A FACILITY LOCATION FORMULATION FOR STABLE POLYNOMIALS AND ELLIPTIC FEKETE POINTS

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Dedicated to my dear friend Mike Shub on the occasion of his 70th birthday.

Abstract. A breakthrough paper written in 1993 by Shub and Smale unveiled the relationship between stable polynomials and points which minimize the discrete logarithmic energy on the Riemann sphere (a.k.a. elliptic Fekete points). This relationship has inspired advances in the study of both concepts, many of whose main properties are not well known yet. In this paper I prove an equivalent formulation for the problem of elliptic Fekete points and some consequences, including a (non-sharp) reciprocal of Shub and Smale's result and some novel nontrivial claims about these classical problems.

1. Introduction

1.1. **Stable polynomials and elliptic Fekete points.** A great deal of the work of Mike Shub during the last decades has been devoted to an understanding of the properties of the condition number of numerical problems. His efforts have produced many fundamental results (see for example [22], [21] or the monograph [6]) and have inspired further advances by many authors. One of the most fascinating outcomes of his work is the relation of the condition number of polynomials to sets of elliptic Fekete points. Introducing this result requires some notation.

The *condition number* of a (complex) homogeneous polynomial

$$h(z_0, z_1) = \sum_{k=0}^{N} a_k z_0^{N-k} z_1^k, \qquad a_k \in \mathbb{C}, \tag{1.1}$$

of degree $N \geq 1$ at a projective point $\zeta = (\zeta_0, \zeta_1) \in \mathbb{P}(\mathbb{C}^2)$ was defined in [22] as

$$\mu(h, \zeta) = N^{1/2} \|(Dh(\zeta) \mid_{\zeta^\perp})^{-1}\| \|h\| \|\zeta\|^{N-1},$$

where $Dh(\zeta) \mid_{\zeta^\perp}$ is the derivative of h at ζ restricted to the orthogonal complement of ζ, $\|\zeta\| = (|\zeta_0|^2 + |\zeta_1|^2)^{1/2}$ and $\|h\|$ is the Bombieri norm (sometimes called Bombieri–Weyl or Kostlan norm) of h given by

$$\|h\|^2 = \sum_{k=0}^{N} \binom{N}{k}^{-1} |a_k|^2.$$

Date: May 7, 2014.

2010 *Mathematics Subject Classification.* Primary: 52A40, 31A10. Secondary: 11K38.

Key words and phrases. Elliptic Fekete points, stable polynomials, condition number, logarithmic energy, Smale's 7th problem, facility location, cap discrepancy, separation distance.

I want to thank Mike Shub for many years of friendship, collaboration and enlightening discussions, and Giuseppe Buttazzo for an inspiring remark. Thanks also to two anonymous referees for many helpful comments.

Communicated by Teresa Krick and James Renegar.

If $Dh(\zeta)\mid_{\zeta^\perp}$ is not invertible we let $\mu(h,\zeta)=\infty$. Note that μ is sometimes denoted by μ_{proj} or μ_{norm} but we keep the most simple notation here.

The Bombieri norm and the condition number of a degree N, one-variable polynomial $f=f(z)$ at some point $z\in\mathbb{C}$ are then defined by

$$\|f\|=\|h\|,\quad \mu(f,z)=\mu(h,(1,z)),$$

where h is the homogeneous version of f, i.e.

$$h(z_0,z)=z_0^N f\left(\frac{z}{z_0}\right).$$

The condition number of a polynomial (and its extension to systems of polynomials) is of essential importance for understanding the stability and complexity of polynomial solving, as proved in [22] and widely studied subsequently. The following lemma (see Section 2.1 for a proof) gives a simple formula for computing μ:

Lemma 1.1. *For a degree N polynomial f and a complex point $z\in\mathbb{C}$ we have*

$$\mu(f,z)=N^{1/2}\frac{\|f\|(1+|z|^2)^{\frac{N}{2}}}{|f'(z)(1+|z|^2)-N\bar{z}f(z)|},$$

where $\bar{z}$ is the complex conjugate of z (or $\mu(f,z)=\infty$ if the denominator equals 0). In particular, if $f(z)=0$ and $f'(z)\neq 0$ then

$$\mu(f,z)=N^{1/2}\frac{\|f\|(1+|z|^2)^{\frac{N}{2}-1}}{|f'(z)|},$$

and $\mu(f,z)=\infty$ if z is a multiple root of f.

An (at first sight) unrelated concept is that of elliptic Fekete points; that is, N spherical points $x_1,\ldots,x_N$ which maximise the product of their mutual affine distances (see for example [15] for an introduction to the problem of distributing points on a sphere). To fix the notation, we will consider points on the Riemann sphere $\mathbb{S}\subseteq\mathbb{R}^3$ that is the sphere of radius $1/2$ centered at $(0,0,1/2)$, and we will let $X=(x_1,\ldots,x_N)\in\mathbb{S}^N$. Maximising the product of the distances is equivalent to minimising what is termed the (discrete) logarithmic energy

$$\mathcal{E}(X)=\sum_{i<j}\log\|x_i-x_j\|^{-1}=-\sum_{i<j}\log\|x_i-x_j\|.$$

If we denote by d_R the Riemannian distance on $\mathbb{S}$ when $\mathbb{S}$ is endowed with the metric inherited from $\mathbb{R}^3$, we have

$$\|x-y\|=\sin d_R(x,y),\quad \forall\, x,y\in\mathbb{S}. \tag{1.2}$$

Denoting

$$m_N=\min_{X\in\mathbb{S}^N}\mathcal{E}(X),$$

a given N–tuple X is called a set of elliptic Fekete points if $\mathcal{E}(X)=m_N$. The value of m_N is not well known for large N, but much effort has been devoted to understanding it. The first progress in this field is sometimes attributed to a general method by Elkies (see the introduction of [12]), but I have not found an appropriate reference. Subsequent works by Wagner [28], Rakhmanov, Saff and Zhou [19], Dubickas [12] and Brauchart [7] have improved the bounds on m_N. We follow [8], but note that the result in that paper is for the unit sphere; we translate it here to the Riemann sphere (which amounts to add $(N^2-N)\log(2)/2$ to the bounds), and do the same for other cited results. Note also that some authors consider the energy as the sum of $-\log\|x_i - x_j\|$ for $i \neq j$, but we consider it just as the sum for $i < j$ which halfs the bounds. Paying attention to these considerations, we present the translation to our notation of the bounds in [8].

Theorem 1.2. *Let C_N be defined by*

$$m_N = \frac{N^2}{4} - \frac{N\log N}{4} + C_N N.$$

Then

$$-0.4593423\ldots \leq \liminf_{N\to\infty} C_N \leq \limsup_{N\to\infty} C_N \leq -0.3700708\ldots$$

If a global bound is wanted, one can use the following one which follows from [28]:

$$m_N \geq \frac{N^2}{4} - \frac{N\log N}{4} - \frac{\log(2\pi)}{4}N. \tag{1.3}$$

In [8], it is conjectured that C_N as defined in Theorem 1.2 does have a limit and that

$$\lim_{N\to\infty} C_N = 2\log 2 + \frac{1}{2}\log\frac{1}{3} + 3\log\frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.4021788\ldots$$

In 1993, a remarkable paper [24] by Mike Shub and Steve Smale outlined a relationship between stable polynomials (i.e., polynomials whose zeros all have a small – polynomial in N – value of the condition number) and points with low energy values:

Theorem 1.3 ([24])**.** *Let $X = (x_1, \ldots, x_N) \in \mathbb{S}^N \setminus \{(0,0,0)\}$ satisfy*

$$\mathcal{E}(X) \leq m_N + K,$$

for some $K \geq 0$. Let $f(z) = (z-z_1)\cdots(z-z_N)$ be the monic polynomial with zeros $z_1, \ldots, z_N \in \mathbb{C}$ obtained from the points $x_1, \ldots, x_N$ by means of the North Pole stereographic projection (1.4)*. Then*

$$\mu(f, z_i) \leq \sqrt{N(N+1)}e^K, \quad 1 \leq i \leq N.$$

The relation between the x_i and the z_i is

$$x_i = \left(\frac{\Re(z_i)}{1+|z_i|^2}, \frac{\Im(z_i)}{1+|z_i|^2}, \frac{1}{1+|z_i|^2}\right), \tag{1.4}$$

where $\Re(z_i)$ and $\Im(z_i)$ are respectively the real and imaginary parts of z_i. Two well-known identities which follow from the algebraic manipulation of (1.4) are

$$\|x_i - x_j\| = \frac{|z_i - z_j|}{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}}, \qquad \|x_i\| = \frac{1}{\sqrt{1+|z_i|^2}}. \tag{1.5}$$

The following result follows immediately from Theorem 1.3.

Corollary 1.4. *Let c be some constant. If for every $N \geq 2$ we can find an N–tuple X_N such that $\mathcal{E}(X_N) \leq m_N + c\log N$, then we can find a sequence of stable polynomials; that is, a sequence of polynomials f_N such that $\mu(f_N, z) \leq N^c$ for every zero z of f_N, for every N.*

This fact motivated the inclusion of the problem of distributing points on the sphere in the list of Smale's problems for the twenty-first century [26]. Finding such a sequence of stable polynomials is still an open problem, unless one is allowed to use randomisation (it has been known since [23] that random polynomials are stable with high probability).

Not much is known about the reverse to Theorem 1.3; namely, *Do stable polynomials produce low energy spherical points?* But, a randomized version was proved in [2]: Random polynomials, which are known to be stable, produce rather low values of the expected logarithmic energy. More precisely:

Theorem 1.5 ([2])**.** *Let $f(z) = \sum_{k=0}^{N} a_k z^k$ be a random polynomial, the coefficients a_k being independent complex random variables, such that the real and imaginary parts of a_k are independent (real) Gaussian random variables centered at* 0 *with variance $\binom{N}{k}$. Then the associated N–tuple X on the Riemann sphere given by* (1.4) *where $z_1, \ldots, z_N$ are the zeros of f have an average value $\mathcal{E}$ of their discrete logarithmic energy such that*

$$\mathcal{E} = \frac{N^2}{4} - \frac{N \log N}{4} - \frac{N}{4}.$$

Another relationship between the condition number and the logarithmic energy comes from the following formula.

Lemma 1.6. *Let $x_1, \ldots, x_N$, $z_1, \ldots, z_N$ and f be as in Theorem 1.3. Then*

$$\mathcal{E}(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^{N} \log \mu(f, z_i) + \frac{N}{2} \log \frac{\prod_{i=1}^{N} \sqrt{1 + |z_i|^2}}{\|f\|} - \frac{N \log N}{4}.$$

Note that the term

$$\frac{\prod_{i=1}^{N} \sqrt{1 + |z_i|^2}}{\|f\|}$$

in Lemma 1.6 is the quotient between the product of the Bombieri–Weyl norm of the factors of f and the Bombieri–Weyl norm of f. This quantity is always greater than 1; see [3]. The formula in Lemma 1.6 was first presented in [2] without a proof, so we include a short proof in Section 2.2 for completeness.

1.2. **A facility location formulation.** Facility location problems are an important family of problems in operations research, and there are hundreds of publications devoted to them; see for example [11] and references therein. Their main feature is: Given some space M (which is usually assumed to have a metric and sometimes a measure), and given some (discrete or continuous) distribution denoting the places where the *customers* live, one has to decide where in M to locate N *facilities* according to some objective function.

My only experience with this field of mathematics comes from a comment by Giuseppe Buttazzo. During the ADORT 2010 meeting in Barcelona, I was presenting the problem of well-distributed spherical points and the main result of [2] when Giuseppe, who was in the audience, pointed out, "This is a facility location problem". His comment made me wonder: which of the many ways of defining well-distributed spherical points might correspond to which location problem? It has been known since [16] that different notions of the energy to be minimised (for example, $\sum_{i<j} \log \|x_i - x_j\|^{-1}$ as in elliptic Fekete points or $\sum_{i<j} \|x_i - x_j\|^{-1}$ as in Thomson's problem – see [30] for an early general review) produce different sets of points and therefore not all of these sets can be expected to solve some particular, fixed facility location problem. Yet Giuseppe's comment kept dancing in my head for some time, and I finally decided that it was too difficult a question or that it had too difficult an answer.

A few years later, Mike Shub and I were talking about the idea of using the heat equation to describe some kind of normalized measure, beyond the condition number, of the hardness of a numerical analysis problem. The same day we had also discussed recent progress on the problem of distributing spherical points and the following question arose in our conversation: what is the best way to distribute points on a sphere in such a way that the heat is "well-distributed" if the points are sources of heat?

It took me several weeks to realize that the answer to this last question is also an answer to Giuseppe's comment; stating and proving this claim and its consequences is the main aim of this article.

Among the many different facility location problems, a frequent choice are what are called *minisum* one-facility problems; that is, given some length space M with some finite measure, one must choose some $m_1 \in M$ such that the expected value of the distance $d(m, m_1)$ when m moves in M is minimal. As the logarithm is an increasing function in $(0, \infty)$, one could also ask for the, say, *minilogsum* problem; that is, looking for the point m_1 in M such that the expected value of $\log d(m, m_1)$ is minimised. This is a similar problem to the classical minisum problem, but different in that very short distances are heavily weighted due to the logarithm in the formula. If we let each facility be not just a point but covering some circle of fixed radius r where no customer can live, then our problem is to find

$$m_1 = \operatorname{argmin} \fint_{\{d(m,m_1)\geq r\}} \log d(m, m_1)\, dm = \operatorname{argmax} \fint_{\{d(m,m_1)\geq r\}} \log d(m, m_1)^{-1}\, dm, \tag{1.6}$$

where for a finite measure space X and an integrable function $f : X \to \mathbb{R}$ we denote by

$$\fint_X f(x)\, dx = \frac{1}{Vol(X)} \int_X f(x)\, dx$$

the average value of f over X.

A realistic example of interest is the problem of situating one source of heat in a two–dimensional Riemannian manifold M. In $\mathbb{R}^2$, the solution of the heat equation $\Delta u = -\delta_0$ (where δ_0 is Dirac's delta and Δ is the usual Laplacian in $\mathbb{R}^2$) is $(2\pi)^{-1} \log \|x\|^{-1}$. This is (idealistically) the steady state temperature of the plane with an infinite heat source at the origin. Neglecting the effect of the curvature in M, we may approximate the steady state temperature at a point $m \in M$ by minus the logarithm of the distance to the source of heat m_1. The point m_1 in (1.6) is then the optimal source location in M if the average temperature is to be maximised, once a "safety radius" r (that could correspond to the physical size of the heat source) around the source of heat has been removed.

Let us now be more precise. Assume that N sources of heat have been placed at some points $x_1, \ldots, x_N$ on the sphere $\mathbb{S}$. Assume moreover that every point of the sphere is cooling at some fixed rate $\lambda > 0$. The heat conduction equation then says that the temperature $u = u(x, t)$, for $x \in \mathbb{S} \setminus \{x_1, \ldots, x_N\}$ and $t \geq 0$, satisfies

$$u_t = \Delta_{\mathbb{S}} u - \lambda, \tag{1.7}$$

where $\Delta_{\mathbb{S}} u$ is the Riemannian Laplacian of u as a function defined on $\mathbb{S}$. The steady state solution of this problem satisfies

$$\Delta_{\mathbb{S}} u = \lambda. \tag{1.8}$$

Recall from [5, Lemma 2.2] that for any i, $\Delta_{\mathbb{S}} \log \|x_i - x\|^{-1} = 2$. Thus the function

$$u(x) = u(x, t) = \frac{\lambda}{2N} \sum_{i=1}^{N} \log \|x_i - x\|^{-1} + u_0,$$

u_0 a constant, is one steady state solution of (1.7), but there are others. For uniqueness, we must impose a condition to specify the limiting rate at which the heat sources raise the temperature of the nearby points; i.e., we must impose some condition on the gradient of u. In our case, we impose the condition that the gradient of u is essentially bounded above by the (unit) tangent vector directed toward x_i divided by the square distance to x_i. We write this more precisely in the following proposition, which will be proved in Section 3.

Proposition 1.7. *Let $\nabla_{\mathbb{S}} u$ denote the gradient of u as a function defined on $\mathbb{S}$. For $x \in \mathbb{S}$, let $\pi_{T_x\mathbb{S}}$ be the orthogonal projection onto the tangent space $T_x\mathbb{S}$ to $\mathbb{S}$ at x. Every C^2 function $u : \mathbb{S} \setminus \{x_1, \ldots, x_n\} \to \mathbb{R}$ which satisfies* (1.8) *and additionally satisfies*

$$\limsup_{\|x_i - x\| \to 0} \left\| \|x_i - x\|^{1-\epsilon} \nabla_{\mathbb{S}} u(x) - \frac{\lambda}{2N} \frac{\pi_{T_x\mathbb{S}}(x_i - x)}{\|x_i - x\|^{1+\epsilon}} \right\| < \infty, \quad 1 \le i \le N, \tag{1.9}$$

for some $\epsilon \in (0,1)$ is of the form

$$u(x) = u(x,t) = \frac{\lambda}{2N} \sum_{i=1}^{N} \log \|x_i - x\|^{-1} + u_0 \tag{1.10}$$

for some constant u_0. Namely, (1.8) *and* (1.9) *define a function which is unique up to an additive constant.*

Remark 1.8. *Note that if u satisfies* (1.9) *for some $\epsilon > 0$ then*

$$\limsup_{\|x_i - x\| \to 0} \frac{\left\| \|x_i - x\| \nabla_{\mathbb{S}} u(x) - \frac{\lambda}{2N} \pi_{T_x\mathbb{S}} \left(\frac{x_i - x}{\|x_i - x\|} \right) \right\|}{\|x_i - x\|^{\epsilon}} < \infty,$$

and as the denominator tends to 0 we conclude that so must the numerator. Thus, informally, the meaning of (1.9) *is:*

$$\nabla_{\mathbb{S}} u(x) = \frac{\lambda}{2N} \pi_{T_x\mathbb{S}} \left(\frac{x_i - x}{\|x_i - x\|^2} \right) + \text{ smaller terms} \quad \text{as } \|x_i - x\| \to 0.$$

A question that I have not explored further is whether the hypotheses (1.9) *can be substituted by some more natural assumption.*

The following facility location problem arises naturally: how should N sources of heat be located on $\mathbb{S}$ in such a way that the average temperature outside some safety radius $r > 0$ around the sources is maximal? More precisely:

Problem 1.9 (Location of heat sources to maximise average temperature)**.** *Fix $r > 0$. Situate N points (sources of heat) $x_1, \ldots, x_N \in \mathbb{S}$ in such a way that the average temperature outside a cap of radius r around each x_i is maximal, assuming a positive loss of heat which is independent of the position. Equivalently, find $x_1, \ldots, x_N \in \mathbb{S}$, $d_R(x_i, x_j) \ge 2r$ for $i \ne j$, in such a way that the following quantity is maximised:*

$$\sum_{i=1}^{N} ⨍_{\{d_R(x,x_j) \ge r \ \forall j\}} \log \|x - x_i\|^{-1} \, dx.$$

Remark 1.10. *It can be seen (see* (4.3) *below) that for u given by* (1.10) *the average temperature over the whole sphere $\mathbb{S}$ is equal to $\lambda/4 + u_0$, independently of where the x_i are located. The problem becomes interesting when a small spherical cap ("safety region") is deleted around each heat source.*

Note also that Problem 1.9 is by nature a facility location problem: the *facilities* are the sources of heat, the *customers* are uniformly distributed outside a given safety radius around each source, and the objective is to enable the customers to enjoy the highest average temperature. Our main result relates the problem of the logarithmic energy to the function in Problem 1.9.

Theorem 1.11 (Main)**.** *Let* $X = (x_1, \dots, x_N) \in \mathbb{S}^N$ *a configuration of distinct points and let* $\delta \in (0,1)$ *be such that*

$$d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\frac{\delta}{N}} \quad \forall\, i \neq j. \tag{1.11}$$

Let

$$B_i = \left\{ x \in \mathbb{S} \colon d_R(x, x_i) \leq \arcsin \sqrt{\frac{\delta}{N}} \right\}, \quad B_0 = \left(\cup_{i=1}^N B_i \right)^c.$$

Thus $B_0 = B_0(\delta)$ *is the complement of the union of the (Riemannian) balls of radius* $\arcsin \sqrt{\delta/N}$ *centered at the* x_i*. Then*

$$\mathcal{E}(X) = C_1(N, \delta) + \frac{N}{2} \fint_{x \in \cup B_j} \sum_{i=1}^N \log \|x - x_i\|^{-1} \, dx \tag{1.12}$$

and

$$\mathcal{E}(X) = C_2(N, \delta) - \frac{1-\delta}{2\delta} N \fint_{x \in B_0} \sum_{i=1}^N \log \|x - x_i\|^{-1} \, dx \tag{1.13}$$

where

$$C_1(N, \delta) = -\frac{N^2}{4} + \frac{N}{4} \log \frac{\delta}{N} - \frac{N(N-1)(N-\delta)}{4\delta} \log \left(1 - \frac{\delta}{N} \right)$$

and

$$C_2(N, \delta) = \frac{N^2}{4\delta} + C_1(N, \delta) = \frac{1-\delta}{4\delta} N^2 + \frac{N}{4} \log \frac{\delta}{N} - \frac{N(N-1)(N-\delta)}{4\delta} \log \left(1 - \frac{\delta}{N} \right).$$

Note that the integration in (1.12) is over the union of all the balls B_j, while in (1.13) it is over the complement of that union. It should be kept in mind that $C_1(N, \delta) \leq 0$, see Lemma 1.16 below. The following result follows directly from Theorem 1.11.

Corollary 1.12 (Facility location formulation for elliptic Fekete points)**.** *The following problems are equivalent:*

- *The* N*–tuple* $X = (x_1, \dots, x_N)$ *is a set of elliptic Fekete points.*
- *For* any $\delta \in (0,1)$ *and* $r = \arcsin \sqrt{\delta/N}$ *such that* $d_R(x_i, x_j) \geq 2r$ *for* $i \neq j$*, the points* $x_1, \dots, x_N$ *solve Problem 1.9.*
- *For* some $\delta \in (0,1)$ *and* $r = \arcsin \sqrt{\delta/N}$ *such that* $d_R(x_i, x_j) \geq 2r$ *for* $i \neq j$*, the points* $x_1, \dots, x_N$ *solve Problem 1.9.*

A trivial remark following from Corollary 1.12 and Theorem 1.3 is:

Corollary 1.13. *Let* $x_1, \dots, x_N$ *solve Problem 1.9. Then the associated polynomial* f *is stable in the sense that the condition number of* f *at all its roots is at most* $\sqrt{N(N+1)}$.

We have thus stated a facility location formulation for stable polynomials and elliptic Fekete points, as the title of this article claimed.

Remark 1.14. *From* [10, Theorem 2], *if X is an N–tuple of elliptic Fekete points, then $\|x_i - x_j\| \geq 1/\sqrt{N}$ for $i \neq j$. Thus, from the convexity of* arcsin,

$$d_R(x_i, x_j) = \arcsin \|x_i - x_j\| \geq \arcsin \frac{1}{\sqrt{N}} \geq 2 \arcsin \frac{1}{2\sqrt{N}} \quad i \neq j,$$

and we conclude that any $\delta \leq 1/4$ satisfies the hypotheses of Theorem 1.11. Moreover, it is known from [14] *that there exist points on the unit sphere which are*

$$\sqrt{\frac{8\pi}{\sqrt{3}N}} + O(N^{-2/3}) = \sqrt{\frac{8\pi}{\sqrt{3}N}} \left(1 + O\left(N^{-1/6}\right)\right)$$

apart (affine distance), and such a bound is asymptotically optimal. For the Riemann sphere we must divide this quantity by 2. We thus have that δ_{max} – the greatest δ for which there exist points satisfying the hypotheses of Theorem 1.11 – asymptotically approaches the greatest δ such that

$$2 \arcsin \sqrt{\frac{\delta}{N}} \leq \arcsin \left(\sqrt{\frac{2\pi}{\sqrt{3}N}} \left(1 + O\left(N^{-1/6}\right)\right) \right).$$

That is,

$$\delta_{max} = \frac{\pi}{2\sqrt{3}} \left(1 + O\left(N^{-1/6}\right)\right) \approx 0.90689\ldots \quad (N >> 1). \tag{1.14}$$

Note also that (1.12) *and* (1.13) *hold for* every δ *satisfying the conditions of the theorem. In particular, the right hand side of either relation is invariant under change of δ.*

Remark 1.15. *A simplified form of C_1 and C_2 can be obtained when noting the expansion*

$$-\log\left(1 - \frac{\delta}{N}\right) = \sum_{k=1}^{\infty} \frac{\delta^k}{kN^k} = \frac{\delta}{N} + \frac{\delta^2}{2N^2} + \frac{\delta^3}{3N^3} + \cdots,$$

which implies that for all $\delta \in (0,1)$ and $N \geq 2$,

$$\frac{\delta}{N} + \frac{\delta^2}{2N^2} + \frac{\delta^3}{3N^3} \leq -\log\left(1 - \frac{\delta}{N}\right) \leq \frac{\delta}{N} + \frac{\delta^2}{2N^2} + \frac{\delta^3}{3N^3} + \frac{\delta^4}{2N^4}.$$

(See Section 2.3 for a proof of the upper bound). With this, we have

$$\left| C_1(N,\delta) - \left(-\frac{N \log N}{4} - \frac{2 + \delta - 2\log\delta}{8} N + \frac{\delta(1 - \delta/3)}{8} \right) \right| \leq \frac{1}{2N}; \tag{1.15}$$

that is,

$$C_1(N,\delta) = -\frac{N \log N}{4} - \frac{2 + \delta - 2\log\delta}{8} N + O(1). \tag{1.16}$$

Similarly,

$$C_2(N,\delta) = \frac{N^2}{4\delta} - \frac{N \log N}{4} - \frac{2 + \delta - 2\log\delta}{8} N + O(1).$$

With some arithmetic one can also get the following bounds (see Section 2.3 for a proof).

Lemma 1.16. *For $\delta \in (0,1)$ and $N \geq 2$, we have*

$$C_1(N,\delta) \leq -\frac{N \log N}{4} - \frac{2 + \delta - 2\log\delta}{8} N + \frac{\delta}{8} \leq -\frac{N \log N}{4} - \frac{3}{8} N + \frac{1}{8} \leq -\frac{N \log N}{4} \leq 0.$$

1.3. **Some consequences.** Theorem 1.11 yields several corollaries about the classical problem of elliptic Fekete points, the classical Tammes problem (i.e. find $x_1, \ldots, x_N \in \mathbb{S}$ which maximise the minimum of their mutual distances) and the problem of stable polynomials. We list them together here and leave the proofs for Section 5.

The first one is an upper bound for the value of the logarithmic energy of N spherical points with the unique assumption that they are not too close to each other.

Corollary 1.17. *Let $\delta \in (0,1)$ and let $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ be such that $d_R(x_i, x_j) \geq 2\arcsin\sqrt{\delta/N}$ for $i \neq j$. Then*

$$\mathcal{E}(X) \leq \frac{1-\log\delta}{4}N^2 + C_1(N,\delta) \leq \frac{1-\log\delta}{4}N^2.$$

If additionally N is even and X is symmetric w.r.t. the origin, then

$$\mathcal{E}(X) \leq \frac{1}{4}\left(1 - \frac{1}{2}\log\frac{\delta}{2} + \frac{2-\delta}{2\delta}\log\left(1-\frac{\delta}{2}\right)\right)N^2 + C_1(N,\delta).$$

We have pointed out that since [16], one cannot expect a precise relationship between the solutions to different spherical point problems. However, from (1.14) and Corollary 1.17 a solution to the Tammes problem cannot have too large a logarithmic energy:

Corollary 1.18. *Let $(X_N)_{N\geq 2}$, $X_N \in \mathbb{S}^N$, be a sequence of solutions to the Tammes problem; so, for $N \geq 2$ and for $1 \leq i \neq j \leq N$ we have*

$$d_R((X_N)_i, (X_N)_j) \geq 2\arcsin\sqrt{\frac{\delta_{max}}{N}},$$

where $\delta_{max} = \pi/(2\sqrt{3}) + O(N^{-1/6})$. Then

$$\mathcal{E}(X_N) \leq \frac{1-\log\delta_{max}}{4}N^2 = 0.27443\ldots\, N^2 + o(N^2).$$

If additionally X_N is symmetric w.r.t. the center of the sphere for all even N, then for all such N we have

$$\mathcal{E}(X_N) \leq \frac{1}{4}\left(1 - \frac{1}{2}\log\frac{\delta_{max}}{2} + \frac{2-\delta_{max}}{2\delta_{max}}\log\left(1-\frac{\delta_{max}}{2}\right)\right)N^2 = 0.25783\ldots\, N^2 + o(N^2).$$

J. P. Dedieu proved in [9, Lemma 2.3] that if $\mu(h,\zeta) \leq c$ for every projective zero ζ of a homogeneous polynomial h of degree N, then for every pair (ζ, ζ') of two different projective zeros of h, we have:

$$d_R(\zeta, \zeta') \geq \arcsin\frac{1}{N^{3/2}c}. \tag{1.17}$$

That is, if the condition number is small then the zeros cannot be too close to each other. Our following result is a reciprocal bound: it bounds the value of the condition number in terms of the minimal distance between the zeros of h.

Corollary 1.19. *Let $\delta \in (0,1)$. Let h be a degree $N \geq 2$ homogeneous polynomial with projective zeros $\zeta_1, \ldots, \zeta_N$ where $d_R(\zeta_i, \zeta_j) \geq 2\arcsin\sqrt{\delta/N}$ for all $i \neq j$. Then*

$$\mu(h, \zeta_i) \leq \frac{\sqrt{N(N+1)}e^{N/4}}{\delta^{N^2/4}}, \quad \forall i, 1 \leq i \leq N.$$

The following result contains a (possibly non-sharp, yet seemingly non-trivial) reciprocal of Theorem 1.3.

Corollary 1.20. *For any degree N homogeneous polynomial h such that* $\max \mu(h,\zeta) \leq c$ *for every zero ζ of h, the associated spherical points $x_1,\ldots,x_N \in \mathbb{S}$ satisfy*

$$\mathcal{E}(x_1,\ldots,x_N) \leq \frac{1+\log(4N^2c^2)}{4}N^2 = \frac{1}{2}N^2\log N + \frac{1}{2}N^2\log c + O(N^2).$$

Note that from (1.17) in the hypotheses of Corollary 1.20 we have

$$\mathcal{E}(x_1,\ldots,x_N) \leq \sum_{i<j}\log(N^{3/2}c) \leq \frac{3}{4}N^2\log N + \frac{1}{2}N^2\log c + O(N^2),$$

while Corollary 1.20 proves that 3/4 in the formula above can be relaxed to 1/2.

In many practical optimisation problems, a useful numerical strategy is *alternating optimisation*. This method first fixes all the variables but one, and then optimises just in that variable (presumably, an easier numerical problem), and repeats the process with the other variables in turn until some equilibrium is reached. The resulting output is sometimes called a coordinate-wise minimum but I prefer the term Nash equilibrium by analogy with the concept of the same name in game theory. Of course, such a point need not be a global or even a local optimum: the point $(0,0)$ is a Nash equilibrium of the problem of minimising x^2+y^2-3xy but it is just a saddle point of the function. In our context, we say that $X=(x_1,\ldots,x_N)$ is a Nash equilibrium of $\mathcal{E}$ if for every $j\in\{1,\ldots,N\}$, we have

$$\mathcal{E}(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_N) \leq \mathcal{E}(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_N), \quad \forall\, x\in\mathbb{S}.$$

Our next result shows that if X has this property, then the value of $\mathcal{E}(X)$ cannot be arbitrarily large. More precisely:

Corollary 1.21. *Let $N\geq 2$ and let $X=(x_1,\ldots,x_N)$ be a Nash equilibrium of $\mathcal{E}$. Then for every $\delta\in(0,1)$ satisfying* (1.11)*, we have*

$$\mathcal{E}(X) \leq \delta C_2(N,\delta) = \frac{N^2}{4} + \delta C_1(N,\delta) \underset{\textit{Lemma 1.16}}{\leq} \frac{N^2}{4} - \delta\frac{N\log N}{4}.$$

In particular we have $\mathcal{E}(X)\leq N^2/4$.

1.4. **Spherical cap discrepancy and logarithmic energy.** A useful quantity to measure how well distributed a configuration $X=(x_1,\ldots,x_N)$ of spherical points is, is the *spherical cap discrepancy* (see for example [7] and references therein):

$$D_C(X) = \sup_{x\in\mathbb{S}, r\in[0,\pi/2]}\left|\frac{\sharp(i\colon x_i\in B(x,r))}{N} - \frac{Area(B(x,r))}{\pi}\right|;$$

that is, $D_C(X)$ is the supremum (extended over all spherical caps) of the difference between the counting measure and the (normalized) area of the cap.

The spherical cap discrepancy is an important topic and it has received much attention. Bounds for the least possible value of $D_C(X)$ were found by Beck [4]:

$$cN^{-3/4} \leq \min_{X\in\mathbb{S}^N} D_C(X) \leq CN^{-3/4}\sqrt{\log N}, \tag{1.18}$$

for some positive constants c, C. In [1] an explicit construction of N–tuples X_N satisfying $D_C(X_N)\leq 44\sqrt{2}N^{-1/2}$ was given. In [7] it was proved that if $(X_N)_{N\geq 2}$ is such that $X_N\in\mathbb{S}^N$ is a set of elliptic Fekete points for all $N\geq 2$, then

$$D_C(X_N) \leq O(N^{-1/4}).$$

Moreover, it follows from [10, Theorem 2] that if $X \in \mathbb{S}^N$ is a set of elliptic Fekete points, then we have $d_R(x_i, x_j) \geq N^{-1/2}$. That is, points with the lowest possible logarithmic energy have a small cap discrepancy and are well separated.

Our next results investigate the reciprocal: we bound the value of $\mathcal{E}(X)$ in terms of $D_C(X)$ and the separation distance, and state that points which are well separated and have a small discrepancy also have a small value of $\mathcal{E}$. See [17] for a similar result for Riesz s–energy instead of the logarithmic energy, and [29, Theorem 4] for a similar bound in the case that instead of $\mathbb{S}$ the points are assumed to lie on the unit circle in $\mathbb{R}^2$.

Theorem 1.22. *Let $X \in \mathbb{S}^N$, $N \geq 2$ and let $\delta \in (0,1)$ be such that $d_R(x_i, x_j) \geq 2 \arcsin \sqrt{\delta/N}$ for $i \neq j$. Then,*

$$\mathcal{E}(X) \leq m_N + \frac{N^2}{4} D_C(X) \log \frac{N}{2\delta} + \frac{N \log(8\pi\delta)}{4}.$$

A direct consequence of Theorem 1.22 and (1.3) is now stated.

Corollary 1.23. *Let $(X_N)_{N\geq 2}$ be a sequence of N–tuples of points on $\mathbb{S}$, and assume that:*

- *X_N is* polynomially separated *in the sense that there exist $c, M > 0$ such that for every $N \geq 2$ and for every $1 \leq i < j \leq N$,*
$$d_R((X_N)_i, (X_N)_j) \geq \frac{c}{N^M}.$$
- *The spherical cap discrepancy of X_N tends to 0 as $N \to \infty$ faster than $\log(N)^{-1}$, that is*
$$\lim_{N\to\infty} D_C(X_N) \log N = 0.$$

Then, $\mathcal{E}(X)$ is, in relative error, asymptotically minimal; that is:

$$\lim_{N\to\infty} \frac{\mathcal{E}(X_N)}{m_N} = 1.$$

It is easy to see that if the separation property is removed from the hypotheses of Corollary 1.23 the claim may fail. Here is a more interesting question:

Problem 1.24. *Can $\lim_{N\to\infty} D_C(X_N) \log N = 0$ be relaxed to $\lim_{N\to\infty} D_C(X_N) = 0$ in Corollary 1.23?*

We also prove the following alternative result:

Proposition 1.25. *Let $(X_N)_{N\geq 2}$ be a sequence of N–tuples of points on $\mathbb{S}$, and assume that there exists a positive constant $c > 0$ such that*

$$\lim_{N\to\infty} \frac{D_C(X_N)^c}{\delta_N} = 0, \tag{1.19}$$

for $\delta_N \in (0,1)$ satisfying $d_R((X_N)_i, (X_N)_j) \geq 2 \arcsin \sqrt{\delta_N/N}$, $i \neq j$. Then,

$$\lim_{N\to\infty} \frac{\mathcal{E}(X_N)}{m_N} = 1.$$

Remark 1.26. *Note that* (1.19) *can also be interpreted as*

$$\lim_{N\to\infty} \frac{D_C(X_N)^c}{N \mathrm{sep}(X_N)^2} = 0,$$

where $\mathrm{sep}(X_N)$ is the separation distance of X_N, that is the minimum distance between two different points of X_N. Compare this to the hypotheses of an "admissible and well–separated" *sequence in* [17] *(with $d = 2$ in the notations of that paper): our hypotheses is less restrictive for if a sequence is well separated then δ_N is bounded above and below for $N \geq 2$.*

Proposition 1.25 readily applies to many well–known collections of points including all those cited in [17] (other interesting references treating the topic of separation and discrepancy of different sequences are [1],[13]). For example, it is known that points minimizing the Coulomb potential energy (i.e. points solving the famous and classical Thomson problem [27]) have discrepancy tending to 0 as $N \to \infty$ and are well separated with a lower bound $cN^{-1/2}$ for some constant $c > 0$. The same qualitative bounds are also valid for N–tuples minimising the Riesz energy for $0 < s < 2$. We thus have:

Corollary 1.27. *If X_N solves Thomson's problem in $\mathbb{S}$ for $N \geq 2$, then $\lim_{N\to\infty} \mathcal{E}(X_N)/m_N = 1$. More generally, fix $s \in (0, 2)$. If X_N minimizes the Riesz s–energy*

$$\sum_{1\leq i<j\leq N} \|(X_N)_i - (X_N)_j\|^{-s}$$

for $N \geq 2$, then $\lim_{N\to\infty} \mathcal{E}(X_N)/m_N = 1$.

Note that Corollary 1.27 is a reciprocal (in the Riemann sphere) to the claim in section "Minimum logarithmic energy" in [17], where points minimizing $\mathcal{E}$ are proved to have an asymptotically minimal Riesz energy. From [18, Theorem 3.3] and [20, p. 199] one can conclude a result analogous to Corollary 1.27 for sets of (non–elliptic) Fekete points. Namely, following the notation in [18], let $L \geq 2$ and let π_L be the dimension of the vector space of spherical harmonics of degree at most L, and let $Q_1^L, \ldots, Q_{\pi_L}^L$ be a basis of that vector space. A collection $X = (x_1, \ldots, x_{\pi_L})$ of points in the unit sphere is called a set of Fekete points (a.k.a. extremal fundamental systems) if it maximises the determinant $|\det(Q_i^L(x_j))_{i,j}|$. These tuples are known to have good properties for cubature and interpolation formulas in the unit sphere, see [25]. Then, we have:

Corollary 1.28. *For every $L \geq 2$, let $X_L = (x_1, \ldots, x_{\pi_L})$ be a set of (non–elliptic) Fekete points in the unit sphere, and let $Y_L = (y_1, \ldots, y_{\pi_L})$ be the associated points in $\mathbb{S}$ (i.e. $x_i = 2y_i - (0, 0, 1)^T$, $1 \leq i \leq \pi_L$). Then*

$$\lim_{L\to\infty} \frac{\mathcal{E}(Y_L)}{m_{\pi_L}} = 1.$$

The proofs of Theorem 1.22 and Proposition 1.25 can be found in Section 6.

2. Proofs of the easy results

2.1. **Proof of Lemma 1.1.** Let $f(z) = \sum_{k=0}^N a_k z^k$, $a_N \neq 0$. The homogeneous counterpart of f is $h(z_0, z) = \sum_{k=0}^N a_k z_0^{N-k} z^k$. Recall that we have defined

$$\mu(f, z) = \mu(h, (1, z)) = N^{1/2} \|(Dh(1, z) \mid_{(1,z)^\perp})^{-1}\| \|h\| \|(1, z)\|^{N-1}$$

and $\|f\| = \|h\|$. Now,

$$Dh(1,z) \mid_{(1,z)^\perp} : \quad \begin{array}{ccc} (1,z)^\perp & \to & \mathbb{C} \\ \lambda(-\bar{z}, 1) & \mapsto & \lambda Dh(1,z)(-\bar{z}, 1). \end{array}$$

We first compute $Dh(1,z)(-\bar{z},1) \in \mathbb{C}$:

$$Dh(1,z)(-\bar{z},1) = \lim_{t\to 0} \frac{h(1-t\bar{z}, z+t) - h(1,z)}{t} = \lim_{t\to 0} \frac{(1-t\bar{z})^N f\left(\frac{z+t}{1-t\bar{z}}\right) - f(z)}{t} =$$

$$\frac{d}{dt}\bigg|_{t=0} \left((1-t\bar{z})^N f\left(\frac{z+t}{1-t\bar{z}}\right)\right) = f'(z)(1+|z|^2) - N\bar{z}f(z).$$

We thus have:

$$Dh(1,z)\mid_{(1,z)^\perp}(\lambda(-\bar{z},1)) = \lambda(f'(z)(1+|z|^2) - N\bar{z}f(z));$$

that is, for $t \in \mathbb{C}$,

$$Dh(1,z)\mid^{-1}_{(1,z)^\perp}(t) = \frac{(-\bar{z},1)}{f'(z)(1+|z|^2) - N\bar{z}f(z)}t$$

and

$$\left\| Dh(1,z)\mid^{-1}_{(1,z)^\perp} \right\| = \left\| \frac{(-\bar{z},1)}{f'(z)(1+|z|^2) - N\bar{z}f(z)} \right\| = \frac{(1+|z|^2)^{1/2}}{|f'(z)(1+|z|^2) - N\bar{z}f(z)|}.$$

We have proved that

$$\mu(f,z) = N^{1/2} \frac{\|f\|\|(1,z)\|^{N-1}(1+|z|^2)^{1/2}}{|f'(z)(1+|z|^2) - N\bar{z}f(z)|}.$$

The lemma follows.

2.2. **Proof of Lemma 1.6.** From Lemma 1.1 we have

$$\frac{1}{2}\sum_{i=1}^N \log\mu(f,z_i) - \frac{N\log N}{4} = \frac{1}{2}\sum_{i=1}^N \log\left(N^{1/2}\frac{\|f\|(1+|z_i|^2)^{\frac{N}{2}-1}}{|f'(z_i)|}\right) - \frac{N\log N}{4} =$$

$$\frac{1}{2}\sum_{i=1}^N \log\frac{\|f\|(1+|z_i|^2)^{\frac{N}{2}-1}}{|f'(z_i)|} = \frac{N\log\|f\|}{2} + \frac{1}{2}\sum_{i=1}^N \log\frac{(1+|z_i|^2)^{\frac{N}{2}-1}}{|f'(z_i)|}.$$

To prove the lemma, we then need:

$$\mathcal{E}(x_1,\ldots,x_N) = \frac{1}{2}\sum_{i=1}^N \log\frac{(1+|z_i|^2)^{\frac{N}{2}-1}}{|f'(z_i)|} + \frac{N}{2}\log\prod_{i=1}^N \sqrt{1+|z_i|^2},$$

or equivalently

$$\mathcal{E}(x_1,\ldots,x_N) = \frac{1}{2}\sum_{i=1}^N \log\frac{(1+|z_i|^2)^{N-1}}{|f'(z_i)|}. \tag{2.1}$$

Indeed,

$$\mathcal{E}(x_1,\ldots,x_N) = \sum_{i<j}\log\|x_i - x_j\|^{-1} \underset{(1.5)}{=} \sum_{i<j}\log\frac{\sqrt{(1+|z_i|^2)(1+|z_j|^2)}}{|z_i - z_j|},$$

and reordering terms we get

$$\mathcal{E}(x_1,\ldots,x_N) = \frac{N-1}{2}\sum_{i=1}^N \log(1+|z_i|^2) + \frac{1}{2}\sum_{i=1}^N \log\prod_{j\neq i}|z_i - z_j|^{-1}.$$

Now relation (2.1) follows from this last formula, using that

$$f'(z_i) = \frac{d}{dz}\Big|_{z=z_i} \prod_{j=1}^{N}(z-z_j) = \prod_{j\neq i}(z_i - z_j).$$

Lemma 1.6 is now proved.

2.3. **Proof of Lemma 1.16.** We first prove that for $\delta\in(0,1)$ and $N\geq 2$ we have

$$-\log\left(1-\frac{\delta}{N}\right) \leq \frac{\delta}{N}+\frac{\delta^2}{2N^2}+\frac{\delta^3}{3N^3}+\frac{\delta^4}{2N^4}.$$

Indeed, the proof follows directly from the fact that

$$-\log\left(1-\frac{\delta}{N}\right) = \sum_{k=1}^{\infty}\frac{\delta^k}{kN^k} = \frac{\delta}{N}+\frac{\delta^2}{2N^2}+\frac{\delta^3}{3N^3}+\frac{\delta^4}{N^4}\sum_{k=0}^{\infty}\frac{\delta^k}{(k+4)N^k} \leq \frac{\delta}{N}+\frac{\delta^2}{2N^2}+\frac{\delta^3}{3N^3}+\frac{\delta^4}{4N^4}\sum_{k=0}^{\infty}\frac{1}{2^k}.$$

Now we prove Lemma 1.16. From the inequality above and the definition of $C_1(N,\delta)$ in Theorem 1.11, we have

$$C_1(N,\delta) \leq -\frac{N^2}{4}+\frac{N}{4}\log\frac{\delta}{N}+\frac{N(N-1)(N-\delta)}{4\delta}\left(\frac{\delta}{N}+\frac{\delta^2}{2N^2}+\frac{\delta^3}{3N^3}+\frac{\delta^4}{2N^4}\right) =$$

$$\frac{N}{4}\log\frac{\delta}{N}-\frac{2+\delta}{8}N+\left(\frac{\delta^3}{24N}+\frac{\delta^2}{24N}-\frac{\delta^2}{24}\right)+\left(\frac{\delta^4}{8N^3}-\frac{\delta^4}{8N^2}\right)+\frac{\delta}{8}-\frac{\delta^3}{24N^2},$$

where the terms have been rearranged. Note that the expressions inside the parentheses in the last formula are bounded above by 0 (use the fact $N\geq 2$). The lemma follows.

3. Proof of Proposition 1.7

We first prove that a function u satisfying (1.8) and (1.9) is unique up to an additive constant. Indeed, let u, v be two solutions. Then the function $w = u - v$ is harmonic (i.e., $\Delta_{\mathbb{S}} w = 0$) in $\mathbb{S}\setminus\{x_1,\ldots,x_N\}$. Moreover, for every $i, 1\leq i\leq N$, if we let

$$A_i(x) = \frac{\lambda}{2N}\frac{\pi_{T_x\mathbb{S}}(x_i - x)}{\|x_i - x\|^{1+\epsilon}},$$

we have:

$$\limsup_{\|x_i-x\|\to 0}\left\|\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}}w(x)\right\| =$$

$$\limsup_{\|x_i-x\|\to 0}\left\|\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}}u(x)-A_i(x)-\left(\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}}v(x)-A_i(x)\right)\right\| \leq$$

$$\limsup_{\|x_i-x\|\to 0}\left\|\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}}u(x)-A_i(x)\right\| + \limsup_{\|x_i-x\|\to 0}\left\|\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}}v(x)-A_i(x)\right\| < \infty.$$

In particular, there exists $C\in\mathbb{R}$ such that for sufficiently small $\|x_i - x\|$ (say, $\|x_i-x\|\leq r_i$) we have:

$$\|\nabla_{\mathbb{S}}w(x)\| \leq \frac{C}{\|x_i-x\|^{1-\epsilon}} \underset{(1.2)}{=} \frac{C}{(\sin d_R(x,x_i))^{1-\epsilon}} \leq \frac{2C}{d_R(x,x_i)^{1-\epsilon}}.$$

Let $\gamma\colon [-c,0)$, $c>0$, be an arclength parametrized geodesic contained in the set $\{x\colon \|x_i-x\|\leq r_i\}$ such that $\gamma(t)\to x_i$ as $t\to 0$. Thus $d_R(\gamma(t),x_i)=|t|$. Applying the Fundamental Theorem of Vector Calculus then yields

$$|w(\gamma(t))| = \left| w(\gamma(-c)) + \int_{-c}^{t} \nabla_{\mathbb{S}} w(\gamma(s))\cdot\dot\gamma(s)\,ds \right| \leq |w(\gamma(-c))| + \int_{-c}^{t} \|\nabla_{\mathbb{S}} w(\gamma(s))\|\,ds \leq$$

$$|w(\gamma(-c))| + 2C\int_{-c}^{t} \frac{1}{d_R(\gamma(s),x_i)^{1-\epsilon}}\,ds = |w(\gamma(-c))| + 2C\int_{-c}^{t}\frac{1}{|s|^{1-\epsilon}}\,ds \leq |w(\gamma(-c))| + \frac{2Cc^{\epsilon}}{\epsilon}.$$

It follows that the $L^\infty(\mathbb{S}\setminus\{x_1,\ldots,x_N\})$ norm of w is bounded:

$$\|w\|_\infty \leq \sup_{x\colon \|x_i-x\|\geq r_i\ \forall i} |w(x)| + \frac{2Cc^{\epsilon}}{\epsilon} < \infty.$$

We thus have a harmonic function which is bounded in $\mathbb{S}\setminus\{x_1,\ldots,x_N\}$. By the maximum principle, this implies w is constant and we have proved that u and v differ by a constant as desired. It remains to prove that the function

$$u(x) = \frac{\lambda}{2N}\sum_{i=1}^{N}\log\|x_i-x\|^{-1}$$

actually satisfies (1.8) and (1.9). Recall from [5, Lemma 2.2] that the function $x\mapsto \log\|x_i-x\|^{-1}$ has, for every i, constant Laplacian equal to 2. Thus

$$\Delta_{\mathbb{S}} u = \frac{\lambda}{2N}\sum_{i=1}^{N} 2 = \lambda,$$

and therefore u satisfies (1.8). As for (1.9), note that (extending u to $\mathbb{R}^3\setminus\{x_1,\ldots,x_N\}$ by the same formula),

$$\nabla_{\mathbb{S}} u(x) = \pi_{T_x\mathbb{S}}\left(\nabla_{\mathbb{R}^3} u\right) = \frac{\lambda}{2N}\sum_{j=1}^{N}\frac{\pi_{T_x\mathbb{S}}(x_j-x)}{\|x_j-x\|^2} = \sum_{j=1}^{N}\frac{A_j(x)}{\|x_j-x\|^{1-\epsilon}}.$$

In particular, for every i, $1\leq i\leq N$ and $\epsilon\in(0,1)$, we have:

$$\lim_{\|x_i-x\|\to 0}\left(\|x_i-x\|^{1-\epsilon}\nabla_{\mathbb{S}} u(x) - A_i(x)\right) = \lim_{\|x_i-x\|\to 0}\sum_{j\neq i}\frac{A_j(x)\|x_i-x\|^{1-\epsilon}}{\|x_j-x\|^{1-\epsilon}} = 0,$$

so (1.9) is trivially satisfied.

4. Proof of Theorem 1.11

For any $x_0\in\mathbb{S}$ and $\epsilon\in[0,\pi/2]$, let

$$B(x_0,\epsilon) = \{x\in\mathbb{S}\colon d_R(x,x_0)\leq\epsilon\}.$$

Recall the following result:

Proposition 4.1. [5, see Proposition 3.2] *Let $x_0, y\in\mathbb{S}$ and let $0<\epsilon\leq d_R(x_0,y)<\pi/2$. Then,*

$$Area(B(x_0,\epsilon)) = \pi\sin^2\epsilon, \tag{4.1}$$

$$\fint_{x\in B(x_0,\epsilon)}\log\|x-y\|^{-1}\,dx = \log\|x_0-y\|^{-1} + \frac{1}{2} + \frac{\log\cos\epsilon}{\tan^2\epsilon}. \tag{4.2}$$

Moreover,

$$\fint_{x\in\mathbb{S}} \log\|x-y\|^{-1}\,dx = \frac{1}{2}. \tag{4.3}$$

An easy consequence of this proposition is:

Lemma 4.2. *Let $x_0\in\mathbb{S}$ and let $\epsilon\in(0,\pi/2)$. Then*

$$\fint_{x\in B(x_0,\epsilon)} \log\|x-x_0\|^{-1}\,dx = \frac{1}{2}-\frac{1}{2}\log\sin^2\epsilon.$$

Proof. Let $\hat{x}_0$ be the point opposite from x_0 in $\mathbb{S}$. Note that

$$\int_{x\notin B(x_0,\epsilon)} \log\|x-x_0\|^{-1}\,dx = \int_{x\in B(\hat{x}_0,\pi/2-\epsilon)} \log\|x-x_0\|^{-1}\,dx \underset{(4.1)}{=}$$
$$\pi\cos^2\epsilon \fint_{x\in B(\hat{x}_0,\pi/2-\epsilon)} \log\|x-x_0\|^{-1}\,dx.$$

By (4.2), this equals

$$\pi\cos^2\epsilon\left(\log 1+\frac{1}{2}+\frac{\log\cos(\pi/2-\epsilon)}{\tan^2(\pi/2-\epsilon)}\right) = \pi\cos^2\epsilon\left(\frac{1}{2}+\tan^2\epsilon\,\log\sin\epsilon\right).$$

Hence,

$$\int_{x\in B(x_0,\epsilon)} \log\|x-x_0\|^{-1}\,dx \underset{(4.3)}{=} \frac{\pi}{2}-\pi\cos^2\epsilon\left(\frac{1}{2}+\tan^2\epsilon\,\log\sin\epsilon\right) = \frac{\pi}{2}\left(\sin^2\epsilon-\sin^2\epsilon\log\sin^2\epsilon\right).$$

The lemma follows from (4.1). □

We now prove Theorem 1.11. First note that

$$\fint_{x\in\cup B_j} \sum_{i=1}^N \log\|x-x_i\|^{-1}\,dx \underset{(4.1)}{=} \frac{1}{\pi\delta}\sum_{i=1}^N\left(\int_{x\in B_i}\log\|x-x_i\|^{-1}\,dx+\sum_{j\neq i}\int_{x\in B_j}\log\|x-x_i\|^{-1}\,dx\right) =$$
$$\frac{1}{N}\sum_{i=1}^N\left(\fint_{x\in B_i}\log\|x-x_i\|^{-1}\,dx+\sum_{j\neq i}\fint_{x\in B_j}\log\|x-x_i\|^{-1}\,dx\right),$$

which, from Proposition 4.1 and Lemma 4.2, equals

$$\frac{1}{N}\sum_{i=1}^N\left(\frac{1}{2}-\frac{1}{2}\log\frac{\delta}{N}+\sum_{j\neq i}\left(\log\|x_i-x_j\|^{-1}+\frac{1}{2}+\frac{N-\delta}{2\delta}\log\left(1-\frac{\delta}{N}\right)\right)\right) =$$
$$\frac{1}{2}-\frac{1}{2}\log\frac{\delta}{N}+\frac{2\mathcal{E}(X)}{N}+(N-1)\left(\frac{1}{2}+\frac{N-\delta}{2\delta}\log\left(1-\frac{\delta}{N}\right)\right) =$$
$$\frac{N}{2}-\frac{1}{2}\log\frac{\delta}{N}+\frac{(N-1)(N-\delta)}{2\delta}\log\left(1-\frac{\delta}{N}\right)+\frac{2\mathcal{E}(X)}{N}.$$

Solving for $\mathcal{E}(X)$, we get (1.12). Now, (1.13) follows from (1.12) using that

$$\fint_{x\in\cup B_j}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx=\frac{1}{\pi\delta}\int_{x\in\cup B_j}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx=$$

$$\frac{1}{\pi\delta}\left(\int_{x\in\mathbb{S}}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx-\int_{x\in B_0}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx\right)\underset{(4.3)}{=}$$

$$\frac{1}{\pi\delta}\left(\frac{\pi N}{2}-\pi(1-\delta)\fint_{x\in B_0}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx\right)=\frac{N}{2\delta}-\frac{1-\delta}{\delta}\fint_{x\in B_0}\sum_{i=1}^{N}\log\|x-x_i\|^{-1}\,dx.$$

5. Proofs of the corollaries of Theorem 1.11

5.1. **Proof of Corollary 1.17.** Let δ and X be as in Corollary 1.17 and note that for every i, $1\le i\le N$, the function

$$x\mapsto\log\|x-x_i\|^{-1}$$

decreases as x moves further from x_i. Thus

$$\int_{x\in\cup B_j}\log\|x-x_i\|^{-1}\,dx\le\int_{x\in\mathbb{S}\colon\, d_R(x,x_i)\le r}\log\|x-x_i\|^{-1}\,dx,$$

where r is chosen in such a way that

$$\mathit{Area}\{x\in\mathbb{S}\colon d_R(x,x_i)\le r\}=\mathit{Area}\,(\cup B_j)=\pi\delta;$$

that is, $\pi\sin^2 r=\pi\delta$ or $\sin^2 r=\delta$. Then

$$\fint_{x\in\cup B_j}\log\|x-x_i\|^{-1}\,dx\le\fint_{x\in\mathbb{S}\colon\, d_R(x,x_i)\le r}\log\|x-x_i\|^{-1}\,dx\underset{\text{Lemma 4.2}}{=}\frac{1}{2}\left(1-\log\sin^2 r\right).$$

From (1.12) we thus have

$$\mathcal{E}(X)\le C_1(N,\delta)+\frac{1-\log\delta}{4}N^2.$$

The first claim of Corollary 1.17 follows from the fact that $C_1(N,\delta)\le 0$; see Lemma 1.16.

For the second claim, we first prove that for any fixed $x_i\in\mathbb{S}$ with opposite point $\hat{x}_i\in\mathbb{S}$, the function

$$\alpha(x)=\log\|x-x_i\|^{-1}+\log\|x-\hat{x}_i\|^{-1}$$

is a decreasing function of $d_R(x,x_i)$ while $0<d_R(x,x_i)<\pi/4$ and an increasing function of $d_R(x,x_i)$ while $\pi/4<d_R(x,x_i)<\pi/2$. Indeed, we can assume that $x_i=(0,0,1)$, $\hat{x}_i=(0,0,0)$, and from (1.5) we then have

$$\alpha(x)=\log\frac{\sqrt{1+|z|^2}}{|z|}+\log\sqrt{1+|z|^2}=\log(1+|z|^2)-\log|z|,$$

where $z\in\mathbb{C}$ is the point associated with x through (1.4). As a function of $|z|$, it is an exercise to check that $\alpha(x)$ is decreasing if $|z|<1$ (which corresponds to $d_R(x,(0,0,1))<\pi/4$) and increasing if $|z|>1$ (which corresponds to $d_R(x,(0,0,1))>\pi/4$).

We thus conclude that for every i, $1\le i\le N$,

$$\fint_{x\in\cup B_j}(\log\|x-x_i\|^{-1}+\log\|x-\hat{x}_i\|^{-1})\,dx\le\fint_{x\in B(x_i,\epsilon)\cup B(\hat{x}_i,\epsilon)}(\log\|x-x_i\|^{-1}+\log\|x-\hat{x}_i\|^{-1})\,dx=$$

$$2\fint_{x\in B(x_i,\epsilon)\cup B(\hat{x}_i,\epsilon)} \log\|x-x_i\|^{-1}\,dx.$$

where ϵ is such that $Area(B(x_i,\epsilon)\cup B(\hat{x}_i,\epsilon)) = Area(\cup B_j) = \pi\delta$; that is, $\sin^2\epsilon = \delta/2$. Now, for such an ϵ, from Lemma 4.2 and Proposition 4.1 respectively we have:

$$\fint_{x\in B(x_i,\epsilon)} \log\|x-x_i\|^{-1}\,dx = \frac{1}{2} - \frac{1}{2}\log\sin^2\epsilon,$$

$$\fint_{x\in B(\hat{x}_i,\epsilon)} \log\|x-x_i\|^{-1}\,dx = \frac{1}{2} + \frac{1}{2}\cot^2\epsilon\log\cos^2\epsilon.$$

We have thus proved:

$$\fint_{x\in\cup B_j} (\log\|x-x_i\|^{-1} + \log\|x-\hat{x}_i\|^{-1})\,dx \le \frac{1}{2} - \frac{1}{2}\log\sin^2\epsilon + \frac{1}{2} + \frac{1}{2}\cot^2\epsilon\log\cos^2\epsilon =$$

$$1 + \frac{1}{2}\left(-\log\frac{\delta}{2} + \frac{2-\delta}{\delta}\log\left(1-\frac{\delta}{2}\right)\right).$$

Hence,

$$\fint_{x\in\cup B_j} \sum_{i=1}^{N} \log\|x-x_i\|^{-1}\,dx \le \frac{N}{2} + \frac{N}{4}\left(-\log\frac{\delta}{2} + \frac{2-\delta}{\delta}\log\left(1-\frac{\delta}{2}\right)\right).$$

The second claim of Corollary 1.17 follows from this last inequality and (1.12).

5.2. **Proof of Corollary 1.19.** Let $x_1,\ldots,x_N$ be the points on $\mathbb{S}$ identified with $\zeta_1,\ldots,\zeta_N \in \mathbb{P}(\mathbb{C}^2)$. Note that $d_R(x_i,x_j)$ equals the Riemannian distance from ζ_i to ζ_j. From Corollary 1.17, we have

$$\mathcal{E}(x_1,\ldots,x_N) \le \frac{1-\log\delta}{4}N^2 + C_1(N,\delta),$$

which from Theorem 1.3 implies

$$\mu(f,\zeta_i) \le \sqrt{N(N+1)}e^{\frac{1-\log\delta}{4}N^2 + C_1(N,\delta) - m_N} \quad \forall\, i\in\{1,\ldots,N\}.$$

Now note that for $N\ge 2$,

$$\frac{N^2}{4} + C_1(N,\delta) - m_N \underset{(1.3)}{\le} \frac{N\log N}{4} + \frac{\log(2\pi)}{4}N + C_1(N,\delta) \underset{\text{Lemma 1.16}}{\le} \frac{2\log(2\pi)-3}{8}N + \frac{1}{8} \le \frac{N}{4}.$$

We thus conclude:

$$\mu(f,\zeta_i) \le \frac{\sqrt{N(N+1)}e^{N/4}}{\delta^{N^2/4}},$$

as wanted. Note that this same proof shows that $N/4$ may be replaced by cN with $c\approx(2\log(2\pi)-3)/8 < 1/4$.

5.3. **Proof of Corollary 1.20.** From (1.17), we have

$$d_R(x_i,x_j) \ge \arcsin\sqrt{\frac{\delta}{N}} \ge 2\arcsin\sqrt{\frac{\delta/4}{N}},$$

where $\delta = (N^2c^2)^{-1}$, and the second inequality comes from the fact that arcsin is a convex function in $[0,1]$, which implies $\arcsin(2t)\ge 2\arcsin(t)$ for $t\in[0,1/2]$. Then Corollary 1.17 implies

$$\mathcal{E}(x_1,\ldots,x_N) \le \frac{1-\log((4N^2c^2)^{-1})}{4}N^2 = \frac{1+\log(4N^2c^2)}{4}N^2,$$

as wanted.

5.4. **Proof of Corollary 1.21.** From the fact that X is a Nash equilibrium, we have that for every $x \in \mathbb{S} \setminus \{x_1, \ldots, x_N\}$ and for every j, $1 \le j \le N$,

$$\sum_{i=1}^{N} \log \|x - x_i\|^{-1} = \log \|x - x_j\|^{-1} + \sum_{i \ne j} \log \|x - x_i\|^{-1} \ge \log \|x - x_j\|^{-1} + \sum_{i \ne j} \log \|x_j - x_i\|^{-1}.$$

Averaging over j we get

$$\sum_{i=1}^{N} \log \|x - x_i\|^{-1} \ge \frac{1}{N} \sum_{j=1}^{N} \log \|x - x_j\|^{-1} + \frac{1}{N} \sum_{j=1}^{N} \sum_{i \ne j} \log \|x_j - x_i\|^{-1} = \frac{1}{N} \sum_{j=1}^{N} \log \|x - x_j\|^{-1} + \frac{2\mathcal{E}(X)}{N}.$$

We thus conclude that for every $x \in \mathbb{S}$,

$$\left(1 - \frac{1}{N}\right) \sum_{i=1}^{N} \log \|x - x_i\|^{-1} \ge \frac{2\mathcal{E}(X)}{N};$$

that is,

$$\sum_{i=1}^{N} \log \|x - x_i\|^{-1} \ge \frac{2\mathcal{E}(X)}{N-1}.$$

From (1.13), this implies

$$\mathcal{E}(X) \le C_2(N, \delta) - \frac{1-\delta}{2\delta} N \frac{2\mathcal{E}(X)}{N-1} \le C_2(N, \delta) - \frac{1-\delta}{\delta} \mathcal{E}(X),$$

and solving for $\mathcal{E}(X)$ gives the desired result.

6. Spherical cap discrepancy

In this section we prove Theorem 1.22. We begin with a lemma.

Lemma 6.1. *Let $x \in \mathbb{S}$ and let $C \subseteq \mathbb{S}$ be a measurable set. Then*

$$\int_{y \in C} \log \|y - x\|^{-1} \, dy = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} Area(C \cap B(x, \theta)) \, d\theta,$$

where $B(x, \theta) = \{y \in \mathbb{S} \colon d_R(x, y) \le \theta\}$.

Proof. Let

$$\mathcal{V} = \{(\theta, y) \in [0, \pi/2] \times \mathbb{S} \colon d_R(x, y) \le \theta\}.$$

From Fubini's theorem we have

$$\int_{\mathcal{V}} \frac{\cos\theta}{\sin\theta} \chi_C(y) \, d(\theta, y) = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} \int_{y \in \mathbb{S} \colon d_R(x,y) \le \theta} \chi_C(y) \, dy \, d\theta = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} Area(C \cap B(x, \theta)) \, d\theta.$$

Now, using Fubini's theorem in the opposite order we also have

$$\int_{\mathcal{V}} \frac{\cos\theta}{\sin\theta}\chi_C(y)\,d(\theta,y) = \int_{y\in\mathbb{S}} \chi_C(y)\int_0^{\pi/2}\frac{\cos\theta}{\sin\theta}\chi_{d_R(y,x)\leq\theta}(\theta)\,d\theta\,dy =$$
$$\int_{y\in\mathbb{S}}\chi_C(y)\int_{d_R(y,x)}^{\pi/2}\frac{\cos\theta}{\sin\theta}\,d\theta\,dy = \int_{y\in C}[\log\sin\theta]_{d_R(y,x)}^{\pi/2}\,dy =$$
$$\int_{y\in C} -\log\sin d_R(y,x)\,dy \underset{(1.2)}{=} \int_{y\in C} -\log\|y-x\|\,dy,$$

and the lemma follows. □

6.1. **Proof of Theorem 1.22.** Let $X\in\mathbb{S}^N$ and let $1>u\geq\delta>0$ be such that $d_R(x_i,x_j)\geq 2\arcsin\sqrt{u/N}$ for $i\neq j$. We use the notations of Theorem 1.11. Note that for every i, $1\leq i\leq N$, we have

$$\text{(6.1)}\qquad \fint_{x\in\cup B_j}\log\|x-x_i\|^{-1}\,dx \underset{\text{Lemma 6.1}}{=} \frac{1}{\pi\delta}\int_0^{\pi/2}\frac{\cos\theta}{\sin\theta}\mathit{Area}((\cup B_j)\cap B(x_i,\theta))\,d\theta.$$

We divide the interval of integration in four pieces where the volume inside the integral is bounded according to different regimes.

- For $\theta\in[0,\arcsin\sqrt{\delta/N}]$ we have
$$\mathit{Area}((\cup B_j)\cap B(x_i,\theta)) = \mathit{Area}(B(x_i,\theta)) = \pi\sin^2\theta.$$
Hence,

$$\text{(6.2)}\quad \frac{1}{\pi\delta}\int_0^{\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}\mathit{Area}((\cup B_j)\cap B(x_i,\theta))\,d\theta = \frac{1}{\pi\delta}\int_0^{\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}\pi\sin^2\theta\,d\theta = \frac{1}{2N}.$$

- For $\theta\in[\arcsin\sqrt{\delta/N}, 2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}]$, using that $d_R(x_i,x_j)\geq 2\arcsin\sqrt{u/N}$ for $i\neq j$ we have
$$\mathit{Area}((\cup B_j)\cap B(x_i,\theta)) = \mathit{Area}(B(x_i,\arcsin\sqrt{\delta/N})) = \frac{\pi\delta}{N}.$$
Hence,

$$\text{(6.3)}\qquad \frac{1}{\pi\delta}\int_{\arcsin\sqrt{\delta/N}}^{2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}\mathit{Area}((\cup B_j)\cap B(x_i,\theta))\,d\theta =$$
$$\frac{1}{\pi\delta}\int_{\arcsin\sqrt{\delta/N}}^{2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}\frac{\pi\delta}{N}\,d\theta = \frac{1}{2N}\log\frac{r_1(u,\delta)}{\delta/N},$$

where
$$r_1(u,\delta) = \sin^2\left(2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}\right)$$
is, for fixed u, a continuous function of $\delta\in[0,u]$.

- For $\theta\in[2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}, \pi/2-\arcsin\sqrt{\delta/N}]$ we have
$$\mathit{Area}((\cup B_j)\cap B(x_i,\theta)) \leq \frac{\pi\delta}{N}q(\theta+\arcsin\sqrt{\delta/N}),$$
where for $s\in[0,\pi/2]$, we define
$$q(s) = \sharp\{j : 1\leq j\leq N, x_j\in B(x_i,s)\}.$$

Now,

$$\left|\frac{q(s)}{N}-\frac{Area(B(x_i,s))}{\pi}\right|\leq D_C(X_N),$$

which implies

$$\frac{q(\theta+\arcsin\sqrt{\delta/N})}{N}\leq D_C(X_N)+\sin^2(\theta+\arcsin\sqrt{\delta/N}).$$

We thus have

$$\frac{1}{\pi\delta}\int_{2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}}^{\pi/2-\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}Area((\cup B_j)\cap B(x_i,\theta))\,d\theta\leq r_2(u,\delta), \tag{6.4}$$

where

$$r_2(u,\delta)=\int_{2\arcsin\sqrt{u/N}-\arcsin\sqrt{\delta/N}}^{\pi/2-\arcsin\sqrt{\delta/N}}\frac{\cos\theta}{\sin\theta}\left(D_C(X_N)+\sin^2(\theta+\arcsin\sqrt{\delta/N})\right)\,d\theta$$

is by Lebesgue's Dominated Convergence Theorem, for fixed u, a continuous function of $\delta\in[0,u)$.

- For $\theta\in[\pi/2-\arcsin\sqrt{\delta/N},\pi/2]$ we have

$$Area((\cup B_j)\cap B(x_i,\theta))\leq Area(\cup B_j)=\pi\delta,$$

which implies

$$\frac{1}{\pi\delta}\int_{\pi/2-\arcsin\sqrt{\delta/N}}^{\pi/2}\frac{\cos\theta}{\sin\theta}Area((\cup B_j)\cap B(x_i,\theta))\,d\theta\leq\log\frac{1}{\sin(\pi/2-\arcsin\sqrt{\delta/N})}. \tag{6.5}$$

If we put together (6.2), (6.3), (6.4) and (6.5) in (6.1), we get:

$$⨍_{x\in\cup B_j}\log\|x-x_i\|^{-1}\,dx\leq\frac{1}{2N}\left(1+\log r_1(u,\delta)-\log\frac{\delta}{N}\right)+r_2(u,\delta)-\log\sin(\pi/2-\arcsin\sqrt{\delta/N}).$$

From Theorem 1.11 we conclude that

$$\mathcal{E}(X)\leq-\frac{N^2}{4}+\frac{N}{4}\log\frac{\delta}{N}-\frac{N(N-1)(N-\delta)}{4\delta}\log\left(1-\frac{\delta}{N}\right)+$$

$$\frac{N}{4}\left(1+\log r_1(u,\delta)-\log\frac{\delta}{N}\right)+\frac{N^2}{2}r_2(u,\delta)-\frac{N^2}{2}\log\sin(\pi/2-\arcsin\sqrt{\delta/N}).$$

One can then simplify the two terms containing $\log(\delta/N)$ obtaining:

$$\mathcal{E}(X)\leq-\frac{N^2}{4}-\frac{N(N-1)(N-\delta)}{4\delta}\log\left(1-\frac{\delta}{N}\right)+$$

$$\frac{N}{4}\left(1+\log r_1(u,\delta)\right)+\frac{N^2}{2}r_2(u,\delta)-\frac{N^2}{2}\log\sin(\pi/2-\arcsin\sqrt{\delta/N}).$$

This last formula holds for every $\delta\leq u$, so we can take the limit as $\delta\to0$ and use the continuity of r_1 and r_2 to get

$$\mathcal{E}(X)\leq\frac{N^2}{2}r_2(u,0)+\frac{N}{4}\log r_1(u,0).$$

Now, note that

$$\begin{aligned}
r_1(u,0) =& \sin^2\left(2\arcsin\sqrt{u/N}\right) < \frac{4u}{N},\\
r_2(u,0) =& \int_{2\arcsin\sqrt{u/N}}^{\pi/2} \frac{\cos\theta}{\sin\theta}\left(D_C(X_N)+\sin^2\theta\right)\,d\theta\\
=& -D_C(X_N)\log(\sin(2\arcsin\sqrt{u/N})) + \frac{1}{2}\left(1-\sin^2(2\arcsin\sqrt{u/N})\right)\\
<& \frac{1}{2} - D_C(X_N)\log\left(2\sqrt{\frac{u}{N}}\sqrt{1-\frac{u}{N}}\right)\\
<& \frac{1}{2} + \frac{1}{2}D_C(X_N)\log\frac{N}{2u},
\end{aligned}$$

where we have used $1-u/N>1/2$. We have thus proved:

$$\mathcal{E}(X) \le \frac{N^2}{4} + \frac{N^2}{4}D_C(X_N)\log\frac{N}{2u} - \frac{N\log N}{4} + \frac{N\log(4u)}{4}.$$

By (1.3) we then have

$$\mathcal{E}(X) - m_N \le \frac{N^2}{4}D_C(X_N)\log\frac{N}{2u} + \frac{N\log(8\pi u)}{4}.$$

The theorem follows after renaming u as δ.

6.2. **Proof of Proposition 1.25.** The proof follows along the lines of that of Theorem 1.22 but uses a different subdivision of the interval of integration and avoids $\delta\to 0$ independently of N. For every $N\ge 2$, let $w_N = D_C(X_N)^{c/2}$ and let $\delta_N\in(0,1)$ satisfy (1.19) and

$$d_R((X_N)_i,(X_N)_j) \ge 2\arcsin\sqrt{\frac{\delta_N}{N}} \quad i\ne j.$$

For any $N\ge 2$ and i, $1\le i\le N$:

- If $\theta\in[0,\arcsin w_N]$ we bound

$$\frac{1}{\pi\delta_N}\int_0^{\arcsin w_N}\frac{\cos\theta}{\sin\theta}Area((\cup B_j)\cap B(x_i,\theta))\,d\theta \le \frac{1}{\pi\delta_N}\int_0^{\arcsin w_N}\frac{\cos\theta}{\sin\theta}\pi\sin^2\theta\,d\theta = \frac{w_N^2}{2\delta_N}.$$

- If $\theta\in[\arcsin w_N, \pi/2-\arcsin\sqrt{\delta_N/N}]$, reasoning as in the proof of (6.4) we get:

$$\frac{1}{\pi\delta_N}\int_{\arcsin w_N}^{\pi/2-\arcsin\sqrt{\delta_N/N}}\frac{\cos\theta}{\sin\theta}Area((\cup B_j)\cap B(x_i,\theta))\,d\theta \le r_3(w_N,\delta_N),$$

  where

$$r_3(w_N,\delta_N) = \int_{\arcsin w_N}^{\pi/2-\arcsin\sqrt{\delta_N/N}}\frac{\cos\theta}{\sin\theta}\left(w_N^{2/c}+\sin^2(\theta+\arcsin\sqrt{\delta_N/N})\right)\,d\theta.$$

  We now compute the limit of $r_3(w_N,\delta_N)$ as $N\to\infty$. Note that

$$\int_{\arcsin w_N}^{\pi/2-\arcsin\sqrt{\delta_N/N}}\frac{\cos\theta}{\sin\theta}w_N^{2/c}\,d\theta \le \int_{\arcsin w_N}^{\pi/2}\frac{\cos\theta}{\sin\theta}w_N^{2/c}\,d\theta = -w_N^{2/c}\log w_N \to 0,$$

  because $w_N\to 0$. On the other hand,

$$\sin(\theta+\arcsin\sqrt{\delta_N/N}) \le \sin\theta + \sqrt{\delta_N/N}.$$

Hence,

$$\int_{\arcsin w_N}^{\pi/2-\arcsin\sqrt{\delta_N/N}} \frac{\cos\theta}{\sin\theta}\sin^2(\theta+\arcsin\sqrt{\delta_N/N})\,d\theta \le A_N+B_N+C_N,$$

with

$$A_N=\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta=\frac{1}{2},$$

$$B_N=2\sqrt{\frac{\delta_N}{N}}\int_0^{\pi/2}\cos\theta\,d\theta\to 0,$$

$$C_N=\frac{\delta_N}{N}\int_{\arcsin w_N}^{\pi/2}\frac{\cos\theta}{\sin\theta}\,d\theta=\frac{\delta_N}{N}\log\frac{1}{w_N}\le\frac{c}{2N}\log\frac{1}{D_C(X_N)}\underset{(1.18)}{\to}0$$

We have proved that

$$\limsup_{N\to\infty} r_3(w_N,\delta_N)\le 1/2.$$

- If $\theta\in[\pi/2-\arcsin\sqrt{\delta_N/N},\pi/2]$, we get again (6.5) with δ replaced by δ_N.

Hence, we have

$$⨍_{x\in\cup B_j}\log\|x-x_i\|^{-1}\,dx\le\frac{w_N^2}{2\delta_N}+r_3(w_N,\delta_N)-\log\sin(\pi/2-\arcsin\sqrt{\delta_N/N}),$$

which holds for every j, $1\le j\le N$. We then have from Theorem 1.11 and Lemma 1.16 that

$$\mathcal{E}(X_N)\le\frac{N^2}{2}\left(\frac{w_N^2}{2\delta_N}+r_3(w_N,\delta_N)-\log\sin(\pi/2-\arcsin\sqrt{\delta_N/N})\right)$$

Using the hypotheses $w_N^2/\delta_N=D_C(X_N)^c/\delta_N\to 0$ we conclude

$$\limsup_{N\to\infty}\frac{\mathcal{E}(X_N)}{N^2/4}\le 2\limsup_{N\to\infty}r_3(w_N,\delta_N)\le 1.$$

On the other hand, from Theorem 1.2 we have

$$\lim_{N\to\infty}\frac{m_N}{N^2/4}=1.$$

We thus conclude

$$\limsup_{N\to\infty}\frac{\mathcal{E}(X_N)}{m_N}=\limsup_{N\to\infty}\frac{\mathcal{E}(X_N)N^2/4}{m_N N^2/4}\le\limsup_{N\to\infty}\frac{\mathcal{E}(X_N)}{N^2/4}\limsup_{N\to\infty}\frac{N^2/4}{m_N}\le 1.$$

Finally, from the definition, $\liminf_{N\to\infty}\mathcal{E}(X_N)/m_N\ge 1$, and the proposition now follows.

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