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A DYNAMIC MAXIMUM PRINCIPLE FOR THE OPTIMIZATION OF RECURSIVE UTILITIES UNDER CONSTRAINTS

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This paper examines the continuous-time portfolio-consumption problem of an agent with a recursive utility in the presence of nonlinear constraints on the wealth. Using backward stochastic differential equations, we state a dynamic maximum principle which generalizes the characterization of optimal policies obtained by Duffie and Skiadas [*J. Math Econ*. **23**, 107–131 (1994)] in the case of a linear wealth. From this property, we derive a characterization of optimal wealth and utility processes as the unique solution of a forward-backward system. Existence of an optimal policy is also established via a penalization method.

0. Introduction. In this paper, we consider the continuous-time portfolio-consumption problem of an agent with a recursive utility when the wealth is supposed to satisfy a nonlinear equation. The case of the large investor or the case of constraints such as taxes can be modeled as special cases of this setting.

The optimization problem in the case of a standard utility and linear wealth in a complete market has been largely studied in the literature. Originally introduced by Merton (1971) in the context of constant coefficients and treated by Markovian methods via the HJB equation, it was developed for general processes by the martingale approach by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989).

In the case of an incomplete market, Duffie, Fleming and Zariphopoulou (1991), Duffie and Zariphopoulou (1993) have provided some existence results and characterization of optimal policies in the Markovian case by considering the HJB equation. In the non-Markovian but still incomplete case, some results of existence and characterization of optimal policies have been obtained using martingale and duality techniques by He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). Similar results have been derived by Cvitanic and Karatzas (1992), who consider the more general case of constraints on the portfolio weights. More recently, Cuoco (1997) considered the optimal consumption problem in the presence of a stochastic endowment and constraints on the portfolio choices. He stated a nice existence result under fairly general assumptions using some fine techniques of analysis and gave also a characterization of optimal consumption policies.

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Duffie and Skiadas (1994) have considered the optimization problem when there are no constraints on the wealth and portfolios but when the utility is nonlinear; the case of a recursive utility [as introduced by Duffie and Epstein (1992)] was included. They presented a martingale version of the Kuhn–Tucker condition for optimal policies (which gives a characterization of optimality). To our knowledge, no existence result has been proved in this case.

In this paper, we consider the optimization problem when the utility is recursive with constraints on the wealth, which include the case of a large investor or the case of taxes. In other terms, the utility and the wealth processes are supposed to satisfy nonlinear equations. In this work, we emphasize the symmetry between utility and wealth and show that the formulation of this problem using BSDEs (backward stochastic differential equations) is the good one. Recall that these BSDEs have been introduced by Pardoux and Peng (1990) and that their properties and their applications to finance have been developed by El Karoui, Peng and Quenez (1997). As it has been noted in this paper, a recursive utility process can be seen as the solution of a BSDE which is not necessarily linear. Furthermore, the comparison theorem for BSDEs gives quite easily that the positive constraint on the wealth process is equivalent to the positivity of the terminal wealth only. Using this property, it is possible to take the terminal wealth instead of the portfolio process as control. Consequently, the problem can be written in a backward formulation which emphasizes the symmetry between utility and wealth; more precisely, for a given consumption process and a given positive terminal wealth, the utility and wealth processes are both solutions of nonlinear BSDEs.

Using BSDE techniques, we generalize the characterization of optimality obtained by Duffie and Skiadas (1994). To do this, we derive a maximum principle which gives a necessary and sufficient condition of optimality. As in Duffie and Skiadas (1994) and Schroder and Skiadas (1997), this characterization can be written in terms of the optimal utility and wealth processes but also their associated optimal deflators. Furthermore, from the maximum principle, we can derive a characterization of the optimal wealth and utility and their associated deflators as the unique solution of a forward-backward system.

Also, we state in this paper the existence and the uniqueness of an optimal strategy. The method consists of approaching our problem by a sequence of penalized optimization problems for which there is existence and uniqueness of an optimal strategy. Then, by using the maximum principle applied to the penalized problems, we show that this sequence is locally weakly compact and we derive the existence of an optimal solution.

The outline of the paper is the following. In Sections 1 and 2, we present the model of recursive utility and nonlinear wealth and give some examples. The agent, endowed with initial wealth x, makes his choice among feasible strategies, in order to maximize his recursive utility. The value function of this optimization problem is called maximal reward. In Section 3, we state a comparison theorem concerning maximal rewards, from which the finiteness of the maximal reward is derived. Then, we give a backward formulation of our problem which emphasizes the symmetry between utility and wealth, both satisfying non linear BSDEs.

In Section 4, using this backward formulation, we obtain a characterization of optimality. First, we derive a maximum principle which gives a necessary condition. Second, in the general case, using concavity properties and the comparison theorem, we state that this condition is sufficient.

In Section 5, we state the existence and the uniqueness of an optimal strategy. We proceed by first approaching our problem by a sequence of penalized optimization problems for which there is existence and uniqueness of an optimal strategy. Then, by using the maximum principle applied to the penalized problems, we show that this sequence is bounded in a space of positive and square-integrable variables and processes. Consequently, there exists a subsequence which converges in a weak sense; we then show that the limit is optimal for our problem.

In Section 6, we derive a characterization of the optimal wealth and utility as the solution of a forward-backward system. In the last section, we give some examples which illustrate this characterization. First, we consider the example of a recursive utility and a linear wealth [studied by Duffie and Skiadas (1994) and also Schroder and Skiadas (1997)]. Second, we consider the case of a generalized recursive utility and linear wealth. In these two examples, the optimal utility process and its associated deflator satisfy a forward-backward system. Third, we analyze the example of a large investor and a standard linear utility function. In this case, the optimal wealth process and its associated deflator satisfy a forward-backward system.

1. The utility process. Recall that in the continuous case, under uncertainty, the notion of recursive utility was first introduced by Duffie and Epstein (1992) to allow a separation between risk aversion and intertemporal substitution.

Let us consider a small agent who can consume between time 0 and time T. Let c_t be the (positive) consumption rate at time t. We suppose that there exists a terminal reward Y at time T. The utility at time t is a function of the instantaneous consumption rate c_t and of the future utility (corresponding to the future consumption). More precisely, the recursive utility at time t is defined by

$$Y_t = E\left[Y + \int_t^T f(s, c_s, Y_s, Z_s)\, ds \,|\mathscr{F}_t\right], \tag{1}$$

where $\mathscr{F}_t$ denotes the natural filtration associated with the n-dimensional Brownian motion W and where f is called a standard driver. Note that the standard driver f can also depend on the "variability" process Z_t (recall that $|Z_.|^2$ is the integrand of the quadratic variation of the process Y). As has been noted by El Karoui, Peng and Quenez (1997), the utility process can be seen as the solution of the BSDE given by

$$-\,dY_t = f(t, c_t, Y_t, Z_t)\, dt - Z_t^* \, dW_t, \qquad Y_T = Y. \tag{2}$$

Recall that the recursive utility defined by Duffie and Epstein (1992) corresponds to the case where the driver f (called also the aggregator) does not depend on Z_t. For example, the standard utility corresponds to a driver of the form

$$f(c, y, z) = u(c) - \beta\, y.$$

It is given by

$$Y_t = E\left[Y\, e^{-\beta(T-t)} + \int_t^T u(c_s)\, e^{-\beta(s-t)}\, ds \,|\mathscr{F}_t\right].$$

Another example is given by the Uzawa utility for which the driver has the same form as the additive utility, but the discounting rate β depends on the consumption rate c_t:

$$f(c, y) = u(c) - \beta(c) y.$$

The additive utility (and also the Uzawa utility) can be generalized quite naturally by considering a driver of the form (in the case of a one-dimensional Brownian motion)

$$f(c, y, z) = u(c) - \beta\, y - \gamma|z|,$$

where γ is a positive constant which can be interpreted as a risk-aversion coefficient or an ambiguity aversion coefficient [see Chen and Epstein (1999)]. In the case of an n-dimensional Brownian motion, the model (called γ-ignorance) becomes

$$f(c, y, z) = u(c) - \beta\, y - \gamma \cdot |z|,$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ and where $|z|$ denotes the n-dimensional vector with ith component $|z^i|$.

We will now specify the notation. Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an $\mathbb{R}^n$-valued Brownian motion W, we denote by $\{(\mathscr{F}_t);\ t \in [0, T]\}$ the filtration generated by the Brownian motion W and augmented, and by $\mathscr{P}$ the σ-field of predictable sets of $\Omega \times [0, T]$. Let $\mathbb{L}^2_T(\mathbb{R}^d)$ be the space of all $\mathscr{F}_T$-measurable random variables $X\colon \Omega \mapsto \mathbb{R}^d$ satisfying $\|X\|^2 = \mathbb{E}(|X|^2) < +\infty$. Also, $\mathbb{H}^2_T(\mathbb{R}^d)$ will denote the space of all predictable processes $\varphi\colon \Omega \times [0, T] \mapsto \mathbb{R}^d$ such that $\|\varphi\|^2 = \mathbb{E}\int_0^T |\varphi_t|^2\, dt < +\infty$. For notational simplicity, we will often write $\mathbb{L}^2_T$ instead of $\mathbb{L}^2_T(\mathbb{R}^d)$, $\mathbb{H}^2_T$ instead of $\mathbb{H}^2_T(\mathbb{R}^d)$.

We make some classical assumptions which ensure that BSDE (2) has a unique solution.

ASSUMPTION A1. *f satisfies the assumptions of* standard drivers. *More precisely, it is a real process defined on $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ which is $\mathscr{P} \times \mathscr{B}(\mathbb{R}^2) \times \mathscr{B}(\mathbb{R}^n)$-measurable. It is supposed to be uniformly Lipschitz with respect to y, z; that is, there exists a constant $K > 0$ such that*

$$\forall t, c, \omega, y_1, y_2, z_1, z_2$$
$$|f(t, \omega, c, y_1, z_1) - f(t, \omega, c, y_2, z_2)| \le K(|y_1 - y_2| + |z_1 - z_2|).$$

ASSUMPTION A2. *We suppose that there exist some constants k_1, k_2 such that*

$$\forall c \in \mathbb{R}^+ \qquad |f(t, c, 0, 0)| \leq k_1 + k_2 \frac{c^p}{p} \quad \textit{a.s.} \quad \textit{with } 0 \leq p < 1 \textit{ and } p \neq 0.$$

Assumptions A1 and A2 ensure that, for each $c \in \mathbb{H}^2$ and each terminal reward $Y \in \mathbb{L}^2$, BSDE (2) has a unique solution (Y, Z) in $\mathbb{H}^2 \times \mathbb{H}^2$.

We also make the following natural assumptions.

ASSUMPTION A3. *f is strictly concave with respect to c, y, z and f is a strictly nondecreasing function with respect to c.*

Assumption A3 ensures, by the comparison theorem, the usual properties of utility functions, that is, monotonicity with respect to the terminal value and to the consumption and concavity with respect to the consumption.

In general, the terminal value Y will measure the utility of terminal wealth, that is, $Y(\omega) = h(X_T(\omega), \omega)$, where X_T is the value of the agent's wealth at terminal time T and where h satisfies the following assumptions.

ASSUMPTION A4. *h is a real function defined on $\mathbb{R} \times \Omega$ which is $\mathscr{F}_T \times \mathscr{B}(\mathbb{R})$-measurable. Furthermore, it is strictly concave and strictly nondecreasing with respect to x and satisfies*

$$\forall x \in \mathbb{R}^+ \qquad |h(x)| \leq k_1 + k_2 \frac{x^p}{p} \quad \textit{a.s.} \quad \textit{with } 0 < p < 1.$$

Assumption A4 ensures that, for each $X_T \in \mathbb{L}^2$, the variable $h(X_T) \in \mathbb{L}^{2/p} \subset \mathbb{L}^2$, and that the recursive utility associated with this terminal value is increasing and concave with respect to terminal wealth.

In the next section, we will specify the dynamics satisfied by the wealth process.

2. The wealth process. The agent can also invest some of his wealth in the market which contains $n+1$ assets. One of them is a nonrisky asset (the money market instrument), with price per unit P_t^0 governed by the equation

$$dP_t^0 = P_t^0 r_t \, dt, \tag{3}$$

where r_t is the short rate.

In addition to the bond, n risky securities (the stocks) are continuously traded. The price process P_t^i for one share of ith stock is modeled by the linear stochastic differential equation

$$dP_t^i = P_t^i \left[b_t^i \, dt + \sum_{j=1}^{n} \sigma_t^{i,j} \, dW_t^j \right]. \tag{4}$$

The predictable volatility matrix $\sigma_t = (\sigma_t^{i,j})$ is supposed to have full rank.

The small agent, whose actions cannot affect market prices, can decide at time $t \in [0, T]$ what amount π_t^i of the wealth X_t to invest in the ith stock, $i = 1, \ldots, n$. Of course, his decisions can only be based on the current information $(\mathcal{F}_t)$, that is, $\pi_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^n)^*$ and $\pi_t^0 = X_t - \sum_{i=0}^n \pi_t^i$ are predictable.

A general setting for the wealth equation can be given by

$$-\,dX_t = b(t, c_t, X_t, \sigma_t^* \pi_t)\,dt - \pi_t^* \sigma_t\,dW_t. \tag{5}$$

Here are some examples of some possible wealth equations.

THE STANDARD LINEAR CASE. This corresponds to the following dynamics:

$$dX_t = (r_t X_t + \pi_t^* \sigma_t \theta_t - c_t)\,dt + \pi_t^* \sigma_t\,dW_t,$$

where θ_t is the risk premium vector, such that $b_t - r_t \mathbf{1} = \sigma_t \theta_t$ $\mathbb{P}$ a.s where $\mathbf{1}$ is the vector whose every component is 1. The driver b is then given by

$$b(t, c_t, x, \sigma_t^* \pi) = -r_t x - \pi^* \sigma_t \theta_t + c.$$

EXAMPLE OF TAXES. We suppose that there is a higher interest rate for borrowing $R_t \geq r_t$ and that there are some taxes which must be paid on the gains made on the risky securities. In this case, the wealth process X satisfies

$$-dX_t = b(t, c_t, X_t, \sigma_t^* \pi_t)\,dt - \pi_t^* \sigma_t\,dW_t,$$

where the driver b of this SDE is given by the convex process

$$b(t, c, x, \sigma_t^* \pi) = -r_t x - \pi^* \sigma_t \theta_t + \alpha(\pi^* \sigma_t \theta_t)^+ + (R_t - r_t)\left(x - \sum_{i=1}^n \pi^i\right)^- + c,$$

where α is a positive constant.

EXAMPLE OF THE LARGE INVESTOR. The example of the nonlinear portfolio-dynamics large investor has been considered by Cuoco and Cvitanic (1995). The prices of the nonrisky assets follow the dynamics

$$\begin{aligned} dP_t^0 &= P_t^0 [r_t + f_0(X_t, \pi_t)]\,dt, \\ dP_t^i &= P_t^i \left[(b_t^i + f_i(X_t, \pi_t))\,dt + \sum_{j=1}^n \sigma_t^{i,j}\,dW_t^j\right]. \end{aligned} \tag{6}$$

Here, $f_i\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, $0 \leq i \leq n$, are some given functions which represent the effect of the strategies chosen by the investor on the prices. The self-financing assumption gives that the dynamics of the total wealth X_t are given by

$$-dX_t = b(t, c_t, X_t, \sigma_t^* \pi_t)\,dt - \pi_t^* \sigma_t\,dW_t,$$

where

$$b(t, c, x, \sigma_t^* \pi) = -r_t x - (x - \pi^* \mathbf{1}) f_0(x, \pi) - \pi^*[b_t - r_t \mathbf{1} + f(x, \pi)] + c, \tag{7}$$

with $f = (f^1, f^2, \ldots, f^n)^*$.

As seen in most of the examples, the driver b of the wealth process generally satisfies similar assumptions as the driver of the utility process. More precisely,

ASSUMPTION A5. *b satisfies the assumptions of a standard driver. In particular, it is Lipschitz with respect to x, π, uniformly with respect to (t, ω, c).*

ASSUMPTION A6. *There exists a positive constant k such that, $\forall c \in \mathbb{R}^+$ $|b(t, c, 0, 0)| \le kc$ a.s.*

ASSUMPTION A7. *The function b is nondecreasing with respect to c and convex with respect to c, x, π.*

ASSUMPTION A8. $\forall c \in \mathbb{R}^+,\ b(t, c, 0, 0) \ge 0$ *a.s.*

The initial wealth $X_0 = x \ge 0$ is taken as a primitive. Let $(X_t^{x,c,\pi}, 0 \le t \le T)$ be the wealth process associated with initial wealth x and strategy $(c, \pi) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$; that is, $(X_t^{x,c,\pi}, 0 \le t \le T)$ is solution of the forward equation (5) with $X_0^{x,c,\pi} = x$. Notice that, since b is Lipschitz, given an initial investment x and a risky portfolio π, there exists a unique wealth process solution. Assumption A7 ensures, by the forward comparison theorem, the concavity of the wealth process $X^{x,c,\pi}$ with respect to (c, π).

The investor, endowed with initial wealth x, has to choose a portfolio-consumption strategy (c, π) *feasible* for the initial wealth x; that is, $(c, \pi) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}^n)$, with $c_t \ge 0$ and $X_t^{x,c,\pi} \ge 0\, d\mathbb{P} \otimes dt$ a.s.

We denote by $\mathscr{A}(x)$ the set of consumption-portfolio strategies (c, π) feasible for the wealth x.

Assumption A8 ensures, by the backward comparison theorem, that if the terminal wealth $X_T^{x,c,\pi}$ is positive, then $X_t^{x,c,\pi}$ is positive at each time $t \in [0, T]$. (Indeed, the pair $(X^{x,c,\pi}, \pi)$ satisfies (5) with terminal value $X_T^{x,c,\pi}$.) This property will be exploited in the optimization problem to transform the positive constraint on the wealth process into a simple positive constraint on the terminal wealth only.

3. Classical and backward formulations of the problem of maximization of recursive utility.

3.1. *Formulation of the problem.* Let us consider a small investor, endowed with initial wealth $x > 0$, who chooses at each time t his stock portfolio π_t and his consumption process c_t. (Hereafter, to simplify notation, we shall suppose that the invertible matrix σ is the identity matrix. It is not restrictive since, in the general case, if we put $q = \sigma^* \pi$, we are led back to the case $\sigma = I$.) Under this assumption, the market value $X^{x,c,\pi}$ of the portfolio satisfies the following equation:

$$-dX_t = b(t, c_t, X_t, \pi_t)\, dt - \pi_t^*\, dW_t. \tag{8}$$

The investor wants to choose a portfolio-consumption strategy belonging to $\overline{\mathscr{A}}(x)$ so that it optimizes the utility of consumption and terminal wealth, $X_T^{x,c,\pi}$. Hence, the classical optimization problem can be written

$$\sup_{(c,\pi)\in\overline{\mathscr{A}}(x)} Y_0^{(x,c,\pi)},$$

where $Y^{(x,c,\pi)}$ is the recursive utility associated with driver $f(t, c_t, y, z)$ and terminal reward $h(X_T^{x,c,\pi})$. Let us define the *maximal reward* by

$$V(x) = \sup_{(c,\pi)\in\overline{\mathscr{A}}(x)} Y_0^{(x,c,\pi)}.$$

It seems natural that the investor prefers to start from a higher wealth than from a smaller one. In other terms, V is nondecreasing with respect to initial wealth x:

PROPOSITION 3.1. *The maximal reward is nondecreasing with respect to initial wealth x. It follows that, for each $x \geq 0$,*

$$V(x) = \sup_{0\leq y\leq x}\ \sup_{(c,\pi)\in\overline{\mathscr{A}}(y)} Y_0^{(y,c,\pi)}.$$

PROOF. The proof is based on the classical comparison theorem for forward SDEs [see, e.g., Karatzas and Shreve (1988), page 291] and the comparison theorem for backward SDEs [see El Karoui, Peng and Quenez (1997), Theorem 2.5]. First, by the forward comparison theorem, the wealth is nondecreasing with respect to the initial wealth; that is, for $0 \leq y \leq x$ and for each portfolio-consumption strategy $(c,\pi) \in \overline{\mathscr{A}}(y)$, we have $X_t^{(y,c,\pi)} \leq X_t^{(x,c,\pi)}, 0 \leq t \leq T$ a.s., and hence, $(c,\pi) \in \overline{\mathscr{A}}(x)$. Since h is nondecreasing, it follows that $h(X_T^{(y,c,\pi)}) \leq h(X_T^{(x,c,\pi)})$. Then, the backward comparison theorem gives that $Y_0^{(y,c,\pi)} \leq Y_0^{(x,c,\pi)}$ and the result follows easily. □

3.2. *Comparison theorem and finiteness of the maximal reward.* We first state a comparison theorem concerning these maximal rewards. If we consider two optimization problems for which the first utility process (respectively wealth process) is smaller (respectively greater) than the second one, then the first maximal reward is smaller than the second one.

THEOREM 3.2. *Let $(b^1, f^1, h^1), (b^2, f^2, h^2)$ be two standard parameters satisfying the above assumptions with*

$$\begin{aligned} &h^1(x) \leq h^2(x);\\ &f^1(t,c,y,z) \leq f^2(t,c,y,z);\\ &b^1(t,c,x,\pi) \geq b^2(t,c,x,\pi). \end{aligned} \tag{9}$$

Let $V^1(x)$ (respectively $V^2(x)$) be the maximal reward associated with (b^1, f^1, h^1) (respectively (b^2, f^2, h^2)). Then

$$V^1(x) \leq V^2(x).$$

PROOF. The arguments are the same as those used in the proof of Proposition 3.1. Let $\overline{\mathscr{A}_1}(x)$ (respectively $\overline{\mathscr{A}_2}(x)$) be the set of feasible strategies associated with $V^1(x)$ (respectively $V^2(x)$). By the forward comparison theorem, since $-b^1 \leq -b^2$, for each portfolio-consumption strategy $(c, \pi) \in \overline{\mathscr{A}_1}(x)$, $X^{1, x, c, \pi} \leq X^{2, x, c, \pi}$, and hence, $(c, \pi) \in \overline{\mathscr{A}_2}(x)$. Also, since h is nondecreasing, $h(X_T^{1, x, c, \pi}) \leq h(X_T^{2, x, c, \pi})$, and by the backward comparison theorem, $Y_0^{1, x, c, \pi} \leq Y_0^{2, x, c, \pi}$, and the result follows. □

REMARK. Note that the same proof shows that, if in addition to the assumptions of Theorem 3.2, we have $0 \leq x_1 \leq x_2$, then $V_1(x_1) \leq V_2(x_2)$. This result generalizes Proposition 3.1.

The finiteness of the maximal reward is usually taken as an assumption, except in the case of a linear wealth and a standard additive HARA utility function [see Karatzas (1989)] for which an explicit formula can be given. The Comparison Theorem 3.2 will allow us to give a sufficient condition for the maximal reward to be finite. First, by the results of Karatzas (1989), we easily derive the following:

PROPOSITION 3.3. *Suppose that $f(t, c, y, z) = U(c) + Cy + b_2(t) \cdot z$, $h(x) = x^p/p$ and $b(t, c, x, \pi) = -r_t x - b_1(t)^* \pi + kc$, where $U(c) = c^p/p$ with $0 < p < 1$, and C (respectively k) is a positive (respectively strictly positive) constant and $b_1(t), b_2(t)$ are predictable bounded coefficients. Then, $V(x) = Kx^p$, where K is a positive constant.*

PROOF. See the Appendix.

From this proposition and the comparison theorem, we derive a sufficient condition for the utility function to be finite, by assuming:

ASSUMPTION A9. *There exists a constant $k > 0$ such that*

$$b(t, c, 0, 0) \geq kc, \qquad c \geq 0.$$

(Notice that this assumption is satisfied in all the examples.)

THEOREM 3.4. *Suppose that Assumptions A1 to A9 are satisfied. Then, the maximum reward is finite and satisfies*

$$V(x) \leq K(x^p + 1), \qquad x \geq 0.$$

PROOF. This theorem is based essentially on assumptions A2, A4 and A9. The convexity and concavity of b and f and Assumptions A2 and A9 give

$$b(t, c, x, \pi) \geq b(t, c, 0, 0) - Cx - b_1(t)\pi \geq kc - Cx - b_1(t)\pi,$$

$$f(t, c, y, z) \leq f(t, c, 0, 0) + Cy + b_2(t)^* z \leq k_1 + k_2 \frac{c^p}{p} + Cy + b_2(t)^* z,$$

where C is the Lipschitz constant of b and f with respect to x and y, where $-b_1(t) \in \partial_\pi b(t, c, 0, 0)$, the (bounded) subdifferential of b with respect to π, and $b_2(t) \in \partial_z f(t, c, 0, 0)$, the (bounded) superdifferential of f with respect to z. The Comparison Theorem 3.2 and Proposition 3.3 lead to the desired result. □

3.3. *Backward formulation of the problem.* Recall that, by Assumption A8 and by the comparison theorem, the positive constraint on the wealth process $X_t^{x, c, \pi} \geq 0, 0 \leq t \leq T$, is equivalent to the positivity constraint on the terminal wealth $X_T^{x, c, \pi} \geq 0$. Using this property, we will show that the set of controls can be changed; more precisely, instead of taking the portfolio process as control, it is possible to take the terminal wealth. According to Duffie and Skiadas's (1994) paper, we take as primitive a consumption space $\mathscr{D}$, the subset of predictable measurable *positive* processes c_t which belong to $\mathbb{H}^2$ (i.e, such that $E\left[\int_0^T c_t^2 dt\right] < +\infty$), and a terminal value space $\mathscr{L}$, the set of square-integrable $\mathscr{F}_T$ measurable positive random variable ξ ($\mathscr{L} = (\mathbb{L}^2)^+$).

DEFINITION 3.5. A couple $(\xi, c) \in \mathscr{L} \times \mathscr{D}$ is called a consumption plan. $(X_t^{(\xi, c)}, \pi_t^{(c, \xi)})$ denotes the wealth and the portfolio associated with a given consumption plan (ξ, c), solution of the BSDE

$$\begin{aligned} -dX_t &= b(t, c_t, X_t, \pi_t)\, dt - \pi_t^* \, dWt, \\ X_T &= \xi. \end{aligned} \tag{10}$$

$(Y^{(\xi, c)}, Z^{(\xi, c)})$ denotes the utility of a consumption plan (ξ, c), solution of the BSDE

$$\begin{aligned} -dY_t &= f(t, c_t, Y_t, Z_t)\, dt - Z_t^* \, dW_t, \\ Y_T &= h(\xi). \end{aligned} \tag{11}$$

The coefficients b, f and h are supposed to satisfy Assumptions A1 to A9.

In the context of consumption plans, the initial wealth x cannot be taken as a primitive but it could naturally be considered as a constraint by considering only consumption plans (ξ, c) such that $X_0^{(\xi, c)} = x$. But the set of such consumption plans is not convex since b is nonlinear. Thus, the nonlinearity leads us to impose a milder constraint: $X_0^{\xi, c} \leq x$, so that we obtain a convex set.

DEFINITION 3.6. A consumption plan $(\xi, c) \in \mathscr{L} \times \mathscr{D}$ is called *feasible for the initial wealth* x if and only if $X_0^{(\xi, c)} \leq x$. We will denote by $\mathscr{A}(x)$ the set of consumption plans feasible for the initial wealth x.

From Proposition 3.1, it follows that the optimization problem can be written in the following *backward formulation*:

$$V(x) = \sup_{(\xi, c) \in \mathscr{A}(x)} Y_0^{(\xi, c)}. \tag{12}$$

Thus, the small investor has to choose a consumption plan (ξ, c) belonging to $\mathscr{A}(x)$ so that it optimizes the recursive utility function given by $Y_0^{(\xi, c)}$.

REMARK 1. It should be noted that this backward formulation emphasizes the symmetry between utility and wealth processes, which are both solutions of non linear BSDEs.

REMARK 2. *Notice that if there exists an optimal consumption plan* (ξ^*, c^*) *for* (12), *then it is unique. Indeed, by the strict backward comparison theorem, the functional* $(\xi, c) \longmapsto Y_0^{(\xi, c)}$ *is strictly concave because* h *and* f *are strictly concave* (*by Assumptions* A3 *and* A4).

4. Maximum principle. This section is devoted to the characterization of optimal consumption plans. As in Duffie and Skiadas's (1994) paper, we will first use a theorem of convex analysis to show that this optimization problem is equivalent to another optimization problem without constraint, with Lagrange multiplier. Recall that the assumption of concavity (respectively convexity) made on f and h (respectively b) ensure that the $\mathbb{R}$-valued functionals defined on $\mathscr{L} \times \mathscr{D}$ by

$$(\xi, c) \longmapsto x - X_0^{(\xi, c)},$$
$$(\xi, c) \longmapsto Y_0^{(\xi, c)},$$

are concave. Thus, we can apply classical results of convex analysis [see, e.g., Luenberger (1969), Corollary 8.31 and Theorem 8.4.2 and also the Appendix for details].

PROPOSITION 4.1. *There exists a constant* $\nu^* > 0$ (*which depends on* x) *such that*

$$V(x) = \sup_{(\xi, c) \in \mathscr{L} \times \mathscr{D}} \left\{ Y_0^{(\xi, c)} + \nu^* \left(x - X_0^{(\xi, c)} \right) \right\}. \tag{13}$$

Also, if the maximum is attained in (12) *by* (ξ^*, c^*), *then it is attained in* (13) *by* (ξ^*, c^*) *with* $X_0^{(\xi^*, c^*)} = x$. *Conversely, if there exists* $\nu^0 > 0$ *and* $(\xi^0, c^0) \in \mathscr{D} \times \mathscr{L}$ *such that the maximum is attained in*

$$\sup_{(\xi, c) \in \mathscr{L} \times \mathscr{D}} \left\{ Y_0^{(\xi, c)} + \nu^0 \left(x - X_0^{(\xi, c)} \right) \right\}, \tag{14}$$

with $X_0^{(\xi^0, c^0)} = x$, *then the maximum is attained in* (12) *by* (ξ^0, c^0).

REMARK. We will give in Section 5 another sufficient condition of optimality for (12) which holds under some additional assumptions (see Proposition 5.4).

We now study, for a fixed constant $\nu > 0$, the following optimization problem:

$$\sup_{(\xi, c) \in \mathscr{L} \times \mathscr{D}} J(\xi, c), \tag{15}$$

where the functional J is defined on $\mathscr{L} \times \mathscr{D}$ by

$$J(\xi, c) = Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)}.$$

Our aim is now to derive a characterization of optimality for (15).

4.1. *Maximum principle, a necessary condition for optimality.* In this section, by using some regularity properties of BSDEs, we derive a maximum principle and thus, we obtain a necessary condition of optimality for (15), under differentiability assumptions on f, b and h. Thus, in this section, we impose some smoothness conditions:

ASSUMPTION A10. *h is supposed to be continuously differentiable and b (respectively f) is continuously differentiable with respect to (c, x, π) (respectively (c, y, z)). Also, the functions h', f_c and b_c (the partial differentials of f and b with respect to c) are supposed to be bounded.*

REMARK. Notice that the Lipschitz property of f and b with respect to $(y, z), (x, \pi)$ involve the boundedness of f_y, f_z, b_x, b_π. Moreover, concerning the derivative of h and the derivative with respect to c of f, instead of the boundedness assumption, which is quite strong, it is possible to suppose only that

$$f_c(t, c, y, z) \leq C(|c| + |y| + |z|), \qquad h_x(x) \leq C|x|. \tag{16}$$

Let (ξ^0, c^0) be an optimal consumption plan for (15), i.e., such that

$$\sup_{(\xi, c)} J(\xi, c) = J(\xi^0, c^0).$$

Let (Y^0, z^0) and (X^0, π^0) be the utility and the wealth processes associated with (ξ^0, c^0). Let (ξ, c) be a consumption plan such that $\xi - \xi^0$ and $c - c_0$ are uniformly bounded (by a constant K). Then, for each $0 \leq \alpha \leq 1$, the pair $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$ is a consumption-plan.

Let (Y^α, Z^α) and (X^α, π^α) be the utility and the wealth processes associated with $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$. Recall that by the results on BSDEs, depending on parameters [see El Karoui, Peng and Quenez (1997)], the pair (Y^α, Z^α) [respectively (X^α, π^α)] is right-differentiable at 0 with respect to α in $\mathbb{H}^2 \times \mathbb{H}^2$, and the derivative $(\partial_\alpha Y^0, \partial_\alpha Z^0)$ is solution of the following linear

BSDE:

$$
\begin{aligned}
-d\partial_\alpha Y_t^0 &= (f_c^0(t)(c_t - c_t^0) + f_y^0(t)\partial_\alpha Y_t^0 + f_z^0(t)\partial_\alpha Z_t^0)\,dt \\
&\quad - (\partial_\alpha Z_t^0)^*\,dW_t, \\
\partial_\alpha Y_T^0 &= h'(\xi^0)(\xi - \xi^0),
\end{aligned}
\tag{17}
$$

where $f_c^0(t) = f_c(t, c_t^0, Y_t^0, Z_t^0)$, $f_y^0(t) = f_y(t, c_t^0, Y_t^0, Z_t^0)$, $f_z^0(t) = f_z(t, c_t^0, Y_t^0, Z_t^0)$. Also, (X^α, π^α) is right-differentiable at 0 with respect to α and the derivative $(\partial_\alpha X^0, \partial_\alpha \pi^0)$ is solution of the following linear BSDE:

$$
\begin{aligned}
-d\partial_\alpha X_t^0 &= (b_c^0(t)(c_t - c_t^0) + b_x^0(t)\partial_\alpha X_t^0 + b_\pi^0(t)\partial_\alpha \pi_t^0)\,dt \\
&\quad -(\partial_\alpha \pi_t^0)^*\,dW_t, \\
\partial_\alpha X_T^0 &= (\xi - \xi^0),
\end{aligned}
\tag{18}
$$

where $b_c^0(t) = b_c(t, c_t^0, X_t^0, \pi_t^0)$, $b_x^0(t) = b_x(t, c_t^0, X_t^0, \pi_t^0)$, $b_\pi^0(t) = b_\pi(t, c_t^0, X_t^0, \pi_t^0)$.

Since (ξ^0, c^0) is an optimal consumption plan, then for each $\alpha \in [0, 1]$,

$$Y_0^\alpha - \nu X_0^\alpha \le Y_0^0 - \nu X_0^0.$$

By dividing the inequality by α and by letting α tend to 0, we obtain

$$\partial_\alpha Y_0^0 - \nu \partial_\alpha X_0^0 \le 0, \tag{19}$$

where $\partial_\alpha Y_0^0, \partial_\alpha X_0^0$ denote the right-derivatives of Y_0^α and X_0^α at $\alpha = 0$.

To derive the maximum principle, we introduce the adjoint process associated with $\partial_\alpha Y^0$ and $\partial_\alpha X^0$. The adjoint process of $\partial_\alpha Y^0$ is given by the process $\Gamma_t^{(f_y^0, f_z^0)}$, solution of the following BDSE

$$
\begin{aligned}
d\Gamma_t &= \Gamma_t\Big(f_y^0(t)\,dt + (f_z^0(t))^*\,dW_t\Big), \\
\Gamma_0 &= 1.
\end{aligned}
$$

Also, the adjoint process of $\partial_\alpha X^0$ is given by the process $H_t^{(b_x^0, b_\pi^0)}$, solution of the following BSDE

$$
\begin{aligned}
dH_t &= H_t\Big(b_x^0(t)\,dt + (b_\pi^0(t))^*\,dW_t\Big), \\
H_0 &= 1.
\end{aligned}
$$

Notice that the process $H_t^{(b_x^0, b_\pi^0)}$ (respectively $\Gamma_t^{(f_y^0, f_z^0)}$) can also be interpreted as a change of numeraire relative to the wealth and corresponding to the coefficients (b_x^0, b_π^0) (respectively a change of numeraire relative to the utility and corresponding to the coefficients (f_y^0, f_z^0)). To simplify notation, hereafter, we will denote $H_t^{(b_x^0, b_\pi^0)}$ (respectively $\Gamma_t^{(f_y^0, f_z^0)}$) by H_t^0 (respectively by Γ_t^0).

Before giving the maximum principle, we introduce a new assumption usually called the *Inada condition*, that is, ASSUMPTION A11. *h (respectively f) satisfies $h^{'}(0) = +\infty$ (respectively $f_c(t, 0, y, z) = +\infty$ for each y, z).*

THEOREM 4.2. *Suppose that Assumptions A1 to A11 are satisfied. Let (ξ^0, c^0) be an optimal consumption plan for (15). Let (Y^0, Z^0) and (X^0, π^0) be the utility and the wealth processes associated with (ξ^0, c^0). Then, the following maximum principle can be written*:

$$\Gamma_T^0 h'(\xi^0) = \nu H_T^0 \quad a.s., \tag{20}$$

$$\Gamma_t^0 f_c(t, c_t^0, y_t^0, z_t^0) = \nu H_t^0 b_c(t, c_t^0, y_t^0, z_t^0), 0 \le t \le T, \quad dt \otimes d\mathbb{P}\text{-}a.s. \tag{21}$$

PROOF. By classical results on linear BSDEs, we have

$$\begin{aligned}\partial_\alpha Y_0^0 - \nu \partial_\alpha X_0^0 &= E(\Gamma_T^0 h'(\xi^0) - \nu H_T^0)(\xi - \xi^0) \\ &\quad + E\int_0^T \Big(\Gamma_t^0 f_c^0(t) - \nu H_t^0 b_c^0(t)\Big)(c_t - c_t^0)\, dt.\end{aligned} \tag{22}$$

This equality with (19) implies

$$E\Big[\Big(\Gamma_T^0 h'(\xi^0) - \nu H_T^0\Big)\big(\xi - \xi^0\big)\Big] \le 0, \tag{23}$$

for each $\xi \in \mathscr{L}$ s.t. $\xi - \xi^0$ is bounded and

$$E\int_0^T \Big(\Gamma_t^0 f_c^0(t) - \nu H_t^0 b_c^0(t)\Big)(c_t - c_t^0)\, dt \le 0$$

for each $c \in \mathscr{D}$ s.t $c - c^0$ is bounded.

Put $A = \{\Gamma_T^0 h'(\xi^0) - \nu H_T^0 > 0\}$. Then, inequality (23) applied to $\xi = \xi^0 + \mathbf{1}_A$ leads to $P(A) = 0$. It follows that

$$\Gamma_T^0 h'(\xi^0) - \nu H_T^0 \le 0 \quad \text{a.s.} \tag{24}$$

Also, one can show easily that, on $\{\xi^0 > 0\}$, (20) is satisfied a.s. by considering, for each $\varepsilon > 0$, the set $B = \{\Gamma_T^0 h'(\xi^0) - \nu H_T^0 < 0, \xi^0 \ge \varepsilon\}$; indeed, inequality (23) applied to $\xi = \xi^0 - \varepsilon \mathbf{1}_B$ implies that $P(B) = 0$.

To derive the desired result, it remains to show that if the Inada condition is satisfied, then $\xi^0 > 0$ a.s. Now, by inequality (23), on $\xi_0 = 0$,

$$\Gamma_T^0 h'(\xi^0) \le \nu H_T^0 < +\infty \quad \text{a.s.},$$

which leads to a contradiction if $h^{'}(0) = +\infty$.

The same arguments hold for the consumption process. □

REMARK. Note that if the Inada condition Assumption A11 is not satisfied, then equality (20) (resp. (21)) is only satisfied on $\{\xi_0 > 0\}$ (resp. $\{c_t^0 > 0\}$). Furthermore, on $\{\xi_0 = 0\}$,

$$\Gamma_T^0 h'(\xi^0) - \nu H_T^0 \le 0, \quad \text{a.s.}, \tag{25}$$

and on $\{c_t^0 = 0\}$,

$$\Gamma_t^0 f_c(t, c_t^0, y_t^0, z_t^0) - \nu H_t^0 b_c(t, c_t^0, y_t^0, z_t^0) \le 0, \qquad dt \otimes d\mathbb{P}\text{-a.s.} \tag{26}$$

4.2. *A sufficient condition for optimality.* In this section, our aim is to obtain that the necessary condition given by the maximum principle is also a sufficient condition for a control (ξ^0, c^0) to be optimal for (15). Recall that if, in addition to this condition, (ξ^0, c^0) satisfies also $X_0^{(\xi^0, c^0)} = x$, then (ξ^0, c^0) is an optimal solution for (12).

In the following, the notation and the assumptions are the same as in the maximum principle (Theorem 4.2).

THEOREM 4.3. *Suppose that Assumptions* A1 *to* A11 *are satisfied. Let* (ξ^0, c^0) *be a consumption plan. Let* (Y^0, Z^0) *and* (X^0, π^0) *be the utility and the wealth processes associated with* (ξ^0, c^0). *Suppose that conditions* (20) *and* (21) *are satisfied. Then,* (ξ^0, c^0) *is optimal.*

REMARK. Actually, the boundedness assumption made on f_c and h' is not necessary for Theorem 4.3. It is sufficient to suppose that $h'(\xi^0)$ is square-integrable and that the process $f_c^0(t) = f_c(t, c_t^0, Y_t^0, Z_t^0)$ is square-integrable, that is, $E\int_0^T (f_c^0(t))^2\, dt < +\infty$.

PROOF OF THEOREM 4.3. The proof is based on the concavity and the convexity properties of f and b and on the comparison theorem for BSDEs. (See the Appendix for details.) □

Thus, Theorems 4.2 and 4.3 give that the maximum principle corresponds to a necessary and sufficient condition of optimality. Let now t be a fixed time between 0 and T. One can wonder what happens if we start from t. It can be derived quite easily that the following holds.

COROLLARY 4.4. *Let* (ξ^0, c^0) *be a consumption plan satisfying conditions* (20) *and* (21). *Then, for each time* t, *it is optimal for the following dynamic control problem*:

$$ess \sup_{(\xi, c)} \left\{ Y_t^{(\xi, c)} - \nu_t X_t^{(\xi, c)} \right\},$$

where the Lagrange multiplier at time t *is given by*

$$\nu_t = \nu H_t^0 (\Gamma_t^0)^{-1}.$$

5. Existence of an optimal consumption plan. This section is devoted to the proof of the existence. First, we will prove the existence of an optimal consumption plan for the optimization problem (15) for each fixed $\nu > 0$. Then, we will derive an existence result for the problem (12). First, notice that the functional J satisfies the following property.

LEMMA 5.1. *Suppose that Assumptions* A1 *to* A10 *are satisfied. Then, the functional* $(\xi, c) \to J(\xi, c); \mathscr{L} \times \mathscr{D} \to \mathbb{R}$ *is strictly concave, strongly continuous and weakly upper-semicontinuous.*

PROOF. First, the strict concavity of $J(\xi, c)$ is involved by the strict concavity of f and convexity of b by applying the strict comparison theorem for BSDEs. Let us show that $(\xi, c) \to Y_0^{\xi,c}$ is continuous. Let (ξ^n, c^n) be a sequence of consumption plans which converge (strongly in $\mathbb{H}^2 \times \mathbb{H}^2$) to (ξ, c). By the a priori estimates for BSDEs [see El Karoui, Peng and Quenez (1997)], we have

$$|Y_0^{(\xi^n, c^n)} - Y_0^{(\xi, c)}| \le K(\|h(\xi^n) - h(\xi)\|_2 + \|f(., c.^n, Y., Z.) - f(., c., Y., Z.)\|_2),$$

where $Y. = Y.^{(\xi, c)}$ and $Z. = Z.^{(\xi, c)}$. By Assumption A10, h (respectively f) is Lipschitz with respect to x (respectively with respect to c), and it follows easily that $Y_0^{(\xi^n, c^n)}$ converges to $Y_0^{(\xi, c)}$. The same argument holds for the wealth. Thus, the continuity of J follows. Since J is concave and strongly upper-semicontinuous, it follows by classical convex analysis results that J is also upper-semicontinuous for the weak convergence [see Brezis (1983), Corollary 3.8]. □

First, the uniqueness follows from the strict concavity of $J(\xi, c)$. To prove the existence, we introduce a penalty function:

$$J^\varepsilon(\xi, c) = J(\xi, c) - \frac{\varepsilon}{2}\left\{\mathbf{E}\int_0^T |c_t|^2 dt + \mathbf{E}|\xi|^2\right\}.$$

For fixed $\varepsilon > 0$, it is clear that, for each constant C, the set given by $\{(\xi, c) \in \mathscr{L} \times \mathscr{D}; J^\varepsilon(\xi, c) \ge C\}$ is bounded (closed, convex and hence weakly compact). By classical results of convex analysis [see Brezis (1983), Corollaire 3.20], this, with the upper continuity and strict concavity of J^ε, yields that the maximum is uniquely attained; that is, there exists a unique pair $(\xi^\varepsilon, c^\varepsilon) \in \mathscr{A}(x)$ such that

$$J^\varepsilon(\xi^\varepsilon, c^\varepsilon) = \sup_{(\xi, c) \in \mathscr{A}(x)} J^\varepsilon(\xi, c).$$

In the following, we assume the following condition:

ASSUMPTION A12. *There exists* $\alpha_0 > 0$ *such that*

$$f_c(t, c, y, z) \le C|c|^{-q} \qquad \forall c \ge \alpha_0, \tag{27}$$

$$b_c(t, c, x, \pi) \le c_1 > 0,$$

$$h_x(x) \le C|x|^{-q} \qquad \forall x \ge \alpha_0, \tag{28}$$

where $q = 1 - p \in]0, 1[$.

We now prove the following:

LEMMA 5.2. *Suppose that Assumptions* A1 *to* A12 *are satisfied. Then, the set of consumption plans* $\{(\xi^\varepsilon, c^\varepsilon)\}_{\varepsilon > 0}$ *is uniformly bounded in* $\mathscr{L} \times \mathscr{D}$.

PROOF. Let $(X^\varepsilon, \pi^\varepsilon)$ (respectively $(Y^\varepsilon, Z^\varepsilon)$) be the wealth-portfolio process (respectively the utility process) associated with the consumption plan $(\xi^\varepsilon, c^\varepsilon)$. Let H^ε be the wealth adjoint process, solution of

$$dH_t^\varepsilon = H_t^\varepsilon[b_x^\varepsilon(t)\,dt + b_\pi^\varepsilon(t)\,dW_t], \qquad H_0^\varepsilon = 1,$$

and let Γ^ε be the utility adjoint process, solution of

$$d\Gamma_t^\varepsilon = \Gamma_t^\varepsilon[f_y^\varepsilon(t)\,dt + f_z^\varepsilon(t)\,dW_t], \qquad \Gamma_0^\varepsilon = 1,$$

where the processes given by

$$b_x^\varepsilon(t) = b_x(t, c_t^\varepsilon, X_t^\varepsilon, \pi_t^\varepsilon), \qquad b_\pi^\varepsilon(t) = b_\pi(t, c_t^\varepsilon, X_t^\varepsilon, \pi_t^\varepsilon),$$
$$f_y^\varepsilon(t) = f_y(t, c_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon), \qquad f_z^\varepsilon(t) = f_z(t, c_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon),$$

are uniformly bounded by the Lipschitz constants of b and f.

The maximum principle (Theorem 4.2) applied to the penalty functions gives that
on $\{\xi^\varepsilon > 0\}$,

$$\Gamma_T^\varepsilon h_x(\xi^\varepsilon) - \nu H_T^\varepsilon - \varepsilon\xi^\varepsilon = 0 \quad \text{a.s.}$$

on $\{c_t^\varepsilon > 0\}$,

$$\Gamma_t^\varepsilon f_c(t, c_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon) - \nu H_t^\varepsilon b_c(t, c_t^\varepsilon, X_t^\varepsilon, \pi_t^\varepsilon) - \varepsilon c_t^\varepsilon = 0, \ dt \otimes d, \mathbb{P}\text{-a.s.}$$

Now, by Assumption A12 [(27) and (28)], it follows that on $\{\xi^\varepsilon \geq \alpha^0\}$,

$$(\xi^\varepsilon)^q + \varepsilon\nu^{-1}(H_T^\varepsilon)^{-1}(\xi^\varepsilon)^{1+q} \leq C\Gamma_T^\varepsilon\nu^{-1}(H_T^\varepsilon)^{-1} \quad \text{a.s.}$$

and on $\{c_t^\varepsilon \geq \alpha^0\}$,

$$(c_t^\varepsilon)^q + \varepsilon\nu^{-1}(H_t^\varepsilon)^{-1}C_1^{-1}(c_t^\varepsilon)^{1+q} \leq CC_1^{-1}\Gamma_t^\varepsilon\nu^{-1}(H_t^\varepsilon)^{-1}, \ dt \otimes d\mathbb{P}\text{-a.s.}$$

Hence,

$$\xi^\varepsilon \leq \Big[C\Gamma_T^\varepsilon\nu^{-1}(H_T^\varepsilon)^{-1}\Big]^{1/q} \vee \alpha_0, \tag{29}$$

$$c_t^\varepsilon \leq \Big[CC_1^{-1}\Gamma_t^\varepsilon\nu^{-1}(H_t^\varepsilon)^{-1}\Big]^{1/q} \vee \alpha_0. \tag{30}$$

Now, the following estimates hold:

$$\mathbf{E}\Bigg[\sup_{t\in[0,T]}\Big[(H_t^\varepsilon)^l + (H_t^\varepsilon)^{-l} + (\Gamma_t^\varepsilon)^l + (\Gamma_t^\varepsilon)^{-l}\Big]\Bigg] \leq C_l \qquad \forall l > 0,$$

where the constant C_l depends only on l. The result easily follows. □

This lemma will allow us to derive the existence of an optimal consumption plan.

THEOREM 5.3. *Suppose that Assumptions* A1 *to* A12 *are satisfied. There exists a unique* $(\xi^0, c^0) \in \mathscr{L} \times \mathscr{D}$ *that attains the maximum of the problem* (15).

PROOF. As seen above, the uniqueness is due to the strict concavity of $J(\xi, c)$. We proceed to prove the existence. Since the above introduced sequence $\{(\xi^\varepsilon, c^\varepsilon)\}$ is bounded in $\mathscr{L} \times \mathscr{D}$, there exists a subsequence which converges weakly to (ξ^0, c^0) in $\mathscr{L} \times \mathscr{D}$. From the upper continuity (strong and weak by concavity) of $J(\xi, c)$, it follows that

$$\overline{\lim_{i\to\infty}} J(\xi^{\varepsilon_i}, c^{\varepsilon_i}) \le J(\xi^0, c^0).$$

But according to the definition of J^ε and the boundedness of $\{(\xi^\varepsilon, c^\varepsilon)\}$, there exists a $C > 0$ such that, for each ε,

$$\sup_{(\xi, c)\in\mathscr{L}\times\mathscr{D}} J(\xi, c) \le J(\xi^\varepsilon, c^\varepsilon) + C\varepsilon.$$

Thus (ξ^0, c^0) must be the optimal feasible plan. The proof is complete. □

From this result, we will derive the existence of an optimal consumption plan for our primal problem (12). First, we state the following property which gives a sufficient condition of optimality for (12).

PROPOSITION 5.4. *Suppose that Assumptions* A1 *to* A12 *are satisfied. Let* ν^* *be such that equality* (13) *is satisfied. Suppose that the maximum is attained in*

$$\sup_{(\xi, c)\in\mathscr{D}\times\mathscr{L}} \left\{ Y_0^{(\xi, c)} - \nu^* X_0^{(\xi, c)} \right\} \tag{31}$$

by (ξ^*, c^*). *Then*, (ξ^*, c^*) *is optimal for* (12).

Theorem 5.3 and Proposition 5.4 give the following existence result:

THEOREM 5.5. *Suppose that Assumptions* A1 *to* A12 *are satisfied. There exists a unique* $(\xi^*, c^*) \in \mathscr{L} \times \mathscr{D}$ *that attains the maximum of the problem* (12).

REMARK. Note that the Inada condition (A11) is not necessary for Lemma 5.2, Proposition 5.4 and Theorem 5.5.

PROOF OF PROPOSITION 5.4. We introduce the dual (convex) functional $\widetilde{V}$ defined on $\mathbb{R}_+^*$ by

$$\widetilde{V}(\nu) = \sup_{(\xi, c)\in\mathscr{D}\times\mathscr{L}} \left\{ Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)} \right\}.$$

Note that by Theorem 5.3, $\widetilde{V}$ is finite on $\mathbb{R}_+^*$. By Luenberger [(1969), Theorem 8.6.1], we have

$$V(x) = \min_{\nu>0} \left\{ \widetilde{V}(\nu) + \nu x \right\}, \tag{32}$$

and the minimum on the right is achieved by ν^*. Now, by the envelope theorem [see Aubin (1984), page 52], the following lemma holds:

LEMMA 5.6. *$\widetilde{V}$ is differentiable at ν^* and $\widetilde{V}'(\nu^*) = -X_0^{(\xi^*, c^*)}$.*

Since the minimum in (32) is achieved at ν^*, it implies that $\widetilde{V}'(\nu^*) = -x$ and hence $X_0^{(\xi^*, c^*)} = x$. By Proposition 4.1, (ξ^*, c^*) is optimal for (12). □

PROOF OF LEMMA 5.6. The differentiability of $\widetilde{V}$ at ν^* is based on the envelope theorem applied to the family of convex functions

$$\nu \to Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)},$$

where (ξ, c) belongs to a well-chosen subset of $\mathscr{L} \times \mathscr{D}$ which will be specified. By Theorem 5.3, the function $(\xi, c) \to Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)}$, defined on $\mathscr{L} \times \mathscr{D}$, attains its maximum at a unique point denoted by (ξ^ν, c^ν); the same arguments as those used in the proof of Lemma 5.2 show that the set of consumption plans $\{(\xi^\nu, c^\nu)\}_{\nu \in]\nu^*/2, +\infty[}$ is uniformly bounded in $\mathscr{L} \times \mathscr{D}$. Hence, there exists a bounded closed convex (and hence weakly compact) subset $\mathscr{P}$ of $\mathscr{L} \times \mathscr{D}$ such that, for each $\nu \in]\nu^*/2, +\infty[$,

$$\widetilde{V}(\nu) = \sup_{(\xi, c) \in \mathscr{P}} \left\{ Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)} \right\}.$$

Recall that, by Lemma 5.1, for each $\nu > 0$, the function $(\xi, c) \to Y_0^{(\xi, c)} - \nu X_0^{(\xi, c)}$ is weakly upper-semicontinuous. Thus, the envelope theorem can be applied and gives the desired result. □

REMARK 1. Recall that Cuoco (1997) has given a proof of existence directly on the primal problem (12) given by

$$V(x) = \sup_{(\xi, c) \in \mathscr{A}(x)} Y_0^{(\xi, c)}$$

in the case of a classical utility function of consumption $Y_0^{(\xi, c)} = E(\xi^p/p + \int_0^T (c_t^p/p)\, dt)$ and constraints on the wealth. If the feasible set $\mathscr{A}(x)$ was weakly-compact, then, by classical results, the maximum would be achieved in (12). But, in any case, $\mathscr{A}(x)$ is not weakly compact in $\mathbb{L}^2 \times \mathbb{H}^2$ since it is not even bounded. The issue is then to see in what sense $\mathscr{A}(x)$ is compact. There is no simple answer to this problem. Cuoco proposes a very nice solution by using some techniques of analysis: the problem is relaxed by extending it on the closure of $\mathscr{A}(x)$ in a well-chosen space. Since this set is compact in a weak sense for a well-chosen topology, standard arguments yield that the maximum of the concave and upper-semicontinuous (in the topology of convergence in measure) functional $(\xi, c) \to Y_0^{(\xi, c)}$ is achieved. It can then be proved that the projection of the above maximizer solves the original problem. In our context of recursive utilities, this approach could be adapted, but it would involve quite a lot of additional work. [In particular, it is not clear that the functional $(\xi, c) \to Y_0^{(\xi, c)}$ is upper-semicontinuous in the topology of convergence in measure.]

REMARK 2. One can wonder if our method can be applied to the case of constraints on the wealth [El Karoui et al. (1997)] or on the portfolio weights [El Karoui and Quenez (1995), Cvitanic and Karatzas (1993), Cuoco (1997)]. In fact, our approach using classical BSDEs cannot be applied directly. However, these problems can be obtained as the limit of constrained penalized problems such as those considered in this paper.

6. Forward-Backward system. In this section, a characterization of the optimum in terms of a backward-forward system is derived from the maximum principle. Suppose that Assumptions A1 to A12 are satisfied. To simplify notation, let us denote (Γ^0, H^0) and (Y^0, Z^0) by (Γ, H) and (Y, Z).

Notice that the maximum principle gives that the optimal terminal wealth ξ^0 satisfies $h^{'}(\xi^0) = \nu H_T \Gamma_T^{-1}$ a.s. and hence,

$$\xi^0 = J(\nu H_T \Gamma_T^{-1}) \quad \text{a.s.}, \tag{33}$$

where the function J is equal to $(h^{'})^{-1}$, the inverse of $h^{'}$.

Furthermore, we will make the following assumption (which is always satisfied in the examples):

ASSUMPTION A13. *The driver of the wealth can be written as* $b(t, c, x, \pi) = \bar{b}(t, x, \pi) + c, \forall (t, c, x, \pi) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n$.

In this case, the wealth driver b satisfies $b_c = 1$ and it follows that the optimal consumption c_t^0 is simply given by

$$c_t^0 = I(t, \nu H_t \Gamma_t^{-1}, Y_t, Z_t), \qquad 0 \le t \le T, \quad d\mathbb{P} \otimes dt\text{-a.s.}, \tag{34}$$

where, for each (t, y, z), the function I is equal to $(f_c)^{-1}(t, ., y, z)$, the inverse of $c \to f_c(t, c, y, z)$.

In the following, we suppose that Assumptions A1 to A13 are satisfied. The maximum principle (i.e., Theorems 4.2 and 4.3) gives

THEOREM 6.1. *Let* $(Y, Z), (X, \pi)$ *and* Γ, H *be predictable square-integrable processes. They coincide with the optimal utility and wealth processes and their associated deflators if and only if they are the unique solution of the following forward-backward system*:

$$\begin{aligned} -dX_t &= b(t, I(t, \nu H_t \Gamma_t^{-1}, Y_t, Z_t), X_t, \pi_t)\, dt - \pi_t^*\, dWt, \\ X_T &= J(\nu H_T \Gamma_T^{-1}), \end{aligned} \tag{35}$$

$$\begin{aligned} -dY_t &= f(t, I(t, \nu H_t \Gamma_t^{-1}, Y_t, Z_t), Y_t, Z_t)\, dt - Z_t^*\, dW_t, \\ Y_T &= h(J(\nu H_T \Gamma_T^{-1})), \end{aligned} \tag{36}$$

$$
\begin{aligned}
d\Gamma_t &= \Gamma_t\Big[f_y(t, I(t, \nu H_t\Gamma_t^{-1}, Y_t, Z_t), Y_t, Z_t)\,dt \\
&\qquad\qquad + f_z(t, I(t, \nu H_t\Gamma_t^{-1}, Y_t, Z_t), Y_t, Z_t)^*\,dW_t\Big],\\
\Gamma_0 &= 1,\\
dH_t &= H_t\Big[b_x\Big(t, I(t, \nu H_t\Gamma_t^{-1}, Y_t, Z_t), X_t, \pi_t\Big)\,dt \\
&\qquad\qquad + b_\pi(t, I(t, \nu H_t\Gamma_t^{-1}, Y_t, Z_t), X_t, \pi_t)^*\,dW_t\Big],\\
H_0 &= 1.
\end{aligned}
$$

In this case, the optimal wealth ξ^0 and consumption c^0 are then given by (33) *and* (34).

Recall that the process $(\nu_t, 0 \le t \le T)$ given for each t by $\nu_t = \nu H_t\Gamma_t^{-1}$ corresponds to the Lagrange multiplier process (see Corollary 4.4). By making the change of variable $A_t = \text{Log}(\nu_t) = \text{Log}(\nu H_t\Gamma_t^{-1})$, we derive easily the following corollary.

COROLLARY 6.2. *Let $(Y, Z), (X, \pi)$ be some predictable square-integrable processes; that is, they belong to $\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R}^n)$. They coincide with the optimal utility and wealth processes if and only if there exists a process $A \in \mathbb{H}^2(\mathbb{R})$ such that $(Y, Z), (X, \pi)$ and A are the unique solution of the following forward-backward system*:

$$
\begin{aligned}
-dX_t &= b(t, I(t, e^{A_t}, Y_t, Z_t), X_t, \pi_t)\,dt - \pi_t^*\,dWt,\\
X_T &= J(e^{A_T}),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
-dY_t &= f(t, I(t, e^{A_t}, Y_t, Z_t), Y_t, Z_t)\,dt - Z_t^*\,dW_t,\\
Y_T &= h(J(e^{A_T})),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
dA_t &= \phi(t, A_t, X_t, Y_t, \pi_t, Z_t)\,dt + \psi(t, A_t, X_t, Y_t, \pi_t, Z_t)^*\,dW_t,\\
A_0 &= \text{Log}(\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
\phi(t, a, x, y, \pi, z) &= b_x(t, I(t, e^a, y, z), x, \pi) - f_y(t, I(t, e^a, y, z), y, z)\\
&\quad - \tfrac{1}{2}|b_\pi(t, I(t, e^a, y, z), x, \pi)|^2\\
&\quad + \tfrac{1}{2}|f_z(t, I(t, e^a, y, z), y, z)|^2
\end{aligned}
$$

and

$$
\psi(t, a, x, y, \pi, z) = b_\pi\big(t, I(t, e^a, y, z), x, \pi\big) - f_z\big(t, I(t, e^a, y, z), y, z\big).
$$

In this case, the optimal wealth ξ^0 and consumption c^0 are then given by $\xi^0 = J(e^{A_T})$ and $c_t = I(t, e^{A_t}, Y_t, Z_t)$.

REMARK. Recall that there are some existence and uniqueness results concerning the forward-backward systems. One has been obtained by Ma, Protter and Yong (1994) in the Markovian case and another one has been obtained by Hu and Peng (1995) in a non-Markovian case. Hence, if we are under the assumptions corresponding to one of these two cases, then the solution of the above forward-backward system satisfies the sufficient conditions (20), (21) of Theorem 4.3 and hence corresponds to the optimal utility and wealth processes. Thus, Theorem 6.1 can give in certain cases a method to derive the existence.

7. Examples. In this section, we give some examples which illustrate the characterization of the optimal utility and wealth process as the solution of a forward-backward system.

7.1. *Example of a recursive utility and a linear wealth.* In this example [see Duffie and Skiadas (1994) and Schroder and Skiadas (1997)], the driver of the recursive utility function does not depend on Z, and hence, the dynamics of the utility function are given by

$$-dY_t = f(c_t, Y_t)\,dt - Z_t^*\,dW_t, \qquad Y_T = h(\xi),$$

with f and h satisfying Assumptions A1 to A11. The wealth process satisfies the classical linear dynamics:

$$-dX_t = (-r_t X_t - \pi_t^*\,\theta_t + c_t)\,dt - \pi_t^*\,dW_t.$$

Then, by Theorem 6.1, the optimal utility (Y, Z) and its associated deflator Γ are the unique solution of the forward-backward system

$$\begin{aligned} d\Gamma_t &= \Gamma_t f_y\big(I(\nu\Gamma_t^{-1}H_t,\, Y_t),\, Y_t\big)\,dt,\\ \Gamma_0 &= 1,\\ -dY_t &= f\big(I(\nu\Gamma_t^{-1}H_t, Y_t),\, Y_t\big)\,dt - Z_t^*\,dW_t,\\ Y_T &= h\big(J(\nu\Gamma_T^{-1}H_T)\big), \end{aligned}$$

with $H_t = \exp\{-\int_0^t r_s\,ds - \int_0^t \theta_s^*\,dW_s - \frac{1}{2}\int_0^t |\theta_s|^2\,ds\}$ and where $I = (f_c)^{-1}$ and $J = (h')^{-1}$.

The wealth process is then given as the solution of the following BSDE:

$$\begin{aligned} -dX_t &= (I(\nu\Gamma_t^{-1}H_t,\, Y_t) - r_t X_t - \theta_t^*\pi_t)\,dt - \pi_t^*\,dW_t,\\ X_T &= J(\nu\Gamma_T^{-1}H_T). \end{aligned}$$

Let us make the change of variable $A_t = \mathrm{Log}(\nu\Gamma_t^{-1}H_t)$, $0 \le t \le T$. Then, the processes (Y, Z) and A are the unique solution of the forward-backward system

$$\begin{aligned} dA_t &= \big(-r_t - f_y\big(I(e^{A_t}, Y_t),\, Y_t\big) - \tfrac{1}{2}|\theta_t|^2\big)\,dt - \theta_t^*\,dW_t,\\ A_0 &= \mathrm{Log}\,\nu, \end{aligned} \tag{39}$$

$$\begin{aligned} -dY_t &= f\big(I(e^{A_t}, Y_t),\, Y_t\big)\,dt - Z_t\,dW_t,\\ Y_T &= h\big(J(e^{A_T})\big). \end{aligned} \tag{40}$$

Suppose now that the coefficients θ_t and r_t are deterministic and that the utility functions $h(x)$ and $f(c, y)$ are deterministic. Since we are in a Markovian case, the solution of the forward-backward system (39), (40) can be computed (under some smoothness conditions on the coefficients) by using the four-step scheme resolution method of Ma, Protter and Yong (1994).

Suppose that the functions B and F, defined by $B(a, y) = f_y(I(e^a, y), y)$ and $F(a, y) = f(I(e^a, y), y)$, are $\mathscr{C}^\infty$, with first-order derivatives with respect to a, y being uniformly bounded. Furthermore, suppose that there exist a continuous function μ, a constant $L > 0$ and $\alpha \in]0, 1[$ such that $\Psi(a) = h(J(e^a))$ is a $\mathscr{C}^{2+\alpha}(\mathbb{R})$, and for all $(a, y) \in \mathbb{R} \times \mathbb{R}$,

$$|B(a, y)| \le \mu(|y|), \qquad F(a, y)^* a \le L(1 + |y|^2).$$

Then, the utility process Y is equal to a deterministic function of t and A_t; that is, $Y_t = \phi(t, A_t)$, where ϕ is solution of the following PDE:

$$\partial_t \phi(t, x) + \mathscr{L}_t \phi(t, x) + f(I(e^x, \phi(t, x)), \phi(t, x)) = 0,$$
$$\phi(T, x) = h(J(e^x)),$$

where $\mathscr{L}_{(t, x)}$ is the generator associated with process A given by

$$\mathscr{L}_{(t, x)} \phi(t, x) = \tfrac{1}{2} |\theta_t|^2 \partial^2_{x^2} \phi(t, x)$$
$$+ [-r - \tfrac{1}{2} |\theta_t|^2 - f_y\big(I(e^x, \phi(t, x)), \phi(t, x)\big)] \partial_x \phi(t, x).$$

Then, the process A is solution of the following classical forward SDE:

$$dA_t = (-r_t - f_y\big(I(e^{A_t}, \phi(t, A_t)), \phi(t, A_t)\big) - \tfrac{1}{2} |\theta_t|^2)\, dt - \theta_t^*\, dW_t,$$
$$A_0 = \operatorname{Log} \nu.$$

Notice that in the Markovian case, Ma, Protter and Yong methodology gives a proof of the existence (without necessarily the boundedness assumption of f_c). Indeed, by Theorem 6.1, the solution Y, A constructed by the four-step scheme corresponds to the optimal utility and its associated deflator (given by $\Gamma_t = \nu H_t e^{-A_t}$).

7.2. *Example of a generalized recursive utility function.* In this example, the recursive utility function satisfies the following dynamics:

$$-dY_t = (-\beta Y_t - g(Z_t) + u(c_t))\, dt - Z_t^*\, dW_t, \qquad Y_T = h(\xi),$$

where u and h are utility functions and where the function g is a differentiable function. Note that the coefficient $g(z)$ (if it is positive) can be interpreted as a risk-aversion coefficient since it penalizes variability of the utility process.

The wealth process satisfies the classical linear dynamics:

$$-dX_t = (-r_t X_t - \pi_t^* \theta_t + c_t)\, dt - \pi_t^*\, dW_t.$$

Then, by Theorem 6.1, the optimal utility (Y, Z) and its associated deflator Γ are the unique-solution forward-backward system:

$$\begin{aligned} d\Gamma_t &= \Gamma_t(-\beta\, dt - g'(Z_t)\, dW_t), \\ \Gamma_0 &= 1, \end{aligned}$$

$$\begin{aligned} -dY_t &= -\beta Y_t - g(Z_t) + u\Big((u')^{-1}(\nu\Gamma_t^{-1}H_t)\Big)\, dt - Z_t\, dW_t, \\ Y_T &= h\Big((h')^{-1}(\nu\Gamma_T^{-1}H_T)\Big), \end{aligned}$$

where $H_t = \exp\{-\int_0^t r_s\, ds - \int_0^t \theta_s\, dW_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds\}$ is the classical deflator associated to the wealth. Notice that the process $-g'(Z_t)$ which appears in the dynamics of the deflator Γ can be interpreted as the optimal risk premium associated with the utility process. Unfortunately, the four-step scheme resolution method of Ma, Protter and Yong cannot be applied in this case since, in the forward equation, the diffusion term $-g'(Z_t)$ depends on Z.

The optimal wealth and the optimal consumption are then simply given by

$$\xi^0 = (h')^{-1}(\nu\Gamma_T^{-1}H_T), \qquad c_t^0 = (u')^{-1}(\nu\Gamma_t^{-1}H_t), \qquad 0 \le t \le T.$$

Moreover, the optimal wealth-portfolio process (X_t, π_t) is then solution of the classical BSDE:

$$\begin{aligned} -dX_t &= (-r_t X_t - \pi_t^*\, \theta_t + (u')^{-1}(\nu\Gamma_t^{-1}H_t))\, dt - \pi_t^*\, dW_t, \\ X_T &= (h')^{-1}(\nu\Gamma_T^{-1}H_T). \end{aligned}$$

7.3. *Example of the large investor.* Recall that in this case, the driver of the wealth is given by (7), that is,

$$\begin{aligned} b(t, c, x, \sigma_t^*\pi) = -r_t x - (x - \pi^*\mathbf{1}) f_0(x, \pi) \\ - \pi^*[b_t - r_t\mathbf{1} + f(x, \pi)] + c. \end{aligned} \tag{41}$$

The utility function is standard and its driver is given by

$$f(t, c, y) = u(c) - \beta y,$$

where β is a positive constant and u is a utility function. The drivers f and b are supposed to satisfy Assumptions A1 to A11. Then, by the previous result, the optimal wealth (X, π) and its associated deflator H are the unique-solution forward-backward system:

$$\begin{aligned} dH_t &= H_t\Big[b_x\Big(t, I(\nu H_t e^{\beta t}), X_t, \pi_t\Big)\, dt + b_\pi\Big(t, I(\nu H_t e^{\beta t}), X_t, \pi_t\Big)^*, \, dW_t\Big], \\ H_0 &= 1, \end{aligned}$$

$$\begin{aligned} -dX_t &= b\Big(t, I(\nu H_t e^{\beta t}), X_t, \pi_t\Big)\, dt - (\pi_t)^*\, dW_t, \\ X_T &= J(\nu H_T e^{\beta T}). \end{aligned}$$

Let us make the simple change of variable $A_t = \mathrm{Log}(\nu e^{\beta t} H_t)$. Then, the processes (X, π) and A are the unique solution of the forward-backward system:

$$\begin{aligned} dA_t &= \phi(t, A_t, X_t, \pi_t)\,dt + b_\pi\big(t, I(e^{A_t}),\ X_t, \pi_t\big)^*\,dW_t,\\ A_0 &= \mathrm{Log}\,\nu, \end{aligned}$$

$$\begin{aligned} -dX_t &= b\big(t, I(e^{A_t}), X_t, \pi_t\big)\,dt - (\pi_t)^*\,dW_t,\\ X_T &= J(e^{A_T}), \end{aligned}$$

where

$$\phi(t, a, x, \pi) = \beta + b_x(t, I(e^a),\ x, \pi) - \tfrac{1}{2}|b_\pi(t, I(e^a),\ x, \pi)|^2.$$

Suppose now that the coefficients b_t and r_t are deterministic and that the functions f_i, for $0 \le i \le n$, are deterministic. Suppose furthermore that the function b_π does not depend on π . Then, under some smoothness assumptions, the solution of the forward-backward system can be computed by using the four step scheme resolution method of Ma, Protter and Yong (1994).

First, the wealth process X coincides with a deterministic function of t and A_t; that is, $X_t = \varphi(t, A_t)$, where φ is solution of the following PDE:

$$\begin{aligned} &\partial_t\varphi(t, a) + \mathscr{L}_t\varphi(t, a) + b(t, I(e^a),\ \varphi(t, a), \partial_a\varphi(t, a)) = 0,\\ &\varphi(T, a) = J(e^a), \end{aligned} \tag{42}$$

where $\mathscr{L}_{(t,a)}$ is the generator associated with process A given by

$$\begin{aligned} \mathscr{L}_{(t,a)}\varphi(t, a) &= \phi\big(t, a,\ \varphi(t, a),\ \partial_a\varphi(t, a)\big)\,\partial_a\varphi(t, a)\\ &\qquad + \tfrac{1}{2}|b_\pi\big(t, I(e^a),\ \varphi(t, a)\big)|^2\,\partial^2_{a^2}\varphi(t, a). \end{aligned}$$

Then, the process A is solution of the following classical forward SDE:

$$\begin{aligned} dA_t &= \phi\big(t, A_t, \varphi(t, A_t),\ \partial_a\varphi(t, A_t)\big)\,dt\\ &\qquad + b_\pi\Big(t, I(e^{A_t}),\ \varphi(t, A_t),\ \partial_a\varphi(t, A_t)\Big)^*\,dW_t,\\ A_0 &= \mathrm{Log}\,\nu. \end{aligned}$$

Remark. Note that, in general, b_π depends on π and hence, the four-step scheme resolution method of Ma, Protter and Yong (1994) cannot be applied.

APPENDIX

Proof of Proposition 3.3. First, recall the classical case corresponding to a linear wealth and a standard additive HARA utility function [see, e.g., Karatzas, Lehoczky and Shreve (1987)].

LEMMA A.1. *Suppose that* $f(t, c, y, z) = c^p/p$ *and* $h(x) = x^p/p$ *with* $0 < p < 1$, *and* $b(t, c, x, \pi) = -r_t x - \theta_t^* \pi + c$. *Then*

$$V(x) = \frac{1}{p}(\chi(1))^{1-p} x^p, \qquad 0 < x < +\infty,$$

where $\chi(y) = y^{1/(p-1)} \mathbb{E} \int_0^T H_t^{p/(p-1)} \, dt + H_T^{p/(p-1)}$ *and where the change of numeraire* H_t *is defined by* $dH_t = H_t(-r_t \, dt - \theta_t^* \, dW_t)$ *with* $H_0 = 1$.

The result of Proposition 3.3 (which is an extension to the case of a utility driver f linear with respect to y, z) can be derived easily from Lemma A.1 by using a change of probability and a discounted factor.

First, note that by considering $e^{Ct} Y_t$ instead of Y_t and $e^{Ct} Z_t$ instead of Z_t, it is clear that this differential utility is equivalent to

$$-dY_t = \left[e^{Ct} U(c_t) + b_2(t) \cdot Z_t\right] dt - Z_t \, dW_t,$$
$$Y_T = e^{CT} h(X_T).$$

We now introduce a probability measure P_Q on $(\Omega, \mathscr{F}_T)$ by

$$\left.\frac{dP_Q}{dP}\right|_{\mathscr{F}_t} = \exp\left[\int_0^T b_2(t) \, dW_t - \frac{1}{2} \int_0^T |b_2(t)|^2 \, dt\right].$$

By Girsanov's theorem, it follows that, under this new probability space $(\Omega, \mathscr{F}, P_Q)$, the process $\widehat{W}_t = -\int_0^t b_2(s) \, ds + W_t$ is a Wiener process. Thus, the above system is equivalent to the following classical maximization of utility where the dynamics of the wealth satisfy

$$-dX_t = [kc_t - r_t X_t - (b_1(t) + b_2(t)) \cdot \pi_t] \, dt - \pi_t d\widehat{W}_t$$

The maximal reward is given by

$$V_0(x) = \sup_{(c, \pi) \in \mathscr{A}(x)} \mathbf{E}_{P_Q}\left[\int_0^T e^{Ct} U(c_t) \, dt + e^{CT} h(X_T^{x, c, \pi})\right].$$

Then, the desired result is a direct consequence of Lemma A.1. □

COMMENTS ON PROPOSITION 4.1. In order to be able to apply the theorem of convex analysis, we have to note that:

LEMMA A.2. *The Slater condition for* (12) *is satisfied*; *that is*, *there exists a consumption plan* $(\xi, c) \in \mathscr{L} \times \mathscr{D}$ *such that* $X_0^{(\xi, c)} < x$.

PROOF OF LEMMA A.2. The assumptions made on b imply

$$b(t, c, x, \pi) \leq C(x + |\pi|) + kc.$$

We now choose $\xi = \frac{x}{2}e^{-CT}$ and $c = 0$. Then, by the comparison theorem, the associated wealth is smaller than the solution $\widetilde{X}_t$ of the deterministic backward equation

$$-d\widetilde{X}_t = C\widetilde{X}_t\,dt, \qquad \widetilde{X}_T = \frac{x}{2}e^{-CT},$$

which satisfies $\widetilde{X}_0 = \frac{x}{2} < x$. $\square$

Then, the result of Proposition 4.1 follows easily from Luenberger (1969), Corollary 8.31.

Note that ν^* cannot be equal to zero since if $\nu^* = 0$, then

$$V(x) = \sup_{(\xi,c)\in\mathscr{D}\times\mathscr{L}} Y_0^{(\xi,c)}$$

would be attained for an infinite consumption plan (since the functional $(\xi, c) \to Y_0^{(\xi,c)}$ is strictly nondecreasing by the backward comparison theorem), which would yield to an infinite optimal wealth, which is impossible.

The second part of Proposition 4.1 follows from Luenberger (1969), Theorem 8.4.2. [Recall that the triple $(\nu^0, (\xi^0, c^0))$ corresponds to a saddle point for the Lagrangian $L(\nu, (\xi, c)) = Y_0^{(\xi,c)} + \nu(x - X_0^{(\xi,c)})$, i.e.,

$$L(\nu^0, (\xi, c)) \le L(\nu^0, (\xi^0, c^0)) \le L(\nu, (\xi^0, c^0))$$

for all $\nu \ge 0$ and $(\xi, c) \in \mathscr{L} \times \mathscr{D}$.]

Proof of Theorem 4.3. Let (ξ, c) be a consumption plan. We denote by $(Y, Z), (X, \pi)$ the associated trajectory. We denote by ΔX (respectively ΔY) the variation of the wealth (respectively of the utility) associated with (ξ, c):

$$\begin{aligned}\Delta X_t &= X_t - X_t^0,\\ \Delta \pi_t &= \pi_t - \pi_t^0,\\ \Delta Y_t &= Y_t - Y_t^0,\\ \Delta Z_t &= Z_t - Z_t^0.\end{aligned}$$

Thus, the problem is to show that the assumption made on (ξ^0, c^0) gives that

$$\Delta Y_0 - \nu\Delta X_0 \le 0 \qquad \forall(\xi, c) \in \mathscr{D} \times \mathscr{L}.$$

Now, the following lemma will allow us to conclude.

Lemma A.3. *The following inequalities are satisfied*:

$$\begin{aligned}\Delta Y_t &\le \partial_\alpha Y_t^0, \qquad P\text{-}a.s,\ 0 \le t \le T,\\ \Delta X_t &\ge \partial_\alpha X_t^0, \qquad P\text{-}a.s,\ 0 \le t \le T.\end{aligned}$$

End of the proof of the theorem. It follows from this lemma that

$$\Delta Y_0^{(c,\xi)} - \nu\Delta X_0^{(c,\xi)} \le (\partial_\alpha Y_0^0)^{(c,\xi)} - \nu(\partial_\alpha X_0^0)^{(c,\xi)} \quad \forall(c, \xi) \in \mathscr{D} \times \mathscr{L}.$$

Now, equality (22) and the assumption made on (ξ^0, c^0) imply

$$(\partial_\alpha Y_0^0)^{(c,\xi)} - \nu(\partial_\alpha X_0^0)^{(c,\xi)} \le 0 \qquad \forall (c,\xi) \in \mathscr{D} \times \mathscr{L},$$

and the desired result follows. □

PROOF OF LEMMA A.3. Those inequalities are consequences of the concavity inequalities and the comparison theorem. Indeed, the pair $(\Delta Y, \Delta z)$ is solution of the following BSDE:

$$-d(\Delta Y)_t = f_1(t, \Delta Y_t, \Delta z_t)\,dt - \Delta z_t^*\,dW_t,$$
$$\Delta Y_T = h(\xi) - h(\xi^0),$$

where $f_1(t,y,z) = f(c_t, Y_t^0 + y, z_t^0 + z) - f(c_t^0, Y_t^0, z_t^0)$. Also, the pair $(\partial_\alpha Y^0, \partial_\alpha Z^0)$ is the solution of the following BSDE:

$$-d\partial_\alpha Y_t^0 = f_2(t, \partial_\alpha Y_t^0, \partial_\alpha Z_t^0)\,dt - (\partial_\alpha Z_t^0)^*\,dW_t,$$
$$\partial_\alpha Y_T^0 = h'(\xi^0)(\xi - \xi^0),$$

where $f_2(t,y,z) = f_c^0(t)(c_t - c_t^0) + f_y^0(t)y + f_z^0(t)z$.

Now, the following concavity inequalities hold:

$$h(\xi) - h(\xi^0) \le h'(\xi^0)(\xi - \xi^0), \qquad P\text{-}a.s,$$
$$f_1(t,y,z) \le f_2(t,y,z) \qquad \forall y, z,\ dP \otimes dt\text{-}a.s.$$

Hence, by the comparison theorem, we have $\Delta Y_t \le \partial_\alpha Y_t^0,\ \forall t\ P\text{-}a.s.$ Using the same arguments, we obtain the second inequality $\Delta X \ge \partial_\alpha X^0$. □

REMARK 1. If the Inada condition Assumption A11 is not satisfied, then the same arguments give the following result. Suppose that conditions (20) and (21) are satisfied respectively on $\{\xi^0 > 0\}$ and $\{c_t^0 > 0\}$ and suppose that conditions (25) and (26) are satisfied respectively on $\{\xi^0 = 0\}$ and $\{c_t^0 = 0\}$. Then, (ξ^0, c^0) is optimal.

REMARK 2. In fact, the result of Theorem 4.3 still remains without the additional smoothness conditions Assumption A10 . It suffices to use the concavity and convexity properties of f and b and to consider sub- and superdifferentials instead of differentials. More precisely, the result and the above proof of Theorem 4.3 still hold in this case with the following notation: $h'(\xi^0)$ denotes in this case a square-integrable $\mathscr{F}_T$-measurable variable belonging to $\partial h(\xi^0)$, where ∂h is the superdifferential of h. The vector $(f_c^0(t), f_y^0(t), f_z^0(t))$ denotes a three-dimensional predictable vector-process with $E\int_0^T (f_c^0(t))^2\,dt < +\infty$ belonging $dP \otimes dt$ almost surely to $\partial f(t, c_t^0, Y_t^0, z_t^0)$, where ∂f is the superdifferential of f with respect to c, y, z. Also, $(b_c^0(t), b_x^0(t), b_\pi^0(t))$ denotes a predictable process with $E\left[\int_0^T (b_c^0(t))^2\,dt\right] < +\infty$ belonging to $\partial b(t, c_t^0, X_t^0, \pi_t^0)$ (subdifferential of b).

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