## International J ournal of Computer Mathematics

Publication details, including instructions for authors and subscription information: http:// www.tandfonline.com/loi/ gcom20

## A family of high order product integration methods for an integral equation of lighthill

Neide Bertoldi Franco ${ }^{a}$ \& Sean Mc Kee ${ }^{\text {b }}$
${ }^{\text {a }}$ ICMSC-USP, Sã o Carlos, SP, Brazil
${ }^{\text {b }}$ Computing Laboratory, 8-11 Keble Road, Oxford, England
Published online: 19 Mar 2007.

To cite this article: Neide Bertoldi Franco \& Sean Mc Kee (1985) A family of high order product integration methods for an integral equation of lighthill, International Journal of Computer Mathematics, 18:2, 173-184

To link to this article: http:// dx. doi.org/ 10.1080/ 00207168508803487

PLEASE SCROLL DOWN FOR ARTICLE
Taylor \& Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor \& Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor \& Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.
This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms \& Conditions of access and use can be found at http:// www.tandfonline.com/page/terms-and-conditions

# A Family of High Order Product Integration Methods for an Integral Equation of Lighthill 

NEIDE BERTOLDI FRANCO<br>ICMSC-USP, São Carlos, SP, Brazil

and

SEAN McKEE<br>Computing Laboratory, 8-11 Keble Road, Oxford, England

(Received November, 1983; in final form August, 1984)


#### Abstract

Lighthill has derived a nonlinear singular Volterra integral equation to describe the temperature distribution of the surface of a projectile moving through a laminar boundary layer at high Mach numbers. This paper presents high order product integration methods for its numerical solution and analyses their convergence. Numerical results are given.


KEY WORDS: Singular Volterra equation, product integration methods, high accuracy.
C.R. CATEGORY: G.1.9.

## 1. INTRODUCTION

Consider the Volterra integral equation

$$
\begin{equation*}
F(z)^{4}=-\frac{1}{2 z^{1 / 2}} \int_{0}^{z} \frac{F^{\prime}\left(z_{1}\right)}{\left(z^{3 / 2}-z_{1}^{3 / 2}\right)^{1 / 3}} d z_{1} ; \quad F(0)=1, \lim _{z \rightarrow \infty} F(z)=0 . \tag{1.1}
\end{equation*}
$$

Lighthill [5] obtained two series solutions, one for small $z$ and one for large $z$ and faired these two curves together.

Following Noble (see Anselone [1], p. 215) using Abel's-type
inversion formula it is not difficult to show that (1.1) can be rewritten in the more convenient form

$$
\begin{equation*}
F(w)=1-\frac{3 \sqrt{3}}{2 \pi} \int_{0}^{w} \frac{z[F(z)]^{4}}{\left(w^{3 / 2}-z^{3 / 2}\right)^{2 / 3}} d z \tag{1.2}
\end{equation*}
$$

Franco, McKee and Dixon [3] have employed a novel Gronwall lemma to demonstrate the convergence of a simple product integration method for solving (1.2). The object of this paper is to present a family of high order methods and show that they are convergent.

## 2. THE NUMERICAL METHOD

Consider the Volterra integral equation

$$
\begin{equation*}
y(t)=1-\frac{3 \sqrt{3}}{2 \pi} \int_{0}^{t} \frac{s[y(s)]^{4}}{\left(t^{3 / 2}-s^{3 / 2}\right)^{2 / 3}} d s, \quad t \in[0,1] . \tag{2.1}
\end{equation*}
$$

The method we shall propose replaces $[y(s)]^{4}$ by some high order interpolating polynomial.

Let $t=t_{i}=i h$ where $h$ is the mesh spacing and is defined to be such that $N h=t_{N}=1$.

We define for all $h \in\left(0, h_{0}\right), h_{0}>0$, the function

$$
\Delta_{h}: C[0,1] \rightarrow R^{N+1}
$$

such that

$$
\Delta_{h} \phi(t)=(\phi(0), \phi(h), \ldots, \phi(1-h), \phi(1))^{T} .
$$

Let $\Phi_{h} z=0$ define all algorithms such that

$$
\Phi_{h}: R^{N+1} \rightarrow R^{N+1}
$$

Rewrite Eq. (2.1) in the form

$$
\begin{equation*}
y\left(t_{i}\right)=1-\frac{3 \sqrt{3}}{2 \pi} \sum_{j=0}^{i-1} \int_{i_{j}}^{t_{j-1}} \frac{s[y(s)]^{4}}{\left(t_{i}^{3 / 2}-s^{3 / 2}\right)^{2 / 3}} d s \tag{2.2}
\end{equation*}
$$

and replace $[y(s)]^{4}$ for $s \in\left[t_{j}, t_{j+1}\right]$ by

$$
P_{n}(s)= \begin{cases}\frac{1}{h^{n}} \sum_{l=-j}^{n-j} \frac{\left[y\left(t_{j+l}\right)\right]^{4}}{(-1)^{n-j-l}(l+j)!(n-j-l)!} \prod_{k}^{n-j}\left(s-t_{j}\right), & 0 \leqq j \leqq n-1 \\ \frac{1}{h^{n}} \sum_{l-1-n}^{1} \frac{\left[y\left(t_{j+l}\right)\right]^{4}}{(-1)^{1-1}(l+n-1)!(1-l)!} \prod_{\substack{k=1 \\ k \neq l}}^{1}\left(s-t_{j+k}\right), & \\ n \leqq j \leqq i-1 .\end{cases}
$$

We can define the algorithm

$$
\left[\Phi_{h} y\right]_{i}=\left\{\begin{array}{l}
y_{i}-\tilde{y}_{i}, \quad 0 \leqq i \leqq n-1  \tag{2.4}\\
y_{i}-1+\frac{3 \sqrt{3}}{2 \pi} h \sum_{j=0}^{i-1} w_{i j} y_{j}^{4}+\frac{3 \sqrt{3}}{2 \pi} w_{i i} y_{i}^{4}, n \leqq i \leqq N
\end{array}\right.
$$

where $\tilde{y}_{i}, 0 \leqq i \leqq n-1$, are given initial values, $y_{i}$ is an approximation to $y\left(t_{i}\right)$ and

$$
\begin{align*}
& \left\{\begin{array}{l}
\frac{1}{h^{n+1}} \frac{1}{(-1)^{n} 0!n!} \sum_{l=0}^{n-1} I_{l}\left(g_{i}(s) \pi_{-l,-l, l}(s)\right), j=0 \\
\frac{1}{h^{n+1}}\left\{\frac{1}{(-1)^{n-j} j!(n-j)!} \sum_{i=0}^{n-1} I_{l}\left(g_{i}(s) \pi_{l,-l+j, l}(s)\right)\right.
\end{array}\right. \\
& \begin{array}{l}
+\sum_{p=0}^{q_{1}} \overline{(-1)^{n-j+p+1}(j-p-1)!(n-j+p+1)!} \\
\times I_{n+p}\left(g_{i}(s) \pi_{1-n . j-n \cdots p . n+p}(s)\right\} \quad 1 \leqq j \leqq n
\end{array}  \tag{2.5}\\
& h^{1} h^{n+1} \sum_{l-j}^{q_{2}} \frac{1}{(-1)^{l-j+1}(n+j-l-1)!(l-j+1)!} \\
& \times I_{t}\left(g_{i}(s) \pi_{1-n,-l+j, l}(s)\right), \quad n+1 \leqq j \leqq i-1 \\
& \frac{1}{h^{n}} \frac{1}{(-1)^{0} n!0!} I_{j-1}\left(g_{i}(s) \pi_{1 \cdots n, 1, j-1}(s)\right), \quad j=i .
\end{align*}
$$

Here we have used and subsequently use the notation

$$
\begin{gathered}
g_{i}(s)=\frac{s}{\left(t_{i}^{3 / 2}-s^{3 / 2}\right)^{2 / 3}} \\
\pi_{\alpha, \gamma, q}(s)=\prod_{\substack{k=\alpha \\
k \neq \gamma}}^{n+\alpha}\left(s-t_{q+k}\right) \\
I_{m}(F(s))=\int_{t_{m}}^{t_{m+1}} F(s) d s \\
q_{1}=\min \{j-1, i-n-1\}
\end{gathered}
$$

and

$$
q_{2}=\min \{i-1, j+n-1\} .
$$

Also here and henceforth we bave used the notation

$$
\sum_{j \in \Theta} D_{j}=0 ; \prod_{j \in \Theta} D_{j}=1
$$

if $\Theta$ denotes the empty set.
In matrix notation (2.4) becomes

$$
\begin{equation*}
\Phi_{h} y=y-e+\frac{3 \sqrt{3}}{2 \pi}\left(h A_{N}+B_{N}\right) y^{4} \tag{2.6}
\end{equation*}
$$

where $y^{4}=\left(y_{0}^{4}, y_{1}^{4}, \ldots, y_{N}^{4}\right)^{T}$

$$
\begin{gathered}
e=\left(\tilde{y}_{0}, \tilde{y}_{1}, \ldots, \tilde{y}_{n-1}, 1,1, \ldots, 1\right)^{r} \\
\left(A_{N}\right)_{i j}= \begin{cases}w_{i j}, & n \leqq i \leqq N, \quad 0 \leqq j \leqq i-1 \\
0, & \text { otherwise }\end{cases}
\end{gathered}
$$

and

$$
B_{N}=\operatorname{diag}\left(0,0, \ldots, 0, w_{n n}, \ldots, w_{N N}\right) \text { with } n \text { zeros. }
$$

## 3. CONSISTENCY

We begin by defining what we mean by convergent starting values.

Definition 3.1. The starting values $\tilde{y}_{i}, i=0,1, \ldots, n-1$ are said to be convergent of order $n+1$ if, for $0 \leqq i \leqq n-1$, there exists $C_{1}$, independent of $h$ and $N$, such that

$$
\left|y_{i}-\tilde{y}_{i}\right| \leqq C_{1} h^{n+1}
$$

Theorem 3.2 The discretization $\Phi_{h}$ is consistent of order $n+1$ if there exists $C$, inde - endent of $h$ and $N$, such that for all $i \geqq n$ we have

$$
\begin{equation*}
\left|\left[\Delta_{h} y(t)-e+\frac{3 \sqrt{3}}{2 \pi}\left(h A_{N}+B_{N}\right) \Delta_{h}[y(t)]^{4}\right]\right|_{i} \leqq C h^{n+1} . \tag{3.1}
\end{equation*}
$$

Proof We need only consider

$$
\begin{aligned}
& \left|\int_{0}^{t_{i}} g_{i}(s)[y(s)]^{4} d s-\sum_{j=0}^{i-1} I_{j}\left(g_{i}(s) P_{n}(s)\right)\right| \\
& =\left|\sum_{j=0}^{i-1} I_{j}\left(g_{i}(s)\left[f(y(s))-P_{n}(s)\right]\right)\right| \\
& \leqq \sum_{j=0}^{n-1}\left|I_{j}\left(g_{i}(s)\left[f(y(s))-P_{n}(s)\right]\right)\right| \\
& \quad+\sum_{j=n}^{i}\left|I_{j}\left(g_{i}(s)\left[f(y(s))-P_{n}(s)\right]\right)\right|
\end{aligned}
$$

where $P_{n}(s)$ is defined by (2.3) and $f(y(s))=[y(s)]^{4}$.
Using the error formula for interpolation (e.g. Isaacson and Keller [4], p. 190) we have

$$
f(y(s))-P_{n}(s)=\left\{\begin{array}{cl}
\pi_{-j}^{n-j}(s) \frac{f^{(n+1)}\left(\xi_{j}^{1}(s)\right)}{(n+1)!}, & 0 \leqq j \leqq n-1  \tag{3.2}\\
\pi_{1}^{1} \quad(s) \frac{f^{(n+1)}\left(\xi_{j}^{2}(s)\right)}{(n+1)!}, & n \leqq j \leqq i-1
\end{array}\right.
$$

where $\zeta_{j}^{1} \in\left(t_{0}, l_{n}\right)$ and $\xi_{j}^{2} \in\left(t_{j-n+1}, t_{j+1}\right)$.

Here we have used and subsequently use the notation

$$
\pi_{\alpha}^{\beta}(s)=\prod_{k=\alpha}^{\beta}\left(s-t_{j+k}\right)
$$

Let

$$
I_{j}^{1}=\left|I_{j}\left(g_{i}(s)\left[f(y(s))-P_{n}(s)\right]\right)\right|, \quad 0 \leqq j \leqq n-1
$$

and

$$
I_{j}^{2}=\left|I_{j}\left(g_{i}(s)\left[f(y(s))-P_{n}(s)\right]\right)\right|, \quad n \leqq j \leqq i-1 .
$$

Then using (3.2) and the fact that for

$$
s \in\left[t_{j}, t_{j+1}\right], s^{1 / 2} \leqq t_{j+1}^{1 / 2}
$$

and that for $0 \leqq j \leqq n-1$,

$$
\max _{s \in\left[f_{j}, t_{j+1}\right]}\left\{\left|\pi_{-j}^{n-j}(s)\right|\right\}=(j+1)!h^{j+1} h(n-j)!h^{n-j-1},
$$

we can write

$$
I_{j}^{1} \leqq t_{j+1}^{1 / 2}(j+1)!(n-j)!h^{n+1} \frac{\left|f^{(n+1)}\left(\xi_{j}^{1}\right)\right|}{(n+1)!} \int_{t_{j}}^{t_{j+1}} \frac{s^{1 / 2}}{\left(t_{i}^{3 / 2}-s^{3 / 2}\right)^{2 / 3}} d s
$$

where $\xi_{j}^{1} \in\left(t_{j}, t_{j+1}\right)$.
Hence on evaluation of the integral

$$
I_{j}^{1} \leqq C^{\prime} t_{j+1}^{1 / 2} h^{n+3 / 2}\left\{\left(i^{3 / 2}-j^{3 / 2}\right)^{1 / 3}-\left(i^{3 / 2}-(j+1)^{3 / 2}\right)^{1 / 3}\right\}
$$

Analogously, for $n \leqq j \leqq i-1$, we obtain

$$
I_{j}^{2} \leqq C^{\prime \prime} t_{j+1}^{1 / 2} h^{n+3 / 2}\left\{\left(i^{3 / 2}-j^{3 / 2}\right)^{1 / 3}-\left(i^{3 / 2}-(j+1)^{3 / 2}\right)^{1 / 3}\right\} .
$$

Therefore

$$
\begin{aligned}
& \sum_{j=0}^{n-1} I_{j}^{1}+\sum_{j=n}^{i-1} I_{j}^{2} \leqq C t_{i}^{1 / 2} h^{n+3 / 2} \sum_{j=0}^{i-1}\left[\left(i^{3 / 2}-j^{3 / 2}\right)^{1 / 3}-\left(i^{3 / 2}-(j+1)^{3 / 2}\right)^{1 / 3}\right] \\
&=C t_{i}^{1 / 2} h^{n+3 / 2} i^{1 / 2}
\end{aligned}
$$

since

$$
\begin{equation*}
\sum_{j=0}^{i-1}\left[\left(i^{3 / 2}-j^{3 / 2}\right)^{1 / 3}-\left(i^{3 / 2}-(j+1)^{3 / 2}\right)^{1 / 3}\right]=i^{1 / 2} \tag{3.3}
\end{equation*}
$$

and so

$$
\sum_{j=0}^{n-1} I_{j}^{1}+\sum_{j=n}^{i-1} I_{j}^{2} \leqq C t_{N}^{1 / 2} h^{n+3 / 2} N^{1 / 2}=C h^{n+1}
$$

since $t_{N}=N h=1$.
This completes the demonstration of consistency.
Before stating a convergence theorem a bound is required on the quadrature weights $w_{i j}$.

Lemma 3.2 The quadrature weights $w_{i j}$ are bounded as follows:

$$
\left|w_{i j}\right| \leqq\left\{\begin{array}{l}
M, \quad j=0 \\
M \frac{j}{\left(i^{3 / 2}-j^{3 / 2}\right)^{2 / 3}}, \quad i \leqq j \leqq i-1 \\
M h^{1 / 3}, \quad j=i
\end{array}\right.
$$

where $M$ is some constant independent of $i, j$ and $h$.
Proof The demonstration of this result is not difficult but is tedious due to the many cases that need to be considered (see Franco [2]).

We are now in a position to prove the convergence of the method.

## 4. CONVERGENCE

Before proving convergence we require a lemma from Franco, McKee and Dixon [3].

Lemma 4.1 Let $x_{i}, i=0,1, \ldots, N$ be a sequence of real numbers satisfying:

$$
\begin{gather*}
\left|x_{0}\right| \leqq \delta, \\
\left|x_{i}\right| \leqq \delta(1+M h)+M h \sum_{j=1}^{i-1} \frac{j}{\left(i^{3 / 2}-j^{3 / 2}\right)^{2 / 3}}\left|x_{j}\right|, \quad i=1,2, \ldots, N \tag{4.1}
\end{gather*}
$$

where $\delta>0, M>0$ is independent of $h$, then

$$
\begin{align*}
\left|x_{i}\right| \leqq & \delta(1+M h)\left\{1+\frac{2}{3} M B\left(\frac{4}{3}, \frac{1}{3}\right)+\frac{4}{9} M^{2} B\left(\frac{1}{3}, \frac{1}{3}\right) B\left(\frac{4}{3}, \frac{2}{3}\right)\right\} \\
& \times \exp \left(\frac{4}{9} M^{3} B\left(\frac{2}{3}, \frac{1}{3}\right) B\left(\frac{1}{3}, \frac{1}{3}\right)\right), \quad i=0,1, \ldots, N . \tag{4.2}
\end{align*}
$$

Theorem 4.2 Suppose that the discretization $\Phi_{h}$ is consistent of order $n+1$ and that the starting values are convergent of order $n+1$. Then the discretization $\Phi_{h}$ is convergent of order $n+1$.

Proof Using (2.6) and adding and subtracting ( $3 \sqrt{3} / 2 \pi$ ) $\left(h A_{N}+\right.$ $\left.B_{N}\right) \Delta_{h}[y(t)]^{4}$ and using the triangle inequality results in

$$
\begin{aligned}
\left|\left[y-\Delta_{h} y(t)\right]_{i}\right| \leqq & \frac{3 \sqrt{3}}{2 \pi}\left|\left[\left(h A_{N}+B_{N}\right)\left(y^{4}-\Lambda_{h}[y(t)]^{4}\right)\right]_{i}\right| \\
& +\left[\left.\left[\Delta_{h} y(t)-e+\frac{3 \sqrt{3}}{2 \pi}\left(h A_{N}+B_{N}\right) \Delta_{h}[y(t)]^{4}\right]_{i} \right\rvert\,\right. \\
\leqq & \frac{3 \sqrt{3}}{2 \pi} h \sum_{j=0}^{i-1}\left|\left(A_{N}\right)_{i j}\right|\left|\left[y^{4}-\Delta_{h}[y(t)]^{4}\right]_{j}\right| \\
& +\frac{3 \sqrt{3}}{2 \pi} \max _{n \leqq i \leqq N}\left|\left(B_{N}\right)_{i i}\right|\left|\left[y^{4}-\Delta_{h}[y(t)]^{4}\right]_{i}\right|+C h^{n+1}
\end{aligned}
$$

using consistency and convergence of the starting values.

Since $y$ is bounded (Lighthill [5]) the function $f(y)=y^{4}$ is Lipschitz continuous with Lipschitz constant $L$, say. Thus

$$
\begin{aligned}
\left|\left[y-\Delta_{h} y(t)\right]_{i}\right| \leqq & \frac{3 \sqrt{3}}{2 \pi} L h \sum_{j=0}^{i-1}\left|\left(A_{N}\right)_{i j}\right|\left|\left[y-\Delta_{h} y(t)\right]_{j}\right| \\
& +\frac{3 \sqrt{3}}{2 \pi} L \max _{n \leqq i \leqq N}\left|\left(B_{N}\right)_{i i}\right|\left|\left[y-\Delta_{h} y(t)\right]_{i}\right|+C h^{n+1}
\end{aligned}
$$

Finally from Lemma 3.1 we have

$$
\left|\left(A_{N}\right)_{i j}\right|=\left|w_{i j}\right| \leqq\left\{\begin{array}{l}
M, \quad j=0 \\
M \frac{j}{\left(i^{3 / 2}-j^{3 / 2}\right)^{2 / 3}}, \quad 1 \leqq j \leqq i-1
\end{array}\right.
$$

and

$$
\left|\left(B_{N}\right)_{i i}\right|=\left|w_{i i}\right| \leqq M h^{1 / 3}, \quad i \geqq n
$$

Therefore for $h$ sufficiently small

$$
\begin{aligned}
\left|\left[y-\Delta_{h} y(t)\right]_{i}\right| \leqq & \frac{h M^{\prime}}{\left(1-M^{\prime} h^{1 / 3}\right)}\left\{\left|y_{0}-\Delta_{h} y\left(t_{0}\right)\right|\right. \\
& \left.+\sum_{j=1}^{i-1} \frac{j}{\left(i^{3 / 2}-j^{3 / 2}\right)^{2 / 3}}\left|\left[y-\Delta_{h} y(t)\right] j\right|\right\}+\delta
\end{aligned}
$$

where $M^{\prime}=(3 \sqrt{3} / 2 \pi) L$ and $\delta=C h^{n+1} /\left(1-M^{\prime} h^{1 / 3}\right)$.
Since $\left|y_{0}-\Delta_{h} y\left(t_{0}\right)\right| \leqq C_{1} h^{n+1}$ we have

$$
\left|\left[y-\Delta_{h} y(t)\right]_{i}\right| \leqq \delta(1+\hat{M} h)+\hat{M} h \sum_{j=1}^{i-1} \frac{j}{\left(i^{3 / 2}-j^{3 / 2}\right)^{2 / 3}}\left|\left[y-\Delta_{h} y(t)\right]_{j}\right|
$$

where $\hat{M}=M^{\prime} /\left(1-M^{\prime} h^{1 / 3}\right)$.
Application of Lemma 4.1 leads to the required results.

## 5. NUMERICAL RESULTS

Equation (2.1) was solved by the product integration methods (2.4) of orders one, two and three. Table 1 displays the values obtained with mesh spacings $h=0.25,0.025$ and 0.0025 and convergence would appear to be being obtained. In Figure 1, the result with $n=2$ (order 3 method) and $h=0.025$ is presented graphically. It is seen to display good agreement with those obtained by Lighthill's asymptotic methods and his somewhat ad hoc approach of fairing together the two curves, one having been obtained from a small $z$ asymptotic expansion and the other from a large $z$ expansion. These results therefore justify Lighthill's approach.


## Acknowledgements

Both authors would like to acknowledge financial support from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Financiadora de Estudos e Projetos (FINEP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico ( CNPq ).
TABLE I


## References

[1] P. M. Anselone, (ed.), Nonlinear Integral Equations, University of Wisconsin Press, Madison (1964).
[2] N. B. Franco, Solução Numérica de Algumas Equações Integrais do Tipo Volterra, Ph.D. thesis, University of São Paulo, Brazil (1982).
[3] N. B. Franco, S. McKee and J. Dixon, A numerical solution of Lighthill's integral equation of the surface temperature distribution of a projectile, Matemática Aplicada e Computacional. Vol. 2, no. 3, pp. 257-271 (1983).
[4] E. Isaacson and H. B. Keller, The Analysis of Numerical Methods, John Wiley and Sons Inc, New York, (1966).
[5] M. J. Lighthill, Contributions to the theory of heat transfer through a laminar boundary layer, Proc. Roy. Soc. pp. 359-377 (1950).

