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α'-expansion of antisymmetric Wilson loops in $\mathcal{N} = 4$ SYM from Fermi gas

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We study the large 't Hooft coupling expansion of 1/2 BPS Wilson loops in the antisymmetric representation in $\mathcal{N} = 4$ super Yang–Mills (SYM) theory at the leading order in the $1/N$ expansion. Via AdS/CFT correspondence, this expansion corresponds to the α' expansion in bulk type IIB string theory. We show that this expansion can be systematically computed by using the low temperature expansion of the Fermi distribution function, known as the Sommerfeld expansion in statistical mechanics. We check numerically that our expansion agrees with the exact result of antisymmetric Wilson loops recently found by Fiol and Torrents.

Subject Index B21

1. Introduction

1/2 BPS circular Wilson loops in 4D $\mathcal{N} = 4$ super Yang–Mills (SYM) theory are interesting observables that can be computed exactly by a Gaussian matrix model [1–3]. Via AdS/CFT correspondence, 1/2 BPS Wilson loops in the fundamental representation correspond to the fundamental string in type IIB string theory on $\mathrm{AdS}_5 \times S^5$ [4,5]. When the rank of the representations becomes large, the corresponding dual objects in the bulk are not fundamental strings but D-branes, and such Wilson loops are sometimes called "giant Wilson loops". In particular, 1/2 BPS Wilson loops in the rank-k symmetric and antisymmetric representations correspond to D3-branes and D5-branes, respectively, with k units of electric flux on their world volumes [6–9]. The leading term in the 't Hooft expansion of $\mathcal{N} = 4$ SYM side is successfully matched with the DBI action of D-branes in the bulk side. For more general representations, a dictionary between Wilson loops in higher-rank representations and bulk D-brane pictures was proposed in Refs. [10,11].

We are interested in the subleading corrections in this correspondence. Recently, there has been some progress in the computation of one-loop corrections in the $1/N$ expansion of giant Wilson loops [12–15]. Here we will focus on the subleading corrections in the large λ expansion (or $1/\lambda$ expansion) with λ being the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$, and we will restrict ourselves to the leading order in the $1/N$ expansion. From the holographic dictionary $R_{\mathrm{AdS}}^2/\alpha' = \sqrt{\lambda}$, the large λ expansion on the SYM side corresponds to the α' expansion in the bulk string theory side.

In this paper, we consider the large λ expansion of the 1/2 BPS Wilson loops in the antisymmetric representation. Using the fact that the generating function of antisymmetric representations can be written as a system of fermions, one can systematically compute the subleading corrections in the large λ expansion by a low-temperature expansion of the Fermi distribution function, known as the Sommerfeld expansion. Here the role of temperature is played by $1/\sqrt{\lambda}$. We have checked

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numerically that the subleading corrections agree with the exact expression of the antisymmetric Wilson loops recently found in Ref. [16].

The rest of the paper is organized as follows. In Sect. 2, we find a systematic large λ expansion of antisymmetric Wilson loops using the Sommerfeld expansion of Fermi distribution function. Our main result is Eq. (2.10). In Sect. 3, we compare our result (2.10) with the exact expression in Ref. [16], and find nice agreement. We conclude in Sect. 4 and discuss some future directions.

2. Large λ expansion of Wilson loops in the antisymmetric representation

We consider the vacuum expectation value (VEV) of 1/2 BPS circular Wilson loops in $\mathcal{N} = 4$ SYM with gauge group $U(N)$. After applying the supersymmetric localization [3], the Wilson loop VEV is reduced to a Gaussian matrix model

$$\left\langle \mathrm{Tr}_R\, P \exp\left[\oint ds\Big(\mathrm{i}A_\mu \dot{x}^\mu + \Phi_I \theta^I |\dot{x}|\Big)\right]\right\rangle = \int dM \exp\left(-\frac{1}{2\pi g_s}\,\mathrm{Tr}\, M^2\right) \mathrm{Tr}_R(e^M). \tag{2.1}$$

Here $x^\mu(s)$ parametrizes a great circle of S^4 on which $\mathcal{N} = 4$ SYM lives, and Φ_I $(I = 1, \ldots, 6)$ denote the adjoint scalar fields in $\mathcal{N} = 4$ SYM and $\theta^I \in S^5$ is a constant unit vector. In Eq. (2.1), g_s denotes the string coupling, which is related to the Yang–Mills gauge coupling g_{YM} by

$$g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}. \tag{2.2}$$

In this paper, we will focus on the Wilson loop VEV in the kth antisymmetric representation $R = A_k$,

$$W_{A_k} = \int dM \exp\left(-\frac{1}{2\pi g_s}\,\mathrm{Tr}\, M^2\right) \mathrm{Tr}_{A_k}(e^M). \tag{2.3}$$

It is convenient to define the VEV of SU(N) part by removing the $U(1)$ contribution:

$$\mathcal{W}_{A_k} = W_{A_k} \exp\left(-\frac{\pi k g_s}{2}\right). \tag{2.4}$$

One can show that $\mathcal{W}_{A_k}$ is symmetric under $k \to N - k$:

$$\mathcal{W}_{A_{N-k}} = \mathcal{W}_{A_k}. \tag{2.5}$$

We are interested in the behavior of the Wilson loop VEV $\mathcal{W}_{A_k}$ in the limit

$$N \to \infty \quad \text{with} \quad \lambda = g_{\mathrm{YM}}^2 N, \quad \frac{k}{N} \text{ fixed}. \tag{2.6}$$

In the large λ limit together with Eq. (2.6), the antisymmetric Wilson loop $\mathcal{W}_{A_k}$ is holographically dual to a D5-brane in AdS$_5 \times S^5$, whose world volume has the form AdS$_2 \times S^4$ [7]. From the computation of the DBI action of the D5-brane, the leading behavior of $\mathcal{W}_{A_k}$ is found to be

$$\log \mathcal{W}_{A_k} = \frac{2N\sqrt{\lambda}}{3\pi} \sin^3\theta_k = \frac{1}{g_s}\frac{(\sqrt{\lambda}\sin\theta_k)^3}{6\pi^2}, \tag{2.7}$$

where θ_k is given by

$$\theta_k - \sin\theta_k \cos\theta_k = \frac{\pi k}{N}. \tag{2.8}$$

From the bulk D5-brane picture, the angle θ_k parametrizes the position of the S^4 part of the world volume inside the S^5 of bulk geometry $\mathrm{AdS}_5 \times S^5$.

We are interested in the subleading corrections of $\mathcal{W}_{A_k}$. There are two expansion parameters g_s and $1/\lambda$. In Ref. [12], it was reported that the one-loop correction in the g_s expansion has the form

$$\log \mathcal{W}_{A_k} = \frac{1}{g_s}\frac{(\sqrt{\lambda}\sin\theta_k)^3}{6\pi^2} + c\log\sin\theta_k, \tag{2.9}$$

where c is an order 1 constant. In this paper, we will consider subleading corrections of the $1/\lambda$ expansion while we focus on the leading order in the g_s-expansion.

As we will show below, the $1/\lambda$ expansion of $\mathcal{W}_{A_k}$ can be computed as

$$\begin{aligned}\log \mathcal{W}_{A_k} = \frac{1}{g_s}\bigg[&\frac{(\sqrt{\lambda}\sin\theta_k)^3}{6\pi^2} + \frac{\sqrt{\lambda}\sin\theta_k}{12} - \frac{\pi^2(19+5\cos 2\theta_k)}{\sqrt{\lambda}\sin^3\theta_k} \\ &- \frac{\pi^4(6788\cos 2\theta_k + 35\cos 4\theta_k + 8985)}{362\,880\lambda^{3/2}\sin^7\theta_k} + \cdots\bigg].\end{aligned} \tag{2.10}$$

This is our main result.

Let us explain how we obtained Eq. (2.10). To study the antisymmetric Wilson loops systematically, it is convenient to introduce the generating function of $\mathcal{W}_{A_k}$ by summing over k with fugacity e^{μ},

$$\sum_{k=0}^{N} e^{k\mu}\mathcal{W}_{A_k} = \langle \det(1+e^{\mu}e^{M})\rangle_{mm}, \tag{2.11}$$

where $\langle \mathcal{O}\rangle_{mm}$ denotes the expectation value in the Gaussian matrix model,

$$\langle \mathcal{O}\rangle_{mm} = \int dM \exp\left(-\frac{1}{2\pi g_s}\operatorname{Tr} M^2\right)\mathcal{O}. \tag{2.12}$$

Using the large N factorization we find

$$\langle \det(1+e^{\mu}e^{M})\rangle_{mm} \approx \exp\left[\langle \operatorname{Tr}\log(1+e^{\mu}e^{M})\rangle_{mm}\right] \tag{2.13}$$

up to $1/N$ corrections, and the right-hand side of Eq. (2.13) in the planar limit becomes

$$\langle \operatorname{Tr}\log(1+e^{\mu}e^{M})\rangle_{mm} = N\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dm\,\rho(m)\log(1+e^{\mu-m}), \tag{2.14}$$

where $\rho(m)$ is the Wigner semicircle distribution of the Gaussian matrix model

$$\rho(m) = \frac{2}{\pi\lambda}\sqrt{\lambda - m^2}. \tag{2.15}$$

Then, as discussed in Ref. [8], the Wilson loop VEV in the kth antisymmetric representation is written as an integral over the chemical potential μ:

$$\mathcal{W}_{A_k} = \int d\mu \exp\left[-k\mu + N\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dm\,\rho(m)\log(1+e^{\mu-m})\right]. \tag{2.16}$$

By rescaling $(m,\mu)\to(\sqrt{\lambda}m, \sqrt{\lambda}\mu)$, we can further rewrite (2.16) as

$$\mathcal{W}_{A_k} = \int d\mu \exp\left[N\left(-\frac{k}{N}\sqrt{\lambda}\mu + J(\mu)\right)\right], \tag{2.17}$$

where

$$J(\mu) = \frac{2}{\pi} \int_{-1}^{1} dm \sqrt{1-m^2} \log\left(1 + \exp\left(\sqrt{\lambda}(\mu - m)\right)\right). \tag{2.18}$$

In the regime of interest (2.6), the μ-integral in Eq. (2.17) can be evaluated by the saddle point approximation since the exponent in Eq. (2.17) is multiplied by the large number N. Thus we conclude that $\log \mathcal{W}_{A_k}$ is essentially given by the Legendre transform of $J(\mu)$:

$$\log \mathcal{W}_{A_k} = -k\sqrt{\lambda}\mu_* + NJ(\mu_*), \tag{2.19}$$

where μ_* is determined by the saddle point equation

$$\partial_\mu J(\mu)\Big|_{\mu=\mu_*} = \frac{k\sqrt{\lambda}}{N}. \tag{2.20}$$

Note that the fluctuation of the μ-integral around the saddle point gives rise to a subleading correction in g_s, as in the case of the ABJM Fermi gas [17], and hence we can safely ignore such corrections for our purpose of studying the leading-order behavior in the g_s expansion.[1]

Noticing that the Fermi distribution function naturally appears in the derivative of $J(\mu)$,

$$\partial_\mu J(\mu) = \frac{2\sqrt{\lambda}}{\pi} \int_{-1}^{1} dm \frac{\sqrt{1-m^2}}{1 + \exp\left(\sqrt{\lambda}(m-\mu)\right)}, \tag{2.21}$$

one can easily compute the $1/\lambda$ expansion by the standard Sommerfeld expansion in statistical mechanics, where $1/\sqrt{\lambda}$ plays the role of temperature.

The large λ expansion of the Fermi distribution function reads

$$\begin{aligned} \frac{1}{1 + \exp\left(\sqrt{\lambda}(m-\mu)\right)} &= \frac{\pi\partial_\mu}{\sqrt{\lambda}\sin(\pi\partial_\mu/\sqrt{\lambda})}\Theta(\mu - m) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n B_{2n}(1/2)}{(2n)!}\left(\frac{4\pi^2\partial_\mu^2}{\lambda}\right)^n \Theta(\mu - m) \\ &= \left(1 + \frac{\pi^2\partial_\mu^2}{6\lambda} + \frac{7\pi^4\partial_\mu^4}{360\lambda^2} + \cdots\right)\Theta(\mu - m), \end{aligned} \tag{2.22}$$

where $B_{2n}(1/2)$ is the value of the Bernoulli polynomial $B_{2n}(z)$ at $z = 1/2$, and $\Theta(\mu - m)$ is the step function

$$\Theta(\mu - m) = \begin{cases} 1 & (\mu > m), \\ 0 & (\mu < m). \end{cases} \tag{2.23}$$

Introducing the variable θ as

$$\mu = -\cos\theta, \tag{2.24}$$

[1] The overall constant of the integral (2.16) and the factor coming from the change of variable $\mu \to \sqrt{\lambda}\mu$ from Eq. (2.16) to Eq. (2.17) are also subleading in the g_s expansion, and we simply ignore them as well.

one can easily show that $\partial_\mu J(\mu)$ is expanded as

$$\partial_\mu J(\mu) = \frac{2\sqrt{\lambda}}{\pi}\left[\frac{1}{2}(\theta - \sin\theta\cos\theta) + \sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}(1/2)}{(2n)!}\left(\frac{4\pi^2}{\lambda}\right)^n\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{2n-1}\sin\theta\right], \tag{2.25}$$

from which the expansion of $J(\mu)$ is found to be

$$\begin{aligned} J(\mu) &= \int d\theta \sin\theta\, \partial_\mu J(\mu) \\ &= \frac{2\sqrt{\lambda}}{\pi}\left[\frac{\sin^3\theta}{3} - \frac{1}{2}(\theta - \sin\theta\cos\theta)\cos\theta \right. \\ &\quad \left. + \sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}(1/2)}{(2n)!}\left(\frac{4\pi^2}{\lambda}\right)^n\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^{2n-2}\sin\theta\right]. \end{aligned} \tag{2.26}$$

Finally, solving the saddle point equation (2.20) order by order in $1/\lambda$ expansion, and plugging the solution μ_* into (2.19), we arrive at our main result (2.10). In this way, we can compute the $1/\lambda$ expansion of $\mathcal{W}_{A_k}$ up to any desired order.

3. Comparison with the exact result

Let us compare our result (2.10) with the exact result of antisymmetric Wilson loops at finite N and k found in Ref. [16].

It is found in Ref. [16] that the generating function of $\mathcal{W}_{A_k}$, Eq. (2.11), is exactly written as a characteristic polynomial of the $N \times N$ matrix A,

$$\sum_{k=0}^{N} e^{k\mu}\mathcal{W}_{A_k} = \det(1 + e^{\mu}A), \tag{3.1}$$

where the matrix element $A_{i,j}$ is given by the generalized Laguerre polynomial

$$A_{i,j} = L_{i-1}^{j-i}(-\pi g_s), \quad (i,j = 1,\ldots,N). \tag{3.2}$$

From expression (3.1), one can extract the exact value of the Wilson loop VEV $\mathcal{W}_{A_k}$ at arbitrary values of N, k, and g_s.

In Fig. 1, we show the plot of $\log\mathcal{W}_{A_k}$ as a function of k/N for $N = 300$ and $\lambda = 100$, corresponding to the value of string coupling $g_s = \lambda/4\pi N = 1/12\pi$. The blue dots are the exact values obtained from Eq. (3.1) while the red dots are the plot of our result (2.10) for the leading term (Fig. 1(a)) and the leading + next-to-leading terms (Fig. 1(b)). One can clearly see that the leading + next-to-leading terms in Fig. 1(b) exhibit nice agreement with the exact result (3.1). Interestingly, the leading term alone is not enough to reproduce the behavior of the exact result (3.1), and the next-to-leading correction has a rather large contribution for this choice of parameters $N = 300$, $\lambda = 100$. Note that the leading and the next-to-leading terms in Eq. (2.10) are of order $\lambda^{3/2}/g_s$ and $\lambda^{1/2}/g_s$, respectively, while the higher-order terms have negative powers of λ; hence in the large λ limit higher-order corrections in Eq. (2.10) are suppressed. Indeed we have checked that the inclusion of higher-order corrections does not change the plot significantly, and the exact result (3.1) is well approximated already at the next-to-leading order. We have performed similar numerical checks for various values of N and λ ($N, \lambda \gg 1$) and find good agreement for all cases.

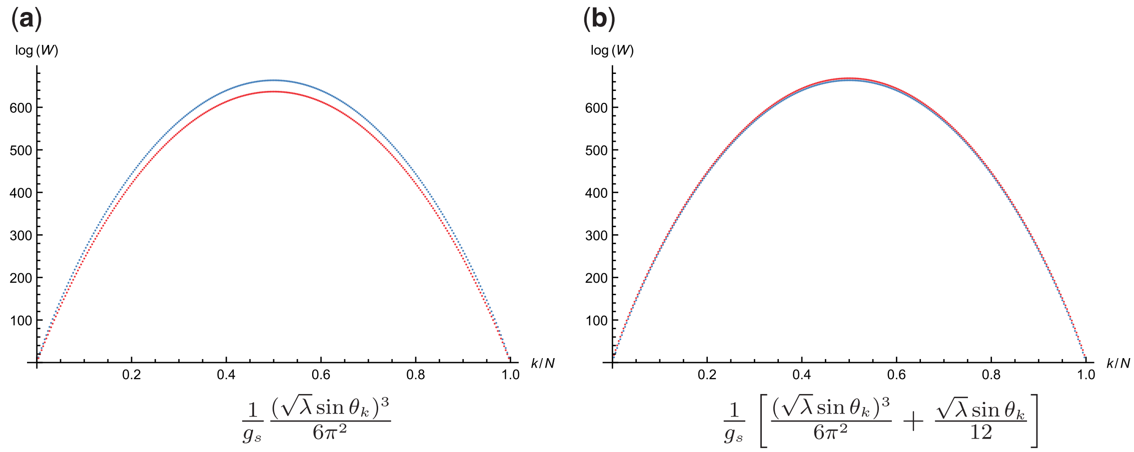

Fig. 1. This is the plot of $\log \mathcal{W}_{A_k}$ as a function of k/N, for $N = 300$, $\lambda = 100$. The blue dots are the exact values obtained from Eq. (3.1), while the red dots represent the behavior of (a) the leading term only and (b) the leading + next-to-leading terms in the expansion (2.10). One can clearly see that the inclusion of the next-to-leading correction improves the matching.

4. Conclusion

We have computed the $1/\lambda$ expansion (or the α'-expansion of bulk type IIB string theory) of the Wilson loop VEV in the antisymmetric representation using the Sommerfeld expansion of the Fermi distribution function. It would be very interesting to reproduce this result from the computation of α'-correction of the D5-brane action in the $\mathrm{AdS}_5 \times S^5$ background.

There are many things to be studied further. It is important to develop a method to compute both the $1/\lambda$ expansion and the g_s expansion systematically. In particular, it would be interesting to find the $1/\lambda$ expansion of the Wilson loop VEV in the symmetric representation by the low-temperature expansion of the Bose distribution. However, the integrand of the μ-integral might have a singularity corresponding to the onset of Bose–Einstein condensation. It would be interesting to understand the analytic structure of the integrand in the case of symmetric representation (see Ref. [8] for a discussion).

Also, it is interesting to understand the convergence property of the expansion. For the 1/2 BPS Wilson loop in the fundamental representation, it is observed that the α'-expansion is *not* Borel summable [2], reflecting the fact that there are corrections of order $\exp\left(-\sqrt{\lambda}\right)$, which is nonperturbative in α'. It would be very interesting to understand the Borel summability of α'-corrections at fixed g_s for Wilson loops in various representations. On the other hand, the g_s expansion of 1/2 BPS Wilson loops with fixed λ seems to have a finite radius of convergence,[2] which is consistent with the absence of Yang–Mills instanton corrections to Wilson loop VEV in $\mathcal{N} = 4$ SYM [3]. We hope that the study of Wilson loops in various representations will provide us with precious information on the nonperturbative structure of string theory.

Finally, we would like to emphasize the importance of our findings. It was reported in Ref. [12] that there is a discrepancy between the computation on the field theory side and the string theory side of the one-loop correction in $1/N$ (2.9) (see [19] for the current status of this problem).[3] However,

[2] For the Wilson loop in the fundamental representation, we have checked the convergence of the g_s expansion numerically using the result in Ref. [18].

[3] Actually, understanding the origin of this discrepancy was one of the motivations for this work.

before settling the issue of this problem of one-loop correction, we have to compute the leading term at large N as a function of λ, including all $1/\lambda$ corrections (2.10). After having found the leading term (2.10), one can try to study the one-loop correction in $1/N$, either numerically or analytically, by subtracting the leading term from the exact result (3.1). In this computation, it is important to subtract the subleading terms in $1/\lambda$ since they are rather large and cannot be neglected, as we saw in Sect. 3. We believe that our result (2.10) is an important first step to resolving the issue of the one-loop discrepancy. We leave the study of the one-loop correction as an interesting future problem.

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