P}$ and yet a time dependence to give the $\mathbb{Q}$ martingale distribution is intractable. The expectation integrals in the $\mathbb{P}$ measure in the standard quasi-Gaussian HJM model are non-integrable, as examined by Pirjol & Zhu [58]. No matter how asymmetric the existing models tried, the natural requirement that these be stationary in their $\mathbb{P}$ distributions (i.e. linear) unavoidably creates non-linearity in their $\mathbb{Q}$ distributions unless a compensating time dependence can be correctly guessed. This defeats the risk-neutral condition required in $\mathbb{Q}$.

The advantage expected of the PYSM arises from its formalism. We adopt a theoretical approach to finding the density and the cumulative probability equations in $\mathbb{Q}$, in section 3, p37, and we show why it must be found before we can justify the exact behaviour of the data generating function in $\mathbb{P}$ that is, the observable yield and spread behaviours. The starting point in the Wiener Ito diffusion martingale of the famous Black Scholes Merton model is to deduce the $\mathbb{Q}$ distribution from its $\mathbb{P}$ dynamic. This can be done in the BSM model since in its diffusion dynamic, the statistical behaviour in Wiener $\mathbb{P}$ probabilities allows for an equivalent risk neutral portfolio to be constructed and solved for its partial differential equation. The starting point for the PYSM logic has to be different since, unlike for diffusion, a risk neutral portfolio for the $\mathbb{P}$ Poisson process in deconvoluted state cannot be readily found. The PYSM's $\mathbb{Q}$ distribution, on the other hand, allows the deconvoluted Poisson distribution to be observed in linear form, from which its nonlinear $\mathbb{P}$ form in dL becomes clear to see.

2.3 Un-spanned Macro Variables within VAR Panel Testing

Economic and associated econometric modelling continues to keep faith with the hypothesis that main macro-economic variables should play a major role in the forecasting of risks, term risk premium included. The DGSE (dynamic stochastic general equilibrium) model is one case in point where a linkage between the diffusional and the time dependence of projections, including the effects of risk premium, is kept to the forefront of the model, Rotemberg & Woodford [62]. The tradition in stochastic derivatives modelling on the other hand downplays the role of exogenous variables in favour of endogenous variables explicit to the model. The nature of the PYSM that we are proposing follows similar logic since its object is to characterise term risk through a time series randomness. The constellation of main economic variables under diffusion dynamic and the cross-sectional paradigm should not influence the term dynamic very much, if at all. Tests on the applicability of the PYSM should then help determine this point.

The search for macro-economic or market scale variables has strongly developed in this field, in Beaglehole & Tenney [11] and recently Bauer & Rudebusch [10]. Traditionally in the development of explanatory mechanisms, the term risk is taken as the long minus short term spread in yields, and a choice of macro variables is additionally tested and screened (Bernanke & Blinder [13]). The study of the expectations hypothesis bias has been modelled using vector autoregression (VAR) methods to provide the main means for searching for, and identifying, the state variables responsible. Examples of such studies are provided by Ang & Piazzesi [5], and Joslin, Priebsch & Singleton [44], with models in which both principal components of level, slope and curvature of the yield curve and macro-market and economic are evaluated as explanatory variables. The issue with this approach is that the regressions include many latent model variables. This may lead to a complexity in parameters, over-specification in models or lack of significance in interpreting results.

Our testing of the PYS Model makes estimations over the model's endogenous variables, to keep the focus on the ability of the model to determine the term risk component. In our studies by PYSM, the control for the macro variables is achieved by the appropriate selection and categorization of data, rather by an explicit econometric estimation over macro variables. To illustrate, the credit state of the economy, and to that effect its monetary state in liquidity, is measured from the variations of the yield spreads for each bond rating AAA, AA, A, BBB compared to the government bond curve. In the PYSM all exogenous macro variable effects are assumed captured therefore time series based. The spread regressions are then carried-out via the PYSM variables, very precisely without modelling the macro variables explicitly. The function of the Poisson Yield Structure Model is to replace the multi-lag autoregressions of VAR by a $\mathbb{Q}$ distribution that can attribute the autoregression risk to a conditional single-event distribution.

2.4 Nelson Siegel and other graphical non-parametric models

Another genus of models has found particular success in the practical world of forecasting and central government bond analysis. The Nelson Siegel models [55] and its variants have provided a series of root solution models which have been found to advance major capabilities in yield curve interpretation, at both practitioner (De Rozende & Ferreira [31]) and government bond office (BIS [17]) levels. The limitation of such methods however exists in the absence of stochastic foundations in such models. Ideally, a linkage between the root solution of the Nelson Siegel model [55] and the dynamic term structure models on stochastic logic is required.

The literature on the use of mathematical functions to fit the yield curves to their shapes and movements started with the use of polynomials fitting by McCulloch [49],[50] This has developed in sophistication and the trend to parsimonious versions of statistical fitting with the Nelson Siegel model [55]. Diebold & Li [32] have presented interpretation in terms of latent factors linked to the Principal Component analysis (PCA) factors: 'level', 'slope' and 'curvature' of the shape variability over time. Variants on this theme are routinely used in practice, as main technique for analysing bond strategies, (Alves, et al. [4]).

In a comparison study between the Diebold & Li [32] model and the PYS Model we suggest some parallels in possible theoretical connection. First, both models have shapes, and therefore factors, which appear to have meaning in their econometric estimation models. Other equilibrium models do not provide these indications. Second, each model is founded on the exponential functional, over 4 coefficients for Nelson Siegel type models and in its dynamic transform representation of the PYSM, over 2 coefficients. The Diebold & Li [32] approach refers to the importance of 'slope' and 'curvature' shapes of yields and their movements. The PYSM in fact shows a graphical interpretation of the yield curve in terms of its lower limit and an inflexion point, as shown in figure 2, to offer a connection between the pure stochastic element dL and the structure of yields.

2.5 Tractability of Solution

Inevitably, the trend of increased complexity in conventional interest rate models imposes additional difficulties when estimating or applying these models. If too simple, models may not include enough features or coefficients to cover the risk correctly. If too complex, they can over-specify results, offer lack of significance in interpretation, or contain specification, out-of-sample or heteroskedasticity errors. One good test of the PYSM is to examine the extent to which its system of equations improves the tractability of analyse of yields and spreads compared with existing models. Firstly, we examine the important results on accuracy and tractability in conventional models in the studies by Duffee [34] and Duffee & Stanton [35]. They show that many of the modelling attempts with panel testing using Efficient Method of Moments (EMM) may be less precise that simpler models such as Maximum Likelihood (ML) or the Linearized Kalman Filter estimation. In early studies, the modelling of the yield curve and its pricing may also have been unable to provide the asymptotic convergences normally expected for stable solutions.

The structure of the PYSM offers two features which are very useful when either estimating the results for particular term risk functions or when pricing particular instruments for interest rate products. Firstly, for the PYSM $\mathbb{Q}$ pricing kernel dL we have is analytic which means that we have closed-form equation for the pricing of any term risk. It follows that we can always write the optimisation for estimation of term risk analytically, therefore use ML rather than GMM estimations. Duffee and Stanton [34] specify that this opportunity should improve the accuracy and reliability of pricing interest rate products. It also provides the linearity needed for portfolio pricing. Secondly, the simple structure of the $\mathbb{Q}$ pricing density dL allows for the easy development of partial differential equations required for pricing the various fixed income contingent claim contracts, in simple differential equations, which are easy to solve.

3. Poisson Yield Spread Model (PYSM) Derivation

This section introduces the formal derivation of the Poisson Yield Spread Model (PYSM). We recall that the distinction of the PYSM is in its new stochastic time dynamic. Access to the new stochastic term risk is sought assuming a different probability paradigm than is generally achieved in the current suit of interest rate models. The rationale of this new probability structure, which originates in the mathematics of phase-space, and its relationship with existing finance equations is best understood by defining the three conditions that have to apply if the new paradigm is to be effective.

The first condition assumes that a new probability paradigm actually exists for the conditionality in time when events occur. Probability theory has developed to interpret the dynamics of the forward rate in the HJM model via stochastic evolution equations in the frequency space of the Fourier transform, for example in Brzezniak & Kok [49]. The conceptual restriction in these models however is the assumption of affine process elements for jumps and diffusion which seems to preclude the conditionality required of term risk. Instead, we might be able to use the representation of risk known in the sciences, where behaviour of time-lagged risk as an auto-correlation of risk is accounted for within spectral analysis. Effectively, the pricing kernel is a spectral function. This simply means that we can apply the Wiener-Khintchine theorem (Strobl [63]) to solve for probability amplitude in frequency space, where frequency is defined as the reciprocal of time. It is interesting to note that models employing frequency transform have already been used extensively in finance. But their purpose has been limited to their particular mathematical advantages in solving the expectation integral more easily. Both the Heston [41] stochastic option pricing model, or the affine dynamic term structure models, see Duffie, Pan & Singleton [37] have used this device to this advantage. The full capabilities of Transform Analysis theory particularly in its special features that we find for term risk in the PYSM, have not however been developed in the Heston [41] or Duffie et al [37] modelling studies. The second and third conditions below as further conditions are also needed however if we are to complete the development of the term risk model though its spectral function.

The second condition concerns the exact choice of underlying dynamic. Since the Gaussian type diffusion model contains no time conditionality element, we cannot rely on it for term risk. Instead, we seek the most obvious and simplest alternative displaying a time dependence for risk, which we find in the AR (1) process of a single lag event model. Furthermore, the AR(1) could be a good choice since, despite its structure as a discrete randomness element in econometric context, it also has a continuous time formulation as the Ornstein Uhlenbeck [64] (OU) process. It thus provides a well understood model for damped Gaussian (Wiener diffusion) motion. Indeed, the OU process used to create the mean reversion property in existing interest rate models. A limitation in such models however is that the OU damping rate to the Wiener diffusion occurs as a deterministic factor, whereas it should encompass the term risk conditionality. In our treatment, the alternative of replacing the Wiener pricing kernel with an entirely new distribution, is made possible if the mathematics of the PYSM transform on the OU is more fully taken into account.

The third condition in the proposed PYSM is to recognise how the possible pathways of the AR(1) driven process should be defined in probability space, if the arbitrage risk free paradigm (martingale) is to allowed to apply as in our PYS Model. Essentially, there are multitude of pathways in the spectral function probability space which exist as possible pathways. Only when a pathway occurs will it be observed as a physical event and have a $\mathbb{P}$ probability. By definition $\mathbb{P}$ probabilities for jumps have $\mathbb{P} = 0$ or 1 as discrete processes. The probability however for systematically priced term risk is defined by expectation over all the possible $\mathbb{Q}$ probability pathways, all of which have yet to occur. Just as in the Wiener Ito martingale a similar methodology can be used following the Harrison Krebs [38] methodology to find the Nikon Radodym differential relating d $\mathbb{P}$ with d$\mathbb{Q}$. From the representation theorem, a complete market martingale is the necessary and sufficient for an arbitrage free pricing where no other functional is required, Harrison & Krebs [38]. Both can then be defined and exploited to price an underlying. Correctly conditioned, the process of finding suitable pricing partial differential equations starting with a new probability paradigm term risk should occur. The model derivation in section 3.1 now follows this structure.

3-1 Modelling the Pricing Equation from its Stochastic Process

The proposition in this study is to model term risk using a new fundamental distribution. The analysis here involves four steps in sequence as follows:

- Step 1, the choice of Ornstein Uhlenbeck distribution in its autocorrelation AR (1) form is made and developed.
- Step 2, the objective probability for the single event AR(1) process is obtained by Fourier Analysis.
- Step 3, the change in measure converting the frequency $PDF^{\mathbb{P}}$ to its risk neutral measure $\mathbb{Q}$ in log frequency is derived. The compliance with complete martingale conditions completes the proof.
- Step 4, the Yield Equation and the Spread Equations are derived from the stochastic differential equations in d$L(\tau)$, assuming the stochasticity is all $\mathbb{Q}$ Poisson term risk driven.

These steps define the PYSM structural equations which provide the foundation for the econometric evaluation of the state variables affecting spreads, see sections 4 and 5. Derivation Steps 1 to 3, are dedicated to developing the Poisson $\mathbb{Q}$ as a pricing distribution kernel for the spot discount rate $z(t;T)$, denoted d$\tilde{L}$. The mechanism assumed for this risk is solely the continuous time version of the AR(1) Ornstein Uhlenbeck process. For space and clarity, we show the full details of the term risk pricing kernel (d$\tilde{L}$) derivation in the separate part at the end of this paper. This helps to show how the elements of the mathematical solution to the unique term risk martingale are introduced and justified. We consider this necessary since in the steps we take in the derivation, although they follow the same principles of current stochastic pricing theory, they also introduce extensions to theory which are innovative, being drawn from other disciplines in probability theory not so familiar in the financial field, certainly not to the highly familiar mean-variance modelling. As precursor of the Appendices 1, we note the examples of key innovations:

(1) The Ornstein Uhlenbeck is a common element in current rate models for the mean reversion property but current models treat this deterministically. Our innovation is to extend the OU to its conditional many pathways form in frequency space (it appears singular in time domain but is actually continuous in frequency domain).

(2) The OU $PDF^{\mathbb{P}}$ is thus found by a Fourier transform (we note that though this technique has been applied very usefully to facilitate SDE integration calculation, its underlying meaning needs extending, see 3). The Wiener-Khintchine theorem is relevant for this purpose.

(3) Using the standard theorem for finding the unique martingale via the Nikon-Radodym differential. Following the derivation Appendices 1 for the martingale pricing kernel, the density as $PDF^{\mathbb{Q}} = \widetilde{\mathrm{d}L}$ and the cumulative distribution function as $CDF^{\mathbb{Q}} = \tilde{L}$ we can complete partial differential equations suited to pricing the yield curve and spreads of forward rates. This derivation is completed in procedure 4 below.

(4) The innovation here is to exploit the simplicity of the $\widetilde{\mathrm{d}L}$ and $\tilde{L}$ functions which because they can be applied in simple differential equation form facilitate easy solution to analytic functions for a) the yield curve and b) the spreads in spot discount rates under defined boundary states. The parametric specification of these models is also analytic, readily facilitated by use of empirical zero rate curve data. Observables for driving the term risk in appropriate instruments is also facilitated. In procedure 4 we develop the Yield Curve and Yield Spread Equations as follows:

First, we recall that the spot rate formula for interest rates as a square root mean reversion dynamic, typically the Cox, Ingersoll & Ross [26], CIR, model has the form

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}.d\widetilde{W}$$

where r is spot interest rate, κ is rate of return to mean rate (θ), and σ is standard deviation of the Wiener diffusion $(\mathrm{d}\widetilde{W})$. Assuming that the yield curve is governed by the Poisson $\mathbb{Q}$ term distribution $\widetilde{L}$ above, then there can be considerable simplification in the nature of the differential equation for its forward interest rate. Under the Poisson $\mathbb{Q}$ dynamic, the whole of the variations in interest rates can be written in terms of the Poisson $\mathbb{Q}$ kernel $\mathrm{d}\widetilde{L}_{\tau}$ as

$$d\ln r_{(t,\tau)} = I.d\widetilde{L}_{\tau}$$

where I is proportionality constant and r is a function of current time (t) and term risk time (τ). Firstly, the mean reversion term in the Wiener CIR model, is no longer required, i.e. $\kappa(\theta - r)dt \to 0$. Also, the coefficient to the Gaussian process is replaced by one for the Poisson Q distribution dL, i.e. $\sigma\sqrt{r} \to I$. The removal of σ as volatility, as the risk term is highly significant. It has been replaced essentially by the coefficient of auto-correlation $\lambda = 1/\tau$ which is the innovation given in the Poisson martingale. Solving this equation as a definite integral within the boundary conditions shown, provides the equation 3.16 below, where the lower bound is defined as $\ln r_0$ when $\ln\tau \to -\infty$ and the upper bound is $\ln r_1$ when $lnt \to \infty$.

$$\int_{-\infty}^{\infty} dlnr_{t,\tau} = \ln r = \ln r_0 + I.\widetilde{L} \qquad (3.16)$$

We can further simplify equation 3.16, knowing its structural features, by instituting an upper interest rate bound as $\ln r_1 = \ln r_0 + I$. Given that the Poisson $\mathbb{Q}$ has by definition a normalized event probability of one, then the cumulative transition probability must be $\widetilde{L}$ = 1 at the upper rate. Substituting in equation 3.16 then obtains our final version of the term structure model, equation 3.17.

$$\ln z = \ln r_0 \,(1 - \widetilde{L}\,) + \ln r_1.\widetilde{L} \qquad (3.17)$$

Equation 3.17 is readily parametrized to empirical zero rate curves. The Levenberg-Marquardt nonlinear curve fitting is used. We can later test and prove the PYSM on the EU debt yield curves for the period 2007 to 2014, in section 4. To determine the PYS Model's factors of activation we examine whether we can factorize its stochastic equation into partial derivatives. Happily, this can be done with first order differentials on the Poisson $\mathbb{Q}$ SDE which simplifies analysis greatly. If PYSM parameters are mutually independent, then equation 3.17 provides the following partial differential solution.

$$\frac{\partial \ln r(t,\tau)}{\partial t} = \frac{\partial \ln r_0}{\partial t}(1 - \widetilde{L}\,) - \ln r_0.\frac{\partial \tilde{L}}{\partial t} + \frac{\partial \ln r_1}{\partial t}\,.\widetilde{L} + \ln r_1.\frac{\partial \tilde{L}}{\partial t}$$

This equation confirms that $\partial \ln z(t,\tau) = f(\partial \ln r_0(t), \partial\tilde{L}(t,\tau))$

where $\ln z = f\,(\ln r_0\,, \ln\widetilde{L}\,)$ and $\widetilde{L} = f\,(\ln t\,, \ln\tau)$. In practice the spreads of yield curve between the risky bonds for each credit rating from the government 'risk-free' bond is justified as ∂r_{it}, which it is driven from $\ln r_0$ and $\ln\tau$ independently. This confirms equation (3.18) for spreads as:

$$\text{spread}_{\text{it}} = \partial z_{it} = \propto + \beta_1 \ln r_{0_{it}} + \beta_2 \ln\tau_{it} + u_{it} \qquad (3.18)$$

Equation 3.18 then determines the spreads capabilities of the Poisson Yield Spread PYSM, where index *i* indicates the different credit rating categories (AAA, AA, A, BBB). We later test and prove for the determination of this spread risk behaviour across the EU debt crisis.

3-2 PYSM, Relevant Data and Graphical Analysis

Here, we introduce the data and methodology required for testing the new PYSM, for its yields term structure and spreads extension formulations in equations 3.17 and 3.18. Acknowledging that most preferred interest rate and credit risk theory depends heavily on reduced form modelling, leaves the question of structural foundations of existing models unanswered. Here we can question whether the term risk component of the PYS Model might offer a solution to this dilemma.

An advantage of this new Poisson model is in its capacity to evaluate interest rates both longitudinally and cross-sectionally when separating correlations, where longitudinality can be specified as long minus short maturities in the zero rate ($\Delta \mathrm{z}_{(t,T)}$), and cross-sectionality can refer to changes in zero rates over different credit rating between curves ($\Delta \mathrm{z}_{(t,T,\text{credit}-\text{risk})}$) as a function of time. The purpose of this PYSM stems from its three time-factor (t, T, τ) structure for determining zero rate effects, z, where t is calendar/current time, T is maturity period and τ is term risk. Until we have proved their meaning experimentally, the factors of the PYSM might still be considered endogenous or latent in effect. But by linking the PYSM to the yield curves in Figure 2, we can already gain some preliminary meaning on the variables as follows. We have two types of factor in the PYS Model:

- The first types of factor are the interest rates at the lower bound r_0 and upper bound r_1 of the yield curve (or the slope *I* of the yield curve). Economic theory already provides some meaning of these factors. The r_0 is linked to the short rate as a key variable in monetary policy, and yield slope has been identified as an indicator of future growth rate in the economy. We should emphasize that at this stage these effects are identifiable suggestions rather than identities. Although our factors may align to some known indicators as proxies yet they may differ significantly in quantitative measure. The value of the model may then be to provide quantitative information on mechanisms, previously known only qualitatively.
- Second, a time constant τ occurs, which has its origins in the intensity factor of the Poisson event risk exponential. τ's structural interpretation then depends on the meaning of the Poisson rate under a martingale change of measure. We can hazard a guess that

this must be some form of rate, or fear of a transition of an event occurring, but this becomes clearer in later analysis.

Relying on the PYSM approach is distinctively different from the majority of yield and credit spread studies undertaken in recent literatures reviewed earlier. We test the extent of the uniqueness and value of this model's factors in the next section. This might be the optimal test if it characterizes the risk premia responsible for spreads during highly stressed crisis conditions. Figure 2 provides an example of the set of curves consisting of the government 'riskless' curve compared to the four corporate risky curves for the investment grade S&P ratings, AAA, AA, A, and BBB from the Merrill Lynch fixed income database and Thompson Reuters Eikon. From the equations 3.16 and 3.17 we can deduce that $\ln r_{i,T} = \mathrm{f}(\ln r_0, \ln r_1, \widetilde{L})$, where i represents credit rating and T is maturity time, also $\widetilde{L} = \mathrm{f}(\ln \tau)$ and $\ln r_1 = \mathrm{f}(\ln r_0, I)$, this gives us the following equation for $\ln r_{i,T}$.

$$\ln r_{i,T} = \mathrm{f}(\ln r_0, \ln r_1, I, \ln \tau)$$

The Poisson Yield Spread Model results can be represented in graphical form: figure 2 for the yield curve, and figures 2 to 4 for the variations in its factors, in quarterly format from 2007 to 2014.

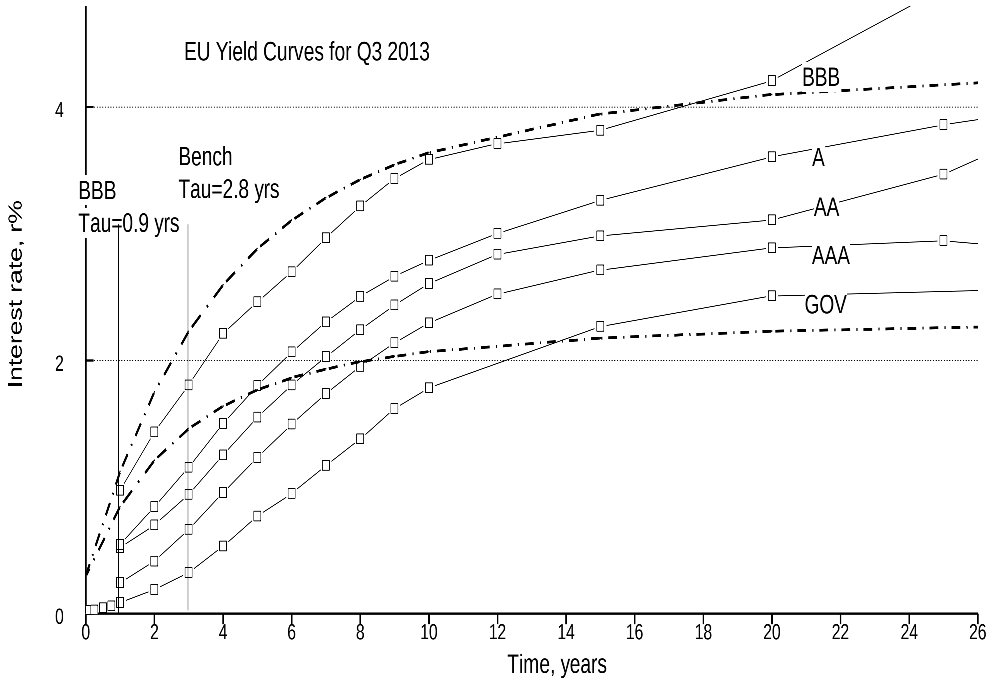
3.2.1 The PYSM yield Curve

Figure 1 shows the term structure of yields treated by classic Cox, Ingersoll & Ross [26] CIR model which specifies the interest rate risk as a square root mean reversion dynamic. The CIR model fits this figure poorly due to, amongst others, the following two reasons:

1) The mean reversion of the interest rate to account for the flattening of these curves at long times is found to be much larger in practice than theory.
2) The market price of risk calibrated from the short end of the yield curves is always too high for the long end.

The failure of models to fit at short times is well known for example in the quasi-Gaussian HJM models, Cuchiero, Fontana & Gnoatto [29], with adverse effects to the pricing of zero-coupon bonds and Eurodollar futures.

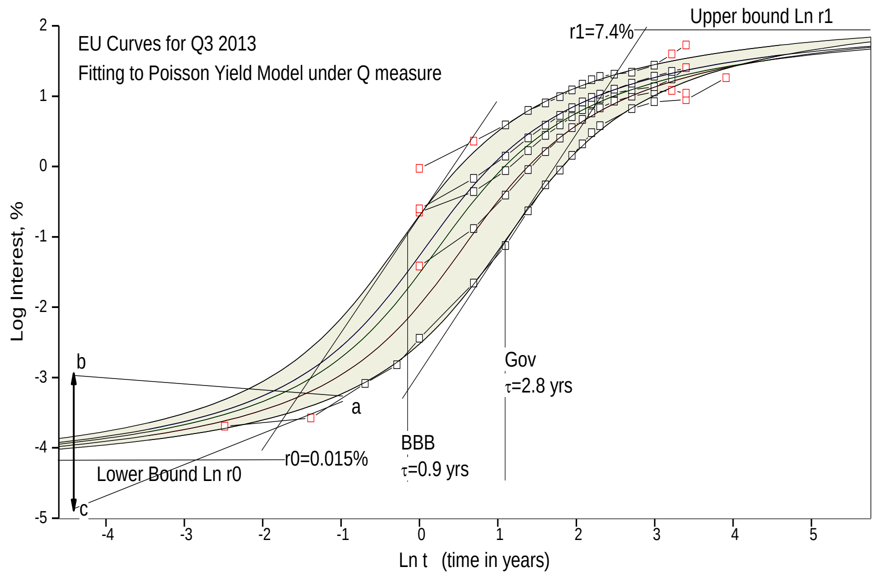
Figure 1- The plot of EU Yield Curves using the Existing Cox, Ingersoll, Ross (1985) Model (CIRM) for 2013 Q3



Note: Data are as described in Figure 2. The CIR curves are plotted assuming the following factors: mean reversion 60% (GOV), 40% (BBB), $\theta = 0.5\%$ (GOV), 2% (BBB), and for both (GOV, BBB) $r_0 = 0.3\%$, $\sigma = 1\%$, market price of risk reversion $\gamma = -1.2$, the average by serial autocorrelation, Ahmad and Wilmott [2].

Furthermore, the next graph for the same yield curve data shows how the new PYS Model proposed here avoids the above problem. This is because the mean reversion effect and the market price of risk are automatically corrected for within the properties of the probability distribution to the Poisson $\mathbb{Q}$ pricing kernel. The graphs show that the PYSM provides a clearer picture of interest rate risk in the short interest rate zone. This we see later is key to interpretation of the driving factors of the whole yield curve, when revealed clearly and fundamentally by the log structure of the PYSM for these factors.

Figure 2- The plot of EU Yield Curves using the Proposed Poisson Yield Model (PYSM) for 2013 Q3 ($\ln r_{i,T}$)



Note: These curves are the daily average interest rates by maturity for Q3 2013 after the EU economic recovery plan in 2012. The government yields maturities are given at 1 m, 3m, 6m, 9m, 1, 2, 3, 4, 5, 6, 7, 8, 10, 15, and 25 years, and risky debt corporate yields at 1, 2, 3, 4, 5, 6, 7, 10, and 20 years for the EU.

Figure 2 shows the increase in interest rates over time. lnr_0 is the asymptotic lower bound (at very short times) of the yield curves located at increasing interest rate levels with credit risk as a set. If each curve is settled by different credit risk which is an inter-temporal risk, then at zero time each curve in the set should be expected to share the same lower bound, lnr_0. The slope is monitored by an intensity factor I interpreted also as the slope of the curves, and τ is the time at the inflexion point (i.e. the midpoint) for each curve. lnr_1 is the upper bound of the set of yield curves. Experimentally it is found that each set of yield curves converges to a common factor at long times. Each of these PYSM factors is now evaluated in the following graphs, figures 3 to 5, for variation and meaning over the cycle. Here we note that each of these graphs, figures 3 to 5, show seven of the main shocks events by the number lined to help appreciate the triggers of high volatility in this analysis during the sample period.

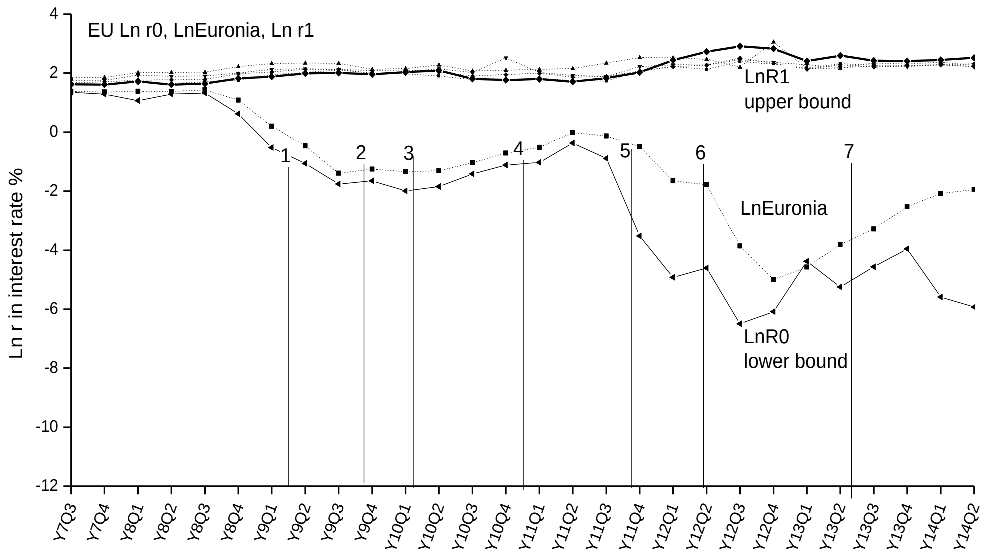
- ***Line 1*** (Q1 2008): a peak impact moment when toxic assets effects, e.g. on mortgage backed derivatives, CDOs) first really registered in market prices.
- ***Line 2*** (Q4 2008): Lehman’s November bankruptcy crash.
- ***Line 3*** (Q2 2009): the EU crisis at the peak of the Greek debt crisis.
- ***Line 4*** (Q2 2010): the EU-IMF bail-out of €100 bn for the Greek crisis.
- ***Line 5*** (Q3 2011): the ECB injection of about €1tn in three-year loans into the EU financial system to correct for contagion fears.

- ***Line 6*** (Q2 2012): Greek elections which challenged the austerity programme and EU economic recovery plan, followed shortly by the Mario Draghi's statement to support the euro 'whatever it takes', Draghi [67].
- ***Line 7*** (Q4 2013): Negative reaction to the US Federal Reserve Board's first suggestion for a QE taper policy, felt internationally.

3.2.2 The PYSM lower bound

Next, figure 4 indicates the variation of $\ln r_0$, with the short rate proxied by the using the overnight rate also shown. The importance of $\ln r_0$ is in its links to the theoretical constructs of the model rather than its justification as a policy rate. Nonetheless, the government policy rate r_{POL}might be guided by the price of money according to the risk-free market indications in the PYSM, therefore the model's lower bound r_0. In practice, a government will of course use its short rate r_{POL} as a lever of monetary policy. The extent to which r_{POL} exceeds r_0 should indicate how much tightening (or if in reverse - loosening) is sought for a given monetary policy. Note that, in this way, inverse yield curves can occur even if on rare occasion. Furthermore, the medium-longer term yield curve, say further-out than 1 year, would be expected to be independent of these short rate variations. The PYSM should be reflective of this, and be stable in the medium term and beyond.

Figure 3- The plot of $\ln r_0$ and $\ln r_1$ for PYS Model over the EU Financial Crisis (from Q3 2007 to Q2 2014)



Note: The $\ln r_0$ curves indicate the asymptotic lower bound (at very short times) of the yield curves for different credit rating, and the $\ln r_1$ shows the upper bound of yield curves for government bond for each quarter from 2007 to 2014.

The evolution of r_0 with time is demonstrated in figure 3 which shows $\ln r_0$ over time as economic circumstances change time. We compare the PYSM factor $\ln r_0$with the variation of the short policy rate proxied by the overnight swap rate. In most of the plot we see how the nominal short rate drops to the PYSM lower bound as the stimulus is tried (it can go to negative values for certain regimes). Under tightening, we should have the short rate widening above the lower bound.

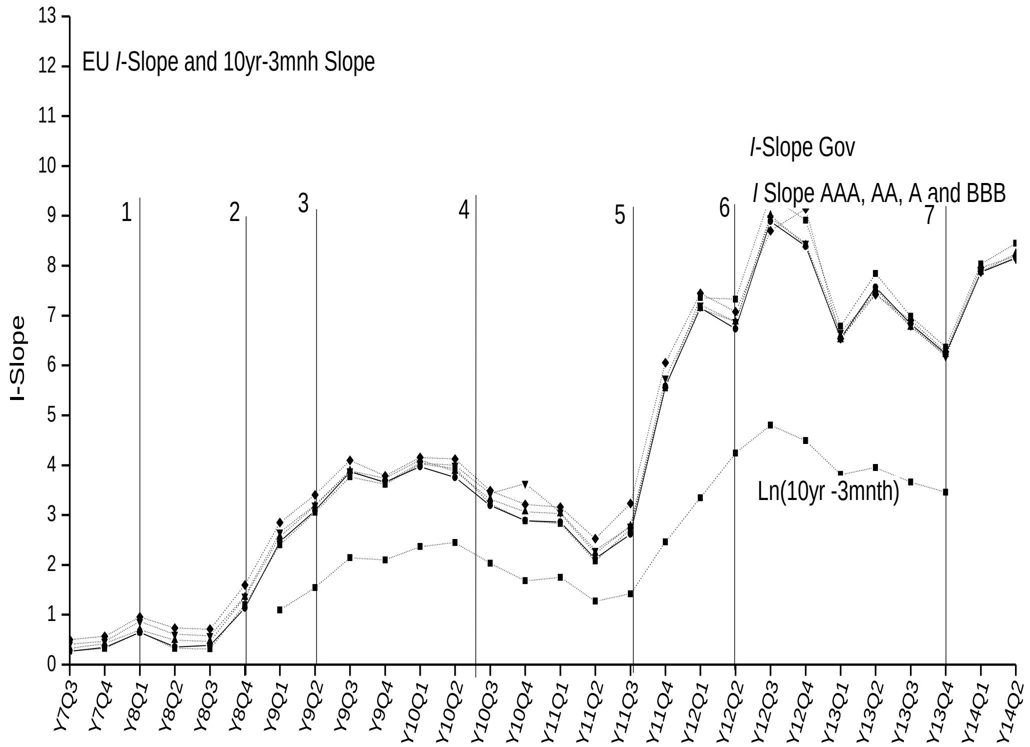
3.2.3 The PYSM upper bound

From Figure 3, we see that the PYSM possesses a well-defined upper bound $\ln r_1$ as the yield curves progresses to long times. If the PYSM is a true model of the yield curve, then its upper bound $\ln r_1$ should be precisely defined by the PYSM's equation (2) as $\ln r_1 = I + \ln r_0$, since at the upper bound one event must have occurred with the upper bound defined when $\widetilde{L} = 1$. The interpretation of $\ln r_1$ is equivalent to a mean reversion steady state interest rate at very long times in our data of 30 years plus. We note that $r_1$is much less sensitive to changes with time, therefore economic circumstances, than $\ln r_0$. This is understandable since, because of the logarithmic scale in r, the changes in $\Delta ln r_1$ are very much less than those in $\Delta ln r_0$ as shown in the results, figure 3. We can later treat $\ln r_1$ as sensibly constant compared with $\ln r_0$ which is important when defining the slope variation, *I*-Slope, in the next section.

3.2.4 The PYSM *I*-Slope

Next, figure 4 shows that the increase in the PYS model's slope coefficient (I) over time. The parallel for I in the conventional interest rate dynamic (CIR model) is in the market price adjustment needed in Gaussian dynamic models (using a diffusion type variance term). Furthermore, the yield curve slope is linked to macro factors as economic indicator of growth or inflation rate in conventional theory. The evolution of the *I*-Slope is demonstrated in figure 4. It is easy to see how the rise in I over time mirrors almost exactly the downward pattern seen in $\ln r_0$, in figure 4. This experimental result is consistent with the PYSM in its constructs, in the following way. PYSM's equation 3.16, $\ln r_1 = I + \ln r_0$, shows that if $\ln r_1$ remains sensibly constant then, since $dI + d\ln r_0 \sim 0$, relationship $d\ln r_0 \sim - dI$ is confirmed This implies that the informational content for both $\ln r_0$ and I factors are equivalent in regards of their term risk effects. Since either can be used equally, from now on we keep to $\ln r_0$ in our parametrization for simplicity.

Figure 4- The plot of slope coefficient (*I*) for PYS Model from 2007 to 2014



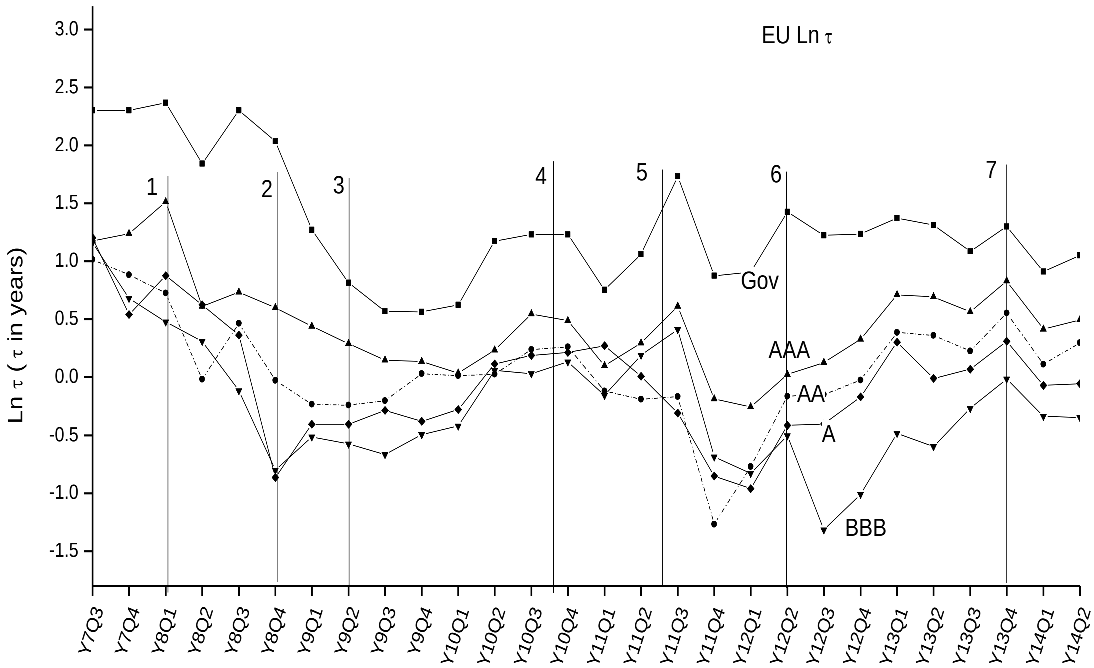
Note: These curves show slope coefficients (*I*) for PYS model of different credit ratings (AAA, AA, A, BBB) and government bond in comparison with 10yr-3month yield curve slop, over the EU Financial Crisis.

We note that in conventional analysis, yield slope is normally calculated as 10 years minus the 2 months rate. To compare the conventional and the PYSM, we plot the conventional formula in figure 4, plotted in logarithm of rates for the comparison. The log conventional slope and the *I*-Slope move in close parallel trend. This provides a level of support for the usual view of the using the yield slope as leading indicator e.g. of economic growth or inflation. The *I*-Slope (or its complementary variable $\ln r_0$) of the PYSM may further quantify relationships linked to slopes in term risk analysis and forecasting, previously known more arbitrarily.

3.2.5 The PYSM Term Factor $\ln\tau$

$\text{Ln}\tau$ is the time related measure of term risk, only accessible when term risk is uniquely attributed to its no-arbitrage Poisson dynamic. The literal interpretation of τ $(= 1/\lambda)$ is the inverse of the Poisson intensity factor λ in original formulation of the Ornstein Uhlenbeck autocorrelation, equation 3.4.

Figure 5- The plot of term risk factor$(\ln\tau)$ for PYS Model from 2007 to 2014



Note: These curves show term risk factor$(ln\tau)$ for PYS model of different credit ratings (AAA, AA, A, BBB) and government bond over the EU financial crisis.

The $\ln\tau$ curves defined at their individual credit risk levels AAA, AA, A, BBB as well as the Government curve (GOV) vary with time, therefore economic states of markets, as seen in figure 5. They follow in a parallel pattern of variations quite closely over the cycle. When events occur which accentuate credit crisis conditions, the tendency is for $\ln\tau$ levels to shorten considerably, across the board. To illustrate see the results for the AA bonds. For these, pre crisis τ moves from 3 years to a during-crisis τ of 7 months, then shorten to a τ of 3 months during the Greek debt crisis, before recovery to a τ of 1 year on implementation of QE properly in Europe, mid 2012. The difference in $\ln\tau$ term risk is clear to see between each level of Rating, ranging from a τ of 5 years for the government bond to a τ of 7 months for BBB risky bonds, at the given point, 2011.

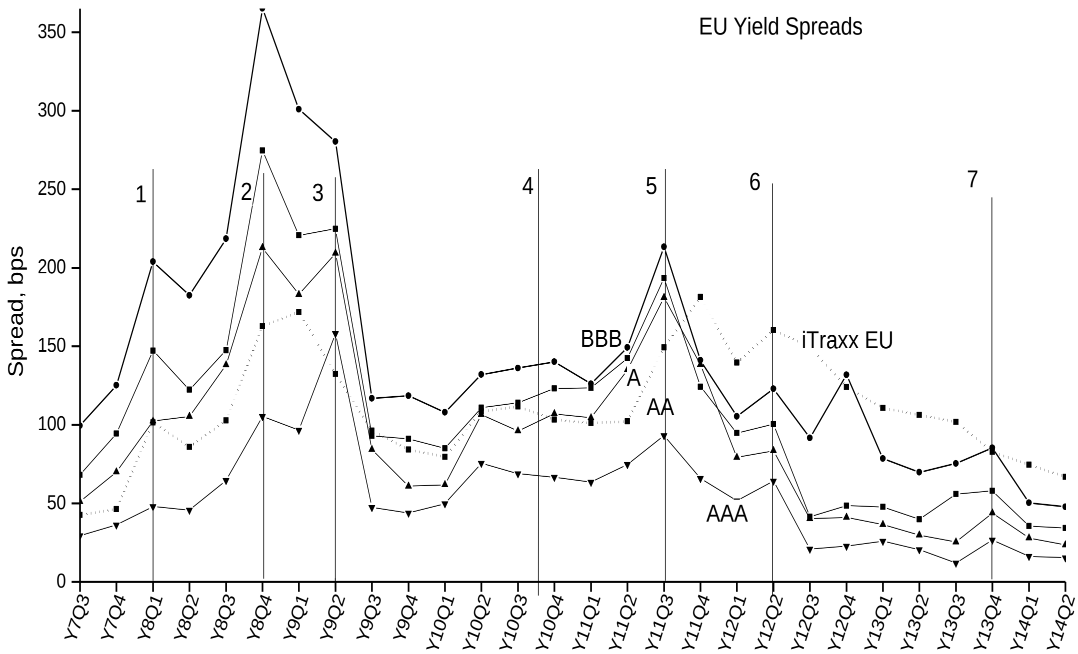
There are some key theoretical features when using $\ln\tau$ as the term risk measure. The difference between risky and less risky bonds is measured systematically across Ratings with high risk in markets registered by a shortening in term risk constant τ. But in $ln\tau$, the adjustments are equally dramatic for all categories of bond. The meaning of riskless interest rate as given by the government bond is given new meaning since at no point is interest rate seen to be free of a term risk. The PYSM allows this to be measured. To understand the meaning of $\ln\tau$ factor, it is useful to consider τ should afresh under its $\mathbb{Q}$ measure as $\tau_{\mathbb{Q}}$. In this representation, τ has been converted to a new observational variable of term risk. It is now under a martingale measure $\mathbb{Q}$, where the martingale is PYSM specific. Although $\tau_{\mathbb{Q}}$ is equivalent to its original time $\tau_{\mathbb{P}}$ where $\mathbb{P}$ denotes physical measure, it has been re-weighted in such a way as to guarantee risk-neutrality when used in pricing formulae. Employing the probability appropriate to $\tau_{\mathbb{Q}}$ removes the need for inputting an artificial term risk drift or market price term which would be necessary if a model on a Gaussian-dynamics were used. Quantitatively, the importance of using $\tau_{\mathbb{Q}}$ in $\mathbb{Q}$ form is apparent in the above data of figure 5. It is clear that the values of $\tau_{\mathbb{Q}}$ factors in figure 4 occur in a given range of times i.e. $5\ yrs > \tau_{\mathbb{Q}} > 3\ mnths$. These times are easily those which we might contemplate as realistic payment terms for debt we might have to repay, for example.

When measuring term risk from an exponential default rate literetately, that is as $\tau_{\mathbb{P}}$ we would have very different results. Taking the Ornstein Uhlenbeck model $exp^{-t/\tau}$, which was our starting model equation 3 before its $\mathbb{Q}$ transform, gives levels for $\tau_{\mathbb{P}}$ in a range $50\ yrs > \tau_{\mathbb{P}} > 10\ yrs$. That is, the physically observed levels of τ, i.e. $\tau_{\mathbb{P}}$, are an order of magnitude different from market delivered times. These results support the observation found in practice that the yield spreads are much greater than physical probabilities predict, with $\Delta z_{\mathbb{Q}} \gg \Delta z_{\mathbb{P}}$ since $\tau_{\mathbb{Q}} \ll \tau_{\mathbb{P}}$, where Δz is the discount spread to be explained. A further perspective can be obtained by assuming that a utility function $U(t)$ for term risk might be definable as a certainty equivalent given on the term risk auto-correlation $\langle X_0, X_t \rangle$. Assuming a simple proportionality then $U(t) = -k\langle X_0, X_t \rangle = -ke^{-\lambda t}$. Form the Arrow-Pratt formalism [6],[59]], the absolute rate of risk aversion $R_a(t) = -\frac{U''}{U'}$ is then obtained as $R_a(t) = \lambda_{\mathbb{Q}} = 1/\tau_{\mathbb{Q}}$. The $\mathbb{Q}$ subscript on $\lambda_{\mathbb{Q}}$ and $\tau_{\mathbb{Q}}$ signifies that these factors can come from a PYSM risk neutral parametrization on yield curves just as applied in our study. This provides a concept of utility of time consistent with at least one risk neutral stochastic measure. Future work applies to see whether this generalises to wider interpretation.

3.2.6 The PYSM Credit Spread Factor Δz

The PYSM spreads (Δz), are measured as the difference between the yields curves for the risky bonds versus the government bond curve as ‘risk-free’ comparator for each rating, by quarter, (see equation 3.18).

Figure 6- the plot of Poisson Yield Structure Spreads from Q3- 2007 to Q2- 2014



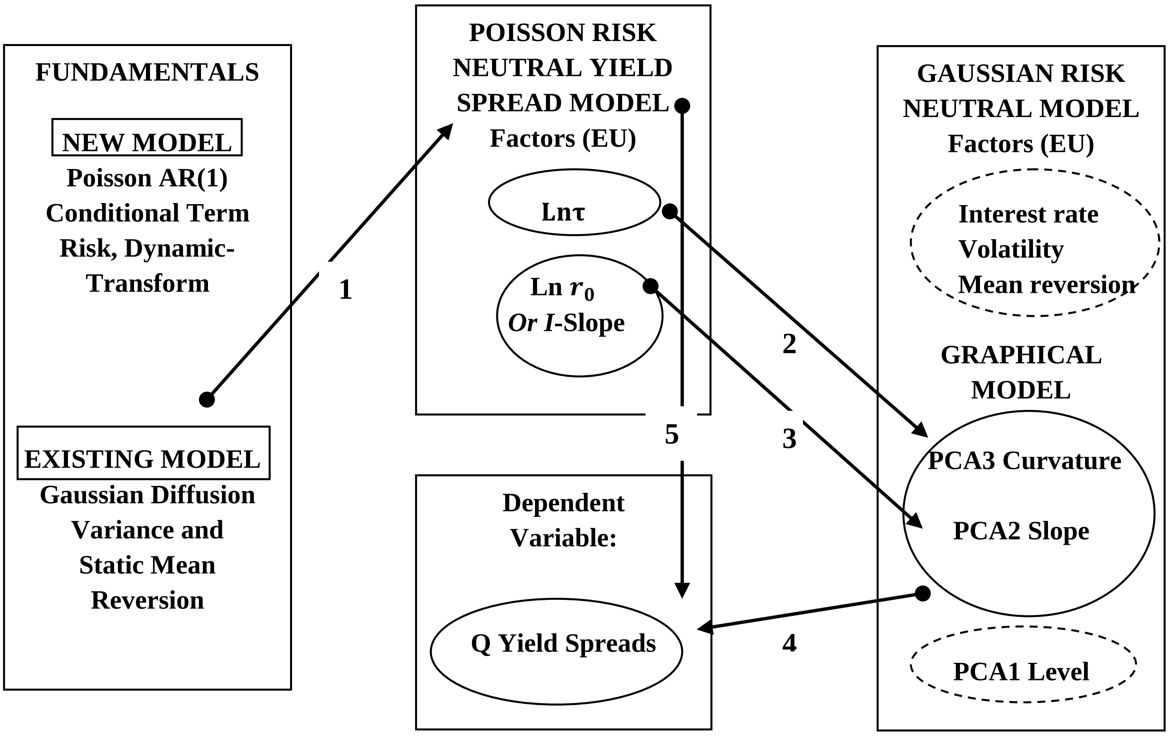
By calculating the spreads at the inflexion point, a number of control features are achieved. First, the spreads at the inflexion point occur when the probability distribution $\mathrm{d}\tilde{L}$ is maximum where at the midpoint of probabilities, the yield curves are most strongly affected by the term risk. Second, there is evidence that the most constant level of the spreads occurs at the inflexion point, see figure 2. Figure 6 shows the PYSM spread variations over time for different credit ratings over the EU financial crisis from 2007 to 2014. These show the extreme levels of risk achieved during the financial crises in 2008 and the intermittent stages of the recovery, including the aftermath of the Greek crisis in 2011. We can also conveniently define the market spread risk in the market dimension. The yield spread $\Delta z_{\mathbb{Q}}$ is determined at the inflexion point $\tau_{\mathbb{Q}}$ of the yield curve. Just as we can have $\tau_{\mathbb{P}}$ matched and defined as $\tau_{\mathbb{Q}}$, similarly the $\Delta z_{\mathbb{P}}$ is redefined for market price as $\Delta z_{\mathbb{Q}}$. This then makes the yield spread meaningful since $\Delta z_{\mathbb{Q}}$ relates to $\tau_{\mathbb{Q}}$ (not $\tau_{\mathbb{P}}$).

Finally, for completeness we include a plot of the iTraxx investment grade 5-year credit derivative index, for comparison with the PYSM modelled spreads. We see that the CDX and PYSM modelled spreads move together but with quite a large approximation to give them equality. Although its reliability has now been questioned, the use of CDS and CDX as leading indicators of market risk was once very fashionable. The PYSM shows a number of legitimate differences to pure credit swap indexes, which the Poisson Q allows us to calculate and measure.

3.3 New PYS Model vs. Existing Yield Structure Models

The new Poisson Yield Spread model proposed here suggest some advantages over existing equilibrium, VAR supported, or parametric yield curve models. Diagram 1 presents a typology in outline of how, from our derivation so far, the PYSM relates to competing models for yield structure, in the relationship of variables within and between models.

Diagram 1. Typology of the constructive variables within the PYS Model and the volatility or empirical graphical models



Note: PCA3 is the Principal Component Analysis at the curvature of the yield curve. PCA2 is the Principal Component Analysis at the slope of the yield curve. PCA1 is the Principal Component Analysis at the level of the yield curve, descriptions given by Litterman & Scheinkman [47]).

In existing models, the pricing kernel of the Gaussian process supplemented with the mean reversion static process of the Ornstein Uhlenbeck is replaced by the pricing kernel of the PYS Model. First, in terms of standard dynamic equilibrium approach, the fundamental difference of the PYS Model approach to conventional models is important. The structural features of the PYSM are attributed to its *a priori* treatment of process, i.e. specific to the Ornstein Uhlenbeck dynamic. This confers the use of deductively specified explanatory variables (i.e. the lower bound short rate r_0 or equivalently *I*-Slope and the term risk time factor τ) for pricing and analysis. Finally, a martingale in analytic form allows an easier estimation in the PYSM framework, for example in regression analysis in respect of its endogenous variables.

This is unlike the formulation in existing VAR models, which uses Gaussian diffusion distribution assumptions, state variables brought from outside, and a static mean reversion. The latter is a literal reversion which is naturally limited by nonlinear no-arbitrage restrictions yet unsolved in Gaussian diffusion. The uncertainty in outside variable VAR formulations, which can grow with number of potential explanatory variables and model structure complexity, limits regression precision. The PYSM may provide an easier modelling hypothesis, number of variables and simpler analytic, in remedy of these known existing problems. Second, the pricing kernel of the PYS Model is based on a martingale is orthogonal to the Gaussian Ito diffusion martingale. This justifies our exclusion of cross-sectional influences and allows more focus on the term structure variations only. Since cross-sectional correlations (e.g. over macro factors) are not therefore important to the dynamic of term risk, we can simplify the econometric analysis to many fewer modelling variables. Third, the diagram 1 above indicates the linkage of the variables between Poisson Yield Structure Model and Principal Component Analysis (PCA). The PCA determine the volatility structure using the eigenvalues and eigenvectors of the covariance matrix between interest rates co-movements at different maturity rates. According to Diebold and Li [32] there is empirical connection between three principle components (PCA1 (level), PCA2 (slope), PCA3 (curvature)) and important information of content which this could explain the structural changes of the yield curve. Whereas in the PYS Model, the two components of term risk, $\ln\tau$ and lower bound of yield curve $\ln r_0$ are associated empirically with PCA2 and PCA3 only.

4. Empirical Analysis

Here, for the further clarity and validation of the PYS Model we provide following empirical results using EU Yield and Spread data. First provided the summary descriptive statistics of Poisson Yield Structure (PYS) spreads, r_0, and τ applied in equations 3.18, using the EU yield curve data from 2007 to 2014 which covers the effect of the 2008 financial crisis on European yields. All data are quarterly from Q_3- 2007 to Q_2- 2014 collected from Thomson Reuters Eikon/DataStream (Credit Curves). Furthermore, we examine the correlation descriptively in table 2.

Table 1 goes here.

From table 1, the mean spreads increase as the standard of credit rating decreases. Similarly, the term risk (τ) worsens as the credit rating decreases since τ is a proxy for time to default**.**

Table 2 goes here.

From the table 2, $\ln r_0$ is positively correlated with yield spread increasing from Spread (AAA) to Spread (BBB). Furthermore, the $\ln\tau$ (GOV) increases as the credit rating rises. For all different level of credit rating there is inverse relationship between $\ln \tau$ and yield spread. The coefficient of variation of all yield spreads are relatively low from 0.53 to 0.61, whereas the relative standard deviation of the term risk for all credit rating are high. This reflects the seriousness of financial crisis in market and economic conditions in Euro zone from 2007 to 2014. Next, we review the data and estimate equations 3.18 over the EU financial crisis from 2007 to 2014. Since in our framework not all variables are treated symmetrically and also they do not influence each other equally therefore we do not use the VAR structure to diagnose cause of the relationship in the underlying PYS model. Hence, we can first test the model under OLS estimation to confirm the validity of the new PYS model and evaluation of the likelihood function in closed form.

Here, first the autocorrelation function (ACF) is applied to determine the appropriate lag order of the ADF tests and the lag truncation for the non-parametric methods. The lag orders are selected for all data with an overall rejection region of 5%. Hence including an appropriate lag order may improve the efficiency of the estimation and alter the subsequent results. The unit root tests applied with individually selected lag lengths on *x(q)* with *(q)* the lag order. Next, we examine the structure of the unit-root test as a technically efficient way of determining non-spurious regression. Since the time series data particularly the data used here are highly time effected and might be non-stationary as a result of technological progress, economic evolution, crises, changes in the consumers' preference and behaviour, policy or regime changes, and organizational or institutional improvement. This can cause significant problems in forecasting and inference; therefore, it is important to find a model that shows a relationship which remains long enough (Hendry & Juselius [40]).

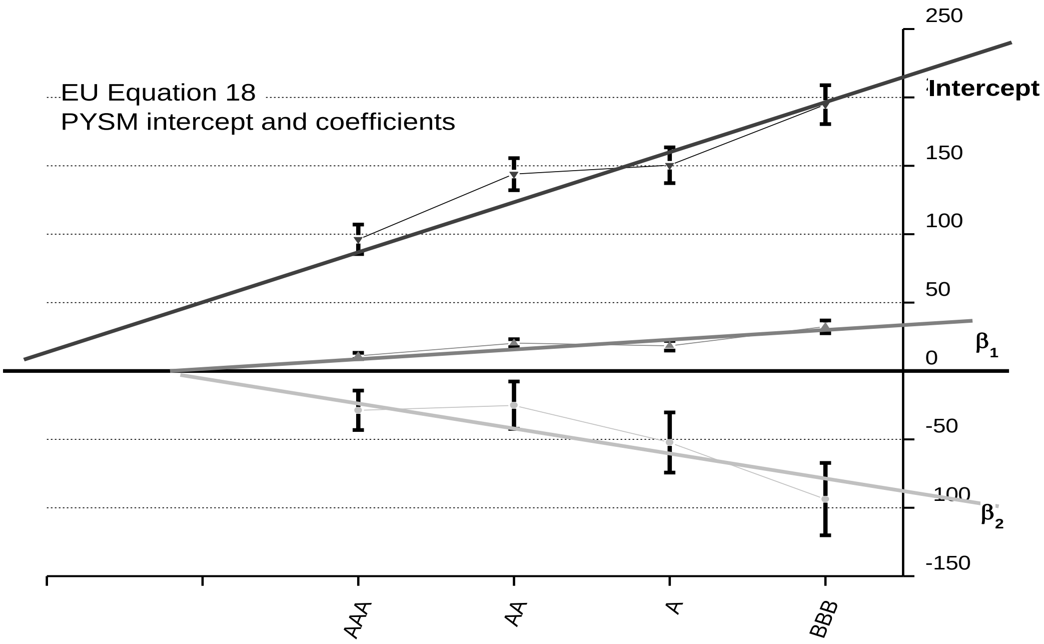
Table 3 goes here.

The result of the stationarity test on PYSM Spread, $\ln r_0$, $\ln \tau$ are presented in table 3 and it confirms the stationarity of all data to permits further estimation for forecasting and long-term analysis. Next, we estimate the PYS Model with different rated debt of AAA, AA, A, BBB using linear regression. The relevant results are presented in Table 4, indicating that changes on lower bound ($\ln r_0$) impact significantly on yield spread. Also, the changes in term risk ($\ln\tau$) impact significantly on yield spread in the case of BBB and A, that is where the credit risk is higher. Whereas, the impact of term risk on yield spread become less significant in the case of AAA and insignificant for credit rating AA. This implies that when the credit risk is highest then the dependence of spread on term risk become less significant.

Table 4 goes here.

To test for a more general relationship between the active variables and spread we plot the intercept and coefficients of the PYS Model to show the sensitivities of spread changes to $\ln r_0$ and $\ln\tau$ changes, figure 7.

Figure 7- Plot of coefficients of PYSM estimation with Different Credit Ratings (AAA, AA, A, BBB)



Note: The PYS Model is driven by equation 3.18, $\partial r_{it} = \propto + \beta_1 \ln r_{0_{it}} + \beta_2 \ln\tau_{it} + u_{it}$.

It is interesting to observe that the effect of ratings dependence appears to be approximately linear in risk model coefficients, though we have no precedence necessarily for this effect. The apparent difference in sensitivities $\beta_2 > \beta_1$ is however illusory. This can be confirmed by calculating the normalized coefficients of β_1 and β_2, i.e. when the coefficients are scaled to zero mean and divided by variance. We obtain the result when averaged over all Ratings AAA, AA, A, and BBB of:

$$\text{by definition} \propto = 0, \quad \beta_1 = 0.659 \pm 0.15 \text{ SE} \quad \text{and} \quad \beta_2 = -0.734 \pm 0.17 \text{ SE}$$

This suggests that the $\ln r_0$ and $\ln\tau$ effects on spread are approximately equal, with an increase in $\ln r_0$ or a reduction in $\ln\tau$ causing an increase in spread, in approximately equal weights. Since on average the mechanism is the same for each Rating, this mechanism is universal. The regression significance tests confirm that we would expect $\ln r_0$ and $\ln\tau$ to operate fully independently. Moreover, the high regression R^2 strengths for spread risks with $\ln r_0$ and $\ln\tau$ in regression equation 3.18, support the isolation of risk in spreads to the term risk mechanism of the PYSM. This dominance is consistent with the view that the effects of the macro variables on term risk should be exogenous to the model, being orthogonal (and cross-sectional) in their effects. Moreover, the standard methodology of Granger causality is applied to evaluate the causation in PYSM model, in its capabilities to characterize risk over the yield curve. The results of Granger Causality test are presented in table 5 which justify that there is strong causation from $\ln r_0$ to yield spread in the case of A, AA and AAA, and also there are significant causation from $\ln\tau$ to yield spread in A and AA. In the case of BBB, it seems that some other factors may cause the changes in yield spread.

Table 5 goes here.

As would be expected, any ARCH/GARCH estimation over the cycle of our estimations, which covers the period of crisis recovery and further crises in the EU period, 2007Q3 to 2014Q2, should reveal extensive heteroskedasticity. The very purpose of the PYSM is to capture these variations in a simple AR(1) dynamic. To see whether we are justified in this assumption, we test for the ARCH/GARCH in the residuals to the equation 3.18 regressions, using the lag structure of 3 or 4 as applied in these primary regressions, shown in table 5. With the results as shown table 6, we conclude that that the autocorrelations in residuals are accordingly very small, indicating that the PYS Model mechanism accurately describes the conditional means of spreads for the different credit ratings AAA, AA, A, and BBB for these data.

5. Conclusion

In this paper, a new system of equations was developed for the pricing of the term risk in yields and spreads in a unified approach. In effect there are three components to conclude:

As the first component, an alternative model for term risk depending on a probability density as new paradigm was proposed. The underlying source of risk was derived as a new martingale for term risk on the Ornstein Uhlenbeck or AR(1) dynamic of damped Brownian motion. In essence, the single event Poisson exponential was found as a probability amplitude optimized in the complex plane of the Fourier transform, with the risk neutral analytic then obtained by a discovered Nikon Radodym change in measure. Although, for reasons of space, the focus in this paper was on the meaning of the distribution in application, rather than in its origins as a probability amplitude in phase space. The mathematical effects of the latter are well worth examining and will be the subject of further research. In this study, analysis is focused on applying the probability density in its new form dL for generating the yield curve differential equations for the yield and spread equations, and for analysing of models for specific yield curve prices – specifically characterizing here the EU corporate bond. The second component was the performance of the PYSM on yield curves and spreads in comparison with classical interest rate modelling. This was done firstly by comparing the nature and rationality of the driving factors in the PYSM compared to the predominantly Gaussian diffusion-based models and multi-factor/multi-element extended versions; secondly, coupled to econometric analysis, how the PYSM's parameters could be used in validation of the interest rate movements and their volatilities, to given shocks over the crisis years in the EU. This gave proof to the following highlighted points:

The PYSM distribution provided a One Factor model for the term risk in the spot rate yield curve and the spread at the probability density maximum. Classical models on the other hand, were characterized by greater complexity in parameters. Besides a major dependence on the stochastic diffusion of Gaussian dynamic models, classical models required a separate market price of risk as descriptor of the systematic interest rate drift, a separate deterministic mean reversion factor, several coefficients required when fitting to time the varying drift in interest rates with maturity, and indeed often a multiplicity of features (jump and stochastic volatility) difficult to fit with any significance. The PYSM kept the driving variables endogenous to the model. Importantly, the PYSM's parameters replaced those of the classical models in a meaningful way. The mean reversion was part of the PYSM density, dependent on the term risk factor ($ln\tau$); further work will be to extend the concept introduced by τ from its connection with the absolute (local) risk averse factor for a term risk utility; the slope of the PYSM yield was directly related to lower bound of the yield curve ($lnr_o = 1/I$). This gave meaning to r_o in comparison with the short rate as policy variable. Multi-factor and multi-process (combining disparate diffusion, jumps and stochastic volatility to get an answer) was avoided. The One Factor character of the PYSM nonetheless provided an analysis of the EU yield curves over their regime changes, the econometric estimations indicated that the PYSM could produce the yields and spreads as outputs to the model within acceptable levels of significance. This compares starkly with the preferred method in LIBOR based modelling which relies on the use of the market curves, principally swap data, as inputs to the modelling of interest rate and spread pricing. The latter method might be more reassuring in being very precisely fitted to the forward data now, but be problematic to the accuracy of assumptions and conditions continuing if predictive powers are important. The dynamic equilibrium credentials of the PYSM might be favoured in the situation where volatility is high, recalibrations very likely and the uncertainty in markets paramount. Its one factor nature also helps in its initial fitting and verification to data in avoidance of overspecification or our-of sample errors. As recommendation, although the range of EU interest rate data we have tested from 2007 to 2014 might be only a beginning in the journey of the model in range of applications. Further research with the PYSM readily centres on the characterization of policy and pricing performance of the yield curve and the variation in swap rates, and how to use futures, swaps and forward rates to manage investments and risks. The markets affected in this case are vast., with a rich source of futures and forwards, interest rate swaps and options data known to exist. Extended further, the PYSM kernel should apply in at least some of the components to the credit risk in CDS and CDX.

The practical application of methods is the third component arising. Results suggested that the PYSM theoretical foundation characterises the term risk represented in the zero yield curves. In principle, the PYSM then provides an interest rate reference point for all positive times, $\forall t > 0$, which should help in interpreting the role of policy rates, including shocks, and also in the filling in of any missing data points. The forwards pricing in the vast and expanding markets for interest rate swaps and futures should then be properly derived rather than imputed from less than complete market data. The replacing of the LIBOR bootstrapping method, very commonly used but recently questioned, should be feasible. Finally, the PYSM parameters may add theoretical meaning to the Nelson Siegel level, slope, curvature graphical interpretations.

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Appendices 1- MATHEMATICAL SOLUTION TO THE PYS MODEL

STEP 1: Ornstein-Uhlenbeck Auto-Correlation $CDF^{\mathbb{P}}$

The Ornstein-Uhlenbeck (OU) model is founded on the concept of an underlying Gaussian Wiener diffusion, coupled to an exponentially damped auto-correlation decay process. As such it can be considered to be a paradigm for time dependence, as a frictionally restrictive effect on a Gaussian randomness. As a first step, we consider the solution assuming the OU is driven by a Levy[2] process.

$$\text{Autocorrelation covariance}(X_t) = \langle X_0, X_t \rangle = \sigma^2(X_0)\exp(-\lambda_t t) \qquad (3.3)$$

This is the randomness defined for the Ornstein Uhlenbeck random process, X_t. This process has zero mean and the exponential correlation function for the time lag t. The constants σ^2 and λ are respectively a variance and a rate of decay factor. Assuming a Levy process, Cont and Tankov [25] show that this equation 3.3 solves to an autocorrelation coefficient.

$$Autocorrelation\ coefficient = \rho(X_t) = \exp(-\lambda_t t) \qquad (3.4)$$

In this form, the OU is aligned to a model for a single jump rate AR(1) process. From the jump process analogy, it is readily seen that the cumulative probability of an event occurring is $\exp(-\lambda t)$. Since in finance we should be interested in risks that could but have yet to happen, the antithesis cumulative probability ($CDF_t^{\mathbb{P}} = 1 - \exp(-\lambda_t t)$) is the usual form needed.

A number of routes have been taken to try to extend the basic OU in autocorrelation factor form, equation 3.4, to model the observed clustering patterns in term risk premia observed in practice. This might involve the jump-diffusion by adding proportional log normal jumps, Bates [7]. Alternatively, in the Barndorff-Nielsen & Shephard model [8], a positive OU process is used directly to represent the square of the volatility process. Another method is proposed by Carr, Geman, Madan [22] in which an exponential Levy model is used with time changes. Interestingly, the latter model uses Levy models without any Brownian component. One guiding feature of such methods is the search for tractable solutions and appropriate risk neutral (martingale) pricing. These models provide some of the asymmetry and the nonlinearity in distributions observed in practice, although not sufficiently for them yet to gain universal acceptance.

Instead, this study uses an original approach which recognizes that the rate constant λ_t in equation 3.4 is adapted to a probability distribution not shown in the time dimension[3]. Although λ_t is applied as a constant in equation 3.4, it consists of a continuous distribution infinite number of rate (or frequency) described pathways, but which need to be integrated for their expectations over all correlations. To find the rate or frequency version of the Poisson exponential the nonparametric solution is obtained by Fourier transform. The theorem for this conversion is the Wiener Khintchine theorem which is well known in the sciences for the analysis of dynamic mechanical molecular transitions, Strobl [63].

Using the Poisson method proposed in this study, this simplifies the findings from the earlier non-Gaussian Ornstein Uhlenbeck literature by excluding the Wiener stochastic and also finding the single event jump transition martingale. This approach leaves an AR(1)

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[2] A Levy process has increments that are stationary, independent and Markovian in their dynamics. Both Gaussian-wiener diffusion and infinitely divisible Poisson jump processes conform to Levy processes.

[3] A single valued intensity factor occurs in the Poisson exponential.

autocorrelation as stochastic differential equation which is a complete martingale[4] for term risk price.

STEP 2: Form of the PDF/CDF in Frequency Measure

The second step is to solve for the Fourier transform of the cumulative probability density of equation 3.3.

As discussed by Strobl for linear and previsible systems, we may use the Weiner-Khintchine theorem (to obtain the marginal probability density by Fourier analysis Rather than characterizing the dynamics of a fluctuating state variable by the time-dependent correlation function $\langle X_0 , X_t \rangle$ between the random variables of X_0 and X_t, one can also describe it by the spectral density, $\langle X_0, X_\omega \rangle$.

The Wiener-Khintchine theorem, a fundamental theorem of statistical physics, states that these two functions represent a pair of Fourier transforms, i.e.

$$\langle X_0 , X_t \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} . \langle {X_\omega}^2 \rangle . d\omega \tag{3.5}$$

and

$$\langle X_0, X_\omega \rangle = \int_{-\infty}^{\infty} e^{i\omega t} . \langle X_0 , X_t \rangle . dt \tag{3.6}$$

where ω represents frequency, which is equal to $1/t$. The result of this conversion to a power spectrum finds a solution as integrand in a complex plane. It is convenient to define X_ω over the stress factor to justify a response function ϕ_ω over the full range of integration. This obtains:

$$\phi_\omega = \phi'_\omega - i . \phi''_\omega = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} e^{-i\omega t}\, PDF_t . dt \tag{3.7}$$

The PDF is obtained from the Ornstein Uhlenbeck CDF above, as

$$PDF_t = (-\lambda). \mathrm{e}^{-\lambda \mathrm{t}}$$

Equation 8 obtains the solution with full range from $-\infty$ to $+\infty$ as:

$$\phi_\omega = \phi'_\omega - i . \phi''_\omega = \frac{\sqrt{2}}{\pi} . \int_{-\infty}^{\infty} \left(\frac{1/\lambda}{1+(\omega/\lambda)^2} - i . \frac{\sqrt{2}}{\pi} . \frac{\omega/\lambda}{1+(\omega/\lambda)^2} \right) . d\omega \tag{3.8}$$

We can solve for $\phi_\omega$by setting the limits correctly. Firstly, we expect in finance that the probability density is a real valued function. Secondly, we can set the upper limit in time as the response function in time ϕ_t , when in equilibrium as

$$\langle X_0 , X_t \rangle = \Delta\phi_t(\infty) - \Delta\phi_t(\mathrm{t})$$

This expresses the fluctuation-dissipation theorem first presented by Onsager [56] and Callen & Welton [20], which states that the return of the response function to equilibrium when disturbed by a small displacement from equilibrium, is determined by the equilibrium fluctuations themselves. The left-hand side represents the effects of spontaneous fluctuations in steady state randomness in prices. The right-hand side defines the reaction of the sample to the imposition of an external force or disturbance. The response of the system denoted by $\Delta\phi_t$ is then linearly related to the forcing function.

Using the boundary condition that the steady state (i.e. final state) of the response function, $\alpha(t \to \infty)$, must agree with the limiting value of the dynamic susceptibility at zero frequency,

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[4] One jump ensures that the martingale is unique over all pathways since probability cumulates to1. Multiple jumps would destroy the complete market condition.

$$\frac{1}{2\pi}\int_0^\infty \langle X_0, X_\omega\rangle . d\omega = \phi'_\omega(at\ \omega = 0) \quad (3.9)$$

Applying the Kramers-Kronig relationship this can be rewritten as:

$$\frac{1}{2\pi}\int_0^\infty \langle X_0, X_\omega\rangle . d\omega = \frac{1}{\pi}\int_0^\infty \frac{\phi''_\omega}{\omega} . d\omega$$

$$PDF_\omega^{\mathbb{P}} = \frac{\sqrt{2}}{\pi} . \frac{1/\lambda}{1+(\omega/\lambda)^2} \quad (3.10)$$

Equation 3.10 provides the pricing kernel for the variations in the Poisson single event process, in frequency measure. The traditional view of the Poisson process is that since each Poisson jump when observed is discrete, the risk neutral and physical probabilities must be the same, i.e. its $\mathbb{P}$ and $\mathbb{Q}$ measures are equal. It is then assumed that the $\mathbb{Q} = \mathbb{P}$ law applies to all jumps, as in the discontinuous process jump-diffusion models of Merton [53], or Zhou [70]. The new feature of a distribution however when the single jump is expanded by frequency transform suggest that a reweighting martingale might be possible after all, which we now try.

First, we notice that the real probability density function for term risk is represented by equation 3.10 as a Cauchy distribution. We cannot however use the Gaussian Ito (and Girsanov theorem) to find its $\mathbb{Q}$ measure martingale, as would be the case possible if term risk were a Gaussian diffusion. The fact that the Cauchy distribution is non-integrable[5] also seems to rule-out its applicability as a pricing distribution. The break through is to observe as we show below that equation 10 can be reweighted using the method in Step 3. A feasible new martingale for term risk, which is integrable by definition, may then be obtained.

STEP 3: Form of $\mathbb{Q}$ risk neutral PDF in log scale frequency measure

To find the correct unique martingale for term risk, we start with the formula under measure $\mathbb{P}$ in equation 3.10. The method we use to explain the martingale employs the theory of Harrison and Kreps [38]. Applying Harrison-Kreps requires finding the Nikon-Radodym ratio, $d\mathbb{Q}/d\mathbb{P}$, which relates the literal $\mathbb{P}$ and the risk-neutral $\mathbb{Q}$ measures, consistent with martingale conditions. If the complete martingale and its technical conditions hold, an implied asymmetric drift-less market pricing model then is obtained in equation 3.11.

$$\int_{-\infty}^{\infty} PDF_\omega^{\mathbb{P}} . d\omega = \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty} \frac{\omega/\lambda}{1+(\omega/\lambda)^2} . \frac{1}{\omega} . d\omega \quad (3.11)$$

To find the martingale measure, we examine how we might be able rewrite the integrand in equation 3.11 to obtain a valid Nikon Radodym martingale differential for equation (3.11) as pricing kernel

$$E^{\mathbb{Q}}(A) = E^P[1_A\xi] = E^P\left[1_A \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

For a martingale, the distribution $\xi = d\mathbb{Q}/d\mathbb{P}$ must be positive and integrable. Here, we test for simplest solution when $\xi = \omega$, $\omega > 0$ and $E[\omega] = 1$ since we have assumed at the outset the one-event Poisson process $P = e^{-\lambda t}$ as our fundamental term risk process.[6] Hence this motivates the rewriting of the integrand of equation 3.11 to equation below:

[5] As an unbounded therefore non-integrable expectation.

[6] The technical conditions for a complete martingale are satisfied when $\omega > 0$ with $\int_{-\infty}^{\infty} d\omega = 1$

$$\int_{-\infty}^{\infty} PDF_{\omega}^{\mathbb{P}} \, . d\omega = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} \frac{1/\lambda}{1+(\omega/\lambda)^2} . d\mathbb{Q}/d\mathbb{P} \, . \frac{d\omega}{\omega} \tag{3.12}$$

A standard condition for a martingale, that the densities in $\mathbb{P}$ and $\mathbb{Q}$ share identical pathways, is of course assumed. To obtain the $\mathbb{Q}$ PDF satisfying all the technical risk neutral conditions above, the integrand in equation 3.12 can be written to a new integration measure and limits, equation 3.13. This is the key step, when the integration measure frequency ω is replaced by lnω

$$\int_{-\infty}^{\infty} PDF_{\omega}^{\mathbb{P}} \, . \omega . d\ln\omega = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} \frac{\omega/\lambda}{1+(\omega/\lambda)^2} . d\ln\omega \tag{3.13}$$

Equation 3.13 confirms that the probability density in ω measure $PDF_{\omega}^{\mathbb{P}}$ (which is equal ϕ_{ω}'' the 'imaginary' component of the complex response function, equation 3.8) should in its martingale be under lnω measure. A few algebraic steps now allow us to make this conversion, to obtain the martingale for term risk.

$$PDF_{\omega}^{\mathbb{P}} = \phi_{\omega}'' = \frac{\sqrt{2}}{\pi} . \frac{\omega/\lambda}{1+(\omega/\lambda)^2}$$

Where the logarithmic form of the ϕ_{ω}'' gives:

$$PDF_{ln\omega}^{\mathbb{Q}} = \phi_{\ln\omega}'' = \frac{\sqrt{2}}{\pi} . \frac{e^{\ln(\omega/\lambda)}}{1+e^{2\ln(\omega/\lambda)}}$$

Substitute x for $ln(\omega/\lambda)$ gives:

$$\phi_{\ln\omega}'' = \frac{\sqrt{2}}{\pi} . \frac{e^{x}}{1+e^{2x}}$$

Multiply through by e^{-x}:

$$\phi_{\ln\omega}'' = \frac{\sqrt{2}}{\pi} . \frac{e^{x} . e^{-x}}{(1+e^{2x}) . e^{-x}} = \frac{\sqrt{2}}{\pi} . \frac{1}{e^{-x}+e^{x}}$$

Expand e^{-x} and e^{x} :

$$\phi_{\ln\omega}'' = \frac{\sqrt{2}}{\pi} . \frac{1}{1-(-x)+\frac{(-x)^2}{2!}-\frac{(-x)^3}{3!}+O.\frac{(-x)^4}{4!}- \dots \quad +1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+O.\frac{x^4}{4!}+ \dots}$$

Odd powers cancel leaving only even functions and the distribution becomes square integrable and symmetric in $ln(\omega/\lambda)$. It is found that $PDF_{ln\omega}^{\mathbb{Q}} = \phi_{\ln\omega}''$ equates to a Cauchy distribution in log frequency or log time (see below), for all values x to leading order. As a means to calculate the expectation integral we now have a new structure $PDF_{ln\omega}^{\mathbb{Q}}$ in log frequency that is martingale to replace the objective probability density $PDF_{\omega}^{\mathbb{P}} \, . \omega$ in equation 3.13. This is then the appropriate distribution for the market pricing $\mathbb{Q}$ of term risk. It is the universal equation for term risk, which we normally identify as $d\widetilde{L}$ from now on. $\widetilde{L}$ will register the cumulative distribution CDF of $d\widetilde{L}$.

$$PDF^{\mathbb{Q}}(term\,risk) = d\widetilde{L} = \frac{\sqrt{2}/\pi}{2+(\ln\omega-\ln\lambda)^2} = \frac{1/\pi\sqrt{2}}{1+\left((\ln\omega-\ln\lambda)/\sqrt{2}\right)^2}$$

Finally, we note that since the pricing kernel for term risk above has had to be developed in a spectral analysis, it occurs in terms of the frequency variable (~1/time). Clearly, frequency is not a well-versed term in the world of finance. If we could re-express the kernel to a function of the time variable we should do so. Fortunately, the conversion for time to frequency and vices versa in the pricing kernel is a trivial matter, given their occurrence in log form. Replacement of $\ln \omega$ with $\ln t$ [7] and also $\ln \lambda$ with its time factor $\ln \tau$ leave the $\widetilde{L}$ and $d\widetilde{L}$ equations unchanged, e.g. to obtain equations 3.14 and 3.15.

$$PDF^{\mathbb{Q}} = d\widetilde{L} = \frac{1/\pi\sqrt{2}}{1+\left(\ln(t)-\ln(\tau)\right)/\sqrt{2}\big)^{2}} \tag{3.14}$$

Since this equation is a Cauchy distribution, it also possesses the Cumulative Probability Distribution $CDF^{\mathbb{Q}}(term\ risk) = \widetilde{L}_{\ln t}$, in Cauchy analytic form.

$$\widetilde{L}_{\ln t} = \left(.5 + (1/\pi).arctan((ln\, t - ln\tau)/\sqrt{2}\right) \tag{3.15}$$

The availability of the term risk martingale PDF and CDFs in analytical forms 3.14 and 3.15 simplifies the pricing of conditional prices in derivatives in general. We shall use these distributions as the pricing kernels for the spot discount rates and spreads extensively in this paper, sections 3 to 5. The problems with existing spot rate or forward rate theories, due to the intractability of models arising from complexity of structures, may also be avoided. The other important advantage is the greater intuition on cause and effect given by specific model factors once the pricing models are obtained, because of the transparency that analytical solutions give. For example, the PDF equations for yield curve and spreads are quite readily obtained in simple differential structures, see section 3.

[7] Since frequency = 1/time, $\ln \omega = -\ln t$. When squared, $(\ln \omega)^2 = (-\ln t)^2$.

Appendices 2
List of Tables:

Table 1- Summary statistics of PYS spreads, $Ln\mathrm{R}_0$ and $Ln\tau$ from 2007 Q3 to 2014 Q2

	Mean	SD	Kurt	Skew
S(AAA)	53.9429	32.8654	2.3326	1.2176
S(AA)	92.1000	55.6732	-0.2232	0.7603
S(A)	109.1964	61.4097	0.8291	1.0179
S(BBB)	143.4536	75.5336	1.9123	1.4047
Ln(t-GOV)	1.3177	0.5421	-0.3957	0.7142
Ln(t-AAA)	0.4626	0.4087	0.6748	0.6524
Ln(t-AA)	0.0729	0.4616	2.0064	-0.5055
Ln(t-A)	-0.0271	0.5042	0.3081	0.3035
Ln(t-BBB)	-0.2383	0.5433	0.4499	0.5200
Ln(R0)	-2.2127	2.5643	-1.2856	-0.1553

Table 2- The Correlation matrix between PYS spreads, LnR_0 and $Ln\tau$ used in equation (18)

	S(AAA)	**S(AA)**	**S(A)**	**S(BBB)**	Ln(R0)
S(AAA)	1				
S(AA)	0.934816	1			
S(A)	0.900015	0.974495	1		
S(BBB)	0.8151	0.915481	0.957856	1	
Ln(R0)	**0.488727**	**0.595855**	**0.643766**	**0.609435**	1
Lnt-Gov	**-0.01365**	**0.173339**	**0.238015**	**0.344077**	**0.542176**
Lnt-AAA	**-0.19011**	-0.07647	0.023293	0.111687	0.46276
Lnt-AA	-0.37076	**-0.36873**	-0.25814	-0.17269	0.299749
Lnt-A	-0.34025	-0.30702	**-0.26588**	-0.22511	0.431787
Lnt-BBB	-0.00508	0.049788	0.075394	**-0.0089**	0.66487

Table 3- Summary of ADF test on S_P, lnR_0, $\mathrm{Ln}\tau$ with different rated debt of AAA, AA, A, and BBB from 2007-Q3 to 2014-Q2

	BBB	A	AA	AAA
S_P(q)	q=3 -3.7872 *** (0.04)	q= 3 -3.9347 *** (0.03)	q=4 -3.3414 * (0.08)	q=3 -3.2467 * (0.10)
Ln τ(q)	q=3 -.4853** (0.02)	q= 4 -4.8616 *** (0.00)	q=4 -3.0313 ** (0.05)	q=2 -3.4765 * (0.02)
Ln R_0(3)	-5.8034 *** (0.00)			

Note: Values without the bracket presents ADF t-statistic and the values with bracket denotes the P-values. ******* Significant at the 99% confidence level, ****** significant at the 95%, and * significant at 90% confidence level.

Table 4- Summary of Regression of Spread on LnR_0 and $Ln\tau$ (PYS Model)

BBB		A		AA		AAA	
α	194.72*** (13.78) [14.13]	**α**	150.42*** (11.59) [12.98]	**α**	143.93*** (14.77) [11.6836]	**α**	96.27*** (8.93) [10.78]
β_1 (q=3)	32.32*** (6.90) [4.68]	**β_1 (q=3)**	18.52*** (5.21) [3.55]	**β_1 (q=3)**	20.40*** (7.04) [2.90]	**β_1 (q=3)**	10.99*** (4.90) [2.24]
β_2 (q=3)	-93.68*** (-3.53) [26.52]	**β_2 (q=4)**	-52.26** (-2.38) [21.92]	**β_2 (q=4)**	-25.04 (-1.45) [17.29]	**β_2 (q=3)**	-28.77* (-1.99) [14.47]
R^2	0.68	R^2	0.65	R^2	0.74	R^2	0.52
SE	46.09	SE	39.05	SE	31.46	SE	24.69
Durbin Watson	1.51	D-W	1.67	D-W	1.9	D-W	1.61
P (F-statistics)	0.00	P (F-stat)	0.00	P (F-stats)	0.00	P (F-stats)	0.00

Note: Estimation of equation 18, PYS Model: $\text{spread}_{\text{it}} = \partial R_{it} = \propto + \beta_1 \ln R_{0_{it}} + \beta_2 \ln\tau_{it} + u_{it}$

Values with () presents t-statistic and the values with [] denotes the Standard Error. ******* Significant at the 99% confidence level, and****** Significant at the 95% confidence level. q is the optimal lag truncation.

Table 5- Summary of Granger Causality Tests between Spread on LnR_0 and $Ln\tau$ in PSY Mode

	Spread causing	**LnR_0 causing**	**$Ln\tau$ causing**
BBB	LnR (0.55)	S (0.12)	S (0.26)
	Lnτ (0.67)	Lnτ (0.73)	LnR_0 (0.53)
A	LnR_0 (0.36)	**S (0.02)****	**S (0.01)****
	Lnτ (0.71)	Lnτ (0.19)	**LnR_0 (0.02)****
AA	LnR_0 (0.25)	**S (0.02)****	**S (0.02)****
	Lnτ (0.15)	**Lnτ (0.01)*****	**LnR_0 (0.03)****
AAA	LnR_0 (0.32)	**S (0.0.04)****	S (0.20)
	Lnτ (0.04)**	**Lnτ (0.02)****	LnR_0 (0.75)

Table 6- Summary of ARCH Estimation in PYS Model Residuals using different rated debt of AAA, AA, A, and BBB from 2007-Q3 to 2014-Q2

BBB		A		AA		AAA	
C	935.26 (0.56) [1681.25]	C	138.08 (0.12) [1155.60]	C	553.44 (0.69) [806.69]	C	30.97 (0.24) [128.95]
$(Resid(-1))^2$	0.20 (0.45) [0.44]	$(Resid(-1))^2$	0.40 (0.82) [0.49]	$(Resid(-1))^2$	0.07 (0.18) [0.41]	$(Resid(-1))^2$	1.18 (1.837) [0.86]
$(Resid(-2))^2$	0.05 (0.16) [0.30]	$(Resid(-2))^2$	0.17 (0.82) [0.49]	$(Resid(-1))^2$	0.07 (0.18) [0.41]	$(Resid(-1))^2$	1.18 (1.837) [0.86]
$(Resid(-3))^2$	-0.24 (-0.69) [0.35]	$(Resid(-3))^2$	-0.03 (-0.02) [1.10]	$(Resid(-3))^2$	-0.11 (-0.69) [0.17]	$(Resid(-3))^2$	-0.02 (-0.02) [0.89]
				$(Resid(-4)^2$	-0.16 (-0.69) [0.20]		
GARCH(-1)	0.46 (0.39) [1.17]	GARCH(-1)	0.34 (0.19) [1.75]	GARCH(-1)	0.34 (0.19) [1.75]	GARCH(-1)	0.22 (0.05) [4.21]
GARCH(-2)	0.02 (0.01) [1.49]	GARCH(-2)	-0.09 (-0.01) [7.14]	GARCH(-2)	0.02 (0.05) [1.35]	GARCH(-2)	-0.10 (-0.08) [1.22]
GARCH(-3)	0.00 (0.00) [1.04]	GARCH(-3)	-0.03 (-0.02) [1.10]	GARCH(-3)	-0.02 (0.01) [1.47]	GARCH(-3)	-0.02 (-0.02) [0.89]
				GARCH(-4)	0.02 (0.25) [0.20]		
R^2 SE Durbin Watson	0.66 47.78 1.62	R^2 SE Durbin Watson	0.65 39.28 1.58	R^2 SE Durbin Watson	0.74 31.57 1.94	R^2 SE Durbin Watson	0.49 25.42 1.53

Note: Values with () presents z-statistic and the values with [] denotes the Standard Error. ******* Significant at the 99% confidence level, ****** Significant at the 95% confidence level, and * significant at the 90% confident level.