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A family of integrable nonlinear equations of hyperbolic type

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A new system of integrable nonlinear equations of hyperbolic type, obtained by a two-dimensional reduction of the anti-self-dual Yang–Mills equations, is presented. It represents a generalization of the Ernst–Weyl equation of General Relativity related to colliding neutrino and gravitational waves, as well as of the fourth order equation of Schwarzian type related to the KdV hierarchies, which was introduced by Nijhoff, Hone, and Joshi recently. An auto-Bäcklund transformation of the new system is constructed, leading to a superposition principle remarkably similar to the one connecting four solutions of the KdV equation. At the level of the Ernst–Weyl equation, this Bäcklund transformation and the associated superposition principle yield directly a generalization of the single and double Harrison transformations of the Ernst equation, respectively. The very method of construction also allows for revealing, in an essentially algorithmic fashion, other integrability features of the main subsystems, such as their reduction to the Painlevé transcendents. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1416488]

I. INTRODUCTION

The main relationship between the anti-self-dual Yang–Mills equations (ASDYM) and integrable systems of partial differential equations (PDEs) stems from the fact that most well-known integrable systems arise as reductions of the ASDYM equations, or higher-dimensional generalizations of them, by imposing appropriate symmetry conditions.[1] We adopt that saying a system of equations is integrable means that the equations under consideration can be linearized directly, or they can be expressed as consistency conditions for the solution of a linear overdetermined system of PDEs of a certain type (Lax pair).

Of particular interest are the two-dimensional reductions, which are constructed using specific two-dimensional subgroups of the full group of conformal isometries of the four-dimensional complex Minkowski space. A prime example of this kind of reduction is provided by the Ernst equation of General Relativity, which forms the basis of stationary axisymmetric, cylindrical or plane symmetric solutions of the Einstein equations.[2–8] A comprehensive review of two-dimensional reductions of the ASDYM equations is presented in Ref. 8, where a general class of a two-dimensional group of conformal transformations, not necessarily translations, is considered. In all of the above reductions to the Ernst equation, at least one of the two conformal Killing vectors (CKVs) has a nontrivial lift to the twistor space and the formulation is adapted to the Yang matrix J.

In the present paper we consider, instead, a two-dimensional reduction of the ASDYM equations based on a pair of commuting CKVs which are left rotations and leave the α-planes through the origin invariant. It leads to a quite general system of integrable equations of hyperbolic type in two independent variables, which represents a generalization not only of the Ernst–Weyl equation for coupled gravitational and neutrino waves in General Relativity, but of the fourth order equation

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of Schwarzian type introduced by Nijhoff, Hone, and Joshi[9] recently, which is related to the KdV hierarchy.

More specifically, our reduction scheme uses $\mathbf{GL}(2,\mathbb{C})$ as the gauge group and connection potentials that are matrices of unit rank. The general background, as well as the specific features, of the reduction process are presented in Sec. II. In the same section we present the end result of the reduction. It consists of a system of equations which, written in an invariant form, reads as follows:

$$\mathrm{d}\,(P+R)-*\mathrm{d}\,(P-R)=\frac{P-R}{\rho}\,\big((P-R)*\mathrm{d}U-\mathrm{d}\beta\big), \tag{1a}$$

$$\rho\mathrm{d}*\mathrm{d}U-\mathrm{d}\beta\wedge\mathrm{d}U=(P-R)\mathrm{d}U\wedge*\mathrm{d}U, \tag{1b}$$

$$\mathrm{d}*\mathrm{d}\rho=0, \quad \mathrm{d}*\mathrm{d}\beta=0. \tag{1c}$$

The complex functions U, P, R depend on the real coordinates u, v. The functions ρ, β are arbitrary real and complex solutions of the wave equation, respectively. The operator $*$ is a two-dimensional Hodge duality operator acting on one-forms as

$$*\mathrm{d}u=\mathrm{d}u, \quad *\mathrm{d}v=-\mathrm{d}v. \tag{1d}$$

The above system of equations can also be considered as the compatibility or integrability condition of a linear pair of equations parametrized by a complex parameter. This Lax pair of equations is presented in Sec. III, where it is shown that it can be derived algorithmically from the Lax pair of the ASDYM equations. In the same section we derive an auto-Bäcklund transformation of system (1), using the above Lax pair and the standard Gauss decomposition of $\mathbf{GL}(2,\mathbb{C})$.

The restrictions

$$\rho=\tfrac{1}{2}(v-u), \quad \mathrm{d}\beta=n\,\mathrm{d}u+m\,\mathrm{d}v, \quad n,m \ \text{ complex parameters}, \tag{2}$$

of the functions ρ and β, reduce system (1) to a potential form of the fourth order equation of Schwarzian type introduced recently by Nijhoff *et al.*,[9] which was called a regular partial differential equation (RPDE) by the above authors. Its importance stems from the fact that it is directly associated with the KdV hierarchies. The exact relation of the system (1) to the RPDE, as well as to the well-known Euler–Poisson–Darboux equation, is the object of Sec. IV. In the same section, we present a new family of fourth order equations which contains the RPDE among its members.

Imposing appropriate conditions on the variables P, R and β, on the other hand, turns the system (1) into the Ernst–Weyl equation,

$$\mathcal{R}e(E)\big(\mathrm{d}(\rho*\mathrm{d}E)-\mathrm{i}\,\mathrm{d}a\wedge\mathrm{d}E\big)=\rho\,\mathrm{d}E\wedge*\mathrm{d}E, \tag{3}$$

for colliding neutrino waves accompanied by gravitational waves.[10] The Ernst equation for colliding pure gravitational waves in a flat background is also obtained in this way, by applying further restrictions on β. The Neugebauer–Kramer involution arises naturally from the conditions imposed on system (1) and defines a map connecting the real components of two Ernst–Weyl equations.

The relation of system (1) to the Ernst and Ernst–Weyl equations is described in full detail in Sec. V. Section VI, on the other hand, is devoted to another integrability feature of the Ernst–Weyl and RPDE subsystems of the system (1), namely their reduction to Painlevé transcendents. More specifically, in Sec. VI we show how the relation between the Ernst–Weyl and the RPDE equations, established by their being members of the same system (1), facilitates the construction of group invariant solutions of the former based on the Painlevé transcendents from similar kinds of solutions of the latter equation.

The final section of the paper consists of the Perspectives, where an overall evaluation of the results obtained in the main body of the paper is presented, along with the description of various avenues for expanding the above results.

II. REDUCTION OF THE ASDYM EQUATIONS

In this section we present the first main result of this paper. It consists of the new integrable system given by (32), which is derived by a specific two-dimensional reduction of the ASDYM equations.

In order to make the presentation of our result self-contained, we first summarize in Sec. II A the general framework of the ASDYM equations. Then, in Sec. II B, we give the details of the reduction scheme that leads to the new integrable system of equations mentioned above.

A. General considerations

Throughout this section we shall follow the notation and conventions of Refs. 8, 11. Let $M=\mathbb{CM}$ denote the four-dimensional complex Minkowski spacetime and G a Lie group, called the gauge group, and $\mathfrak{g}$ the corresponding Lie algebra. In the finite-dimensional case G can be taken to be $\mathbf{GL}(N,\mathbb{C})$.

Let $P(M,G)$ be a principal bundle, $\{U_i\}$ an open covering of M and s_i a local section defined on each U_i. The Lie algebra valued one-form $\omega\in\mathfrak{g}\otimes T^*P$, called the connection one-form, and the two-form $\Omega\in\mathfrak{g}\otimes\Omega^2(P)$, called the curvature two-form, satisfy the Cartan structure equation,

$$\Omega=\mathrm{d}_P\omega+\omega\wedge\omega,$$

where d_P is the exterior derivative on P. The $\mathfrak{g}$-valued one-form (gauge potential) Φ_i is defined locally as the pull-back $\Phi_i=s_i^*\omega$ of the connection one-form ω and the $\mathfrak{g}$-valued two-form F_i, also called curvature two-form or (Yang–Mills) field strength, is defined by $\mathrm{F}_i=s_i^*\Omega$. If s, s' are local sections over U such that $s'(p)=s(p)g(p)$, $p\in U$, $g\in G$ then the corresponding local one-forms Φ and Φ' are related by

$$\Phi'=g^{-1}\Phi g+g^{-1}\,\mathrm{d}g, \tag{4}$$

where d is the exterior derivative on M. The potentials Φ and Φ' are said to be related through a gauge transformation and they are regarded as being equivalent. From the Cartan structure equation it follows that the curvature F can be expressed in terms of the gauge potential Φ as

$$\mathrm{F}=\mathrm{d}\Phi+\Phi\wedge\Phi. \tag{5}$$

Under gauge transformations (4) the local two-forms F and F$'$ are related by

$$\mathrm{F}'=g^{-1}\mathrm{F}g.$$

In double null coordinates $x^a=(w,z,\tilde{w},\tilde{z})$ the metric on $\mathbb{CM}$ is given by

$$\mathrm{d}s^2=2(\mathrm{d}z\,\mathrm{d}\tilde{z}-\mathrm{d}w\,\mathrm{d}\tilde{w}). \tag{6}$$

In this coordinate system the gauge potential Φ may be written as

$$\Phi=\Phi_w\,\mathrm{d}w+\Phi_z\,\mathrm{d}z+\Phi_{\tilde{w}}\,\mathrm{d}\tilde{w}+\Phi_{\tilde{z}}\,\mathrm{d}\tilde{z}, \tag{7}$$

where the components are $\mathfrak{g}$-valued functions. Φ is said to be anti-self-dual iff F is Hodge anti-self-dual with respect to the metric (6), i.e.,

$$\mathrm{F}=-*\mathrm{F}. \tag{8}$$

Choosing an orientation, condition (8) is equivalent to the ASDYM equations,

$$\partial_z \Phi_w - \partial_w \Phi_z + [\Phi_z, \Phi_w] = 0, \tag{9a}$$

$$\partial_{\tilde{z}} \Phi_{\tilde{w}} - \partial_{\tilde{w}} \Phi_{\tilde{z}} + [\Phi_{\tilde{z}}, \Phi_{\tilde{w}}] = 0, \tag{9b}$$

$$\partial_z \Phi_{\tilde{z}} - \partial_{\tilde{z}} \Phi_z - \partial_w \Phi_{\tilde{w}} + \partial_{\tilde{w}} \Phi_w + [\Phi_z, \Phi_{\tilde{z}}] - [\Phi_w, \Phi_{\tilde{w}}] = 0. \tag{9c}$$

These equations are the integrability conditions of the overdetermined linear system (Lax pair),[12,13]

$$(\partial_w + \Phi_w - \zeta(\partial_{\tilde{z}} + \Phi_{\tilde{z}}))\Psi = 0, \tag{10a}$$

$$(\partial_z + \Phi_z - \zeta(\partial_{\tilde{w}} + \Phi_{\tilde{w}}))\Psi = 0, \tag{10b}$$

where $\Psi(x^a;\zeta)$ is a G-valued function of the spacetime coordinates and the spectral parameter ζ.

B. The reduced equations

For a two-dimensional reduction of the ASDYM equations, one first chooses a two-dimensional subgroup H of the full group of conformal isometries of the Minkowski space. Then, one can reduce the number of the dependent variables by imposing algebraic constraints on the components of Φ, in a way which is consistent with the equations.

A general class of two-dimensional reductions is considered in Ref. 8 where H is generated by two conformal Killing vectors:

$$X = a\partial_w + b\partial_z + \tilde{a}\partial_{\tilde{w}} + \tilde{b}\partial_{\tilde{z}}, \quad Y = c\partial_w + d\partial_z + \tilde{c}\partial_{\tilde{w}} + \tilde{d}\partial_{\tilde{z}}, \tag{11}$$

where a, b, c, d and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ depend only on w, z and $\tilde{w}$, $\tilde{z}$, respectively. Both of the quadruples $\{X, Y, \partial_w, \partial_z\}$ and $\{X, Y, \partial_{\tilde{w}}, \partial_{\tilde{z}}\}$ should be linearly independent and the reduced metric on the orbits of H should be nondegenerate. These conditions assure a compatible reduction.

The most straightforward reduction of this form arises when the corresponding algebra $\mathfrak{h}$ is *Abelian*. We assume that this is the case and we further limit the choices of the components of X and Y by demanding that X and Y leave invariant the α-planes through the origin (the meaning of this requirement will become clear later) and are not a combination of translations. It turns out that these requirements are satisfied only by the commuting null CKVs,

$$X = w\partial_w + \tilde{z}\partial_{\tilde{z}}, \quad Y = z\partial_z + \tilde{w}\partial_{\tilde{w}}. \tag{12}$$

The invariant spacetime coordinates on the orbits of the two-dimensional group of conformal transformations generated by X, Y are arbitrary functions of the fractions $w/\tilde{z}$, $z/\tilde{w}$. Without loss of generality we choose the coordinates of the space of orbits S to be

$$u = \frac{w}{\tilde{z}}, \quad v = \frac{z}{\tilde{w}}, \tag{13}$$

and restrict ourselves to the ultrahyperbolic slice of $\mathbb{CM}$ where the spacetime coordinates are real. The metric induced on S is conformal to two-dimensional Minkowski spacetime in null coordinates, i.e.,

$$ds^2 = \frac{2}{v-u}\, du\, dv. \tag{14}$$

The invariance conditions of the potential Φ with respect to the algebra generated by X, Y are

$$\mathcal{L}_X \Phi = \mathcal{L}_Y \Phi = 0, \tag{15}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X. Under these conditions, one can write the components of the gauge potential Φ in the form

$$\Phi_w = \frac{1}{w} A(u,v), \quad \Phi_z = \frac{1}{z} B(u,v), \quad \Phi_{\tilde{w}} = \frac{1}{\tilde{w}} \tilde{A}(u,v), \quad \Phi_{\tilde{z}} = \frac{1}{\tilde{z}} \tilde{B}(u,v). \tag{16}$$

We choose to work with the invariant gauge where $\tilde{A}$ and $\tilde{B}$ become the Higgs fields of X and Y, respectively. This means that $\tilde{A}$ and $\tilde{B}$ are contractions of the invariant gauge potential Φ with the vector fields X and Y, respectively, i.e., $\tilde{A} = X \lrcorner \Phi$, $\tilde{B} = Y \lrcorner \Phi$. In this gauge one can put $A = B = 0$, whereupon the ASDYM equations (9) become

$$v \tilde{B}_{,v} - u \tilde{A}_{,u} + [\tilde{B}, \tilde{A}] = 0, \tag{17a}$$

$$\tilde{B}_{,v} - \tilde{A}_{,u} = 0. \tag{17b}$$

Equation (17b) implies the existence of a matrix function $K(u,v)$ such that

$$\tilde{B} = K_{,u}, \quad \tilde{A} = K_{,v}, \tag{18}$$

and hence Eq. (17a) becomes

$$(v-u) K_{,u\,v} + [K_{,u}, K_{,v}] = 0. \tag{19}$$

The remaining gauge freedom is $\tilde{A} \to g^{-1} \tilde{A} g$ and $\tilde{B} \to g^{-1} \tilde{B} g$ or, equivalently, $K \to g^{-1} K g + c$ where g, c constant matrices.

Alternatively, one can look at (17a) as a sufficient condition of the existence of the matrix function J such that

$$\tilde{A} = -v J^{-1} J_{,v}, \quad \tilde{B} = -u J^{-1} J_{,u}. \tag{20}$$

Then (17b) takes the following form:

$$(u J^{-1} J_{,u})_{,v} - (v J^{-1} J_{,v})_{,u} = 0. \tag{21}$$

Introducing the functions ρ, σ by

$$\rho = \tfrac{1}{2}(v-u), \quad \sigma = \tfrac{1}{2}(v+u), \tag{22}$$

one can now write Eq. (19) in an invariant form, namely

$$\rho\, \mathrm{d}{*}\mathrm{d}K - \mathrm{d}K \wedge \mathrm{d}K = 0. \tag{23}$$

In a similar fashion Eq. (21) takes the coordinate free form,

$$\mathrm{d}(\rho J^{-1} {*}\mathrm{d}J) = \mathrm{d}(\sigma J^{-1}\, \mathrm{d}J). \tag{24}$$

From the way the coordinates (u,v) were introduced one sees that a more general coordinate system is obtained via the coordinate transformation

$$u \to f(u), \quad v \to g(v), \tag{25}$$

i.e., by relabeling the null coordinates (u,v). Within this more general setting it follows that

$$\rho = \tfrac{1}{2}(g(v) - f(u)), \quad \sigma = \tfrac{1}{2}(g(v) + f(u)), \tag{26}$$

and the functions ρ, σ may be invariantly defined as conjugate solutions of the wave equation $\mathrm{d}*\mathrm{d}\rho=0$.

Now, Eq. (17) imply that

$$\partial_u(\operatorname{tr}\tilde{A}^k)=\partial_v(\operatorname{tr}\tilde{B}^k)=0, \text{ where } k=1,2,...,N-1. \tag{27}$$

Hence

$$\operatorname{tr}\tilde{A}^k=m_k(v), \quad \operatorname{tr}\tilde{B}^k=n_k(u). \tag{28}$$

To reduce the number of the dependent variables we restrict to the case where $N=2$ and $\tilde{A},\tilde{B}$ are matrix functions with

$$\operatorname{rank}\tilde{A}=\operatorname{rank}\tilde{B}=1. \tag{29}$$

With these algebraic constraints, $\tilde{A}$, $\tilde{B}$ may be written as

$$\tilde{A}=\begin{pmatrix} m(v)-RQ & Q \\ R(m(v)-RQ) & RQ \end{pmatrix}, \quad \tilde{B}=\begin{pmatrix} n(u)-PS & S \\ P(n(u)-PS) & PS \end{pmatrix}, \tag{30}$$

where P, Q, R, S are complex functions of (u,v). Inserting $\tilde{A}$ and $\tilde{B}$ given by (30) into the matrix equation (17b), one finds that the upper right element gives

$$Q_{,u}=S_{,v}, \tag{31}$$

which implies the existence of a function U such that $Q=U_{,v}$ and $S=U_{,u}$. In virtue of these relations, Eq. (17a) yields the following system of PDEs:

$$(g(v)-f(u))P_{,v}=(R-P)(m(v)+(P-R)U_{,v}), \tag{32a}$$

$$(g(v)-f(u))R_{,u}=(R-P)(n(u)-(P-R)U_{,u}), \tag{32b}$$

$$(g(v)-f(u))U_{,uv}=m(v)U_{,u}-n(u)U_{,v}+2(P-R)U_{,u}U_{,v}. \tag{32c}$$

This system will be denoted by

$$\Sigma(u,v,U,P,R;m(v),n(u))=0 \tag{33}$$

in the following. We close this section by pointing out that the remaining equation of system (17), namely Eq. (17b), is trivially satisfied when (33) holds.

III. THE REDUCED LAX PAIR AND AN AUTO-BÄCKLUND TRANSFORMATION

Linear (Lax pairs) and nonlinear (Bäcklund transformations) deformation problems are invaluable techniques for generating solutions of integrable equations. As a matter of fact these problems are so interrelated that one can in general derive Bäcklund transformations from the corresponding Lax pair (see, for example, Refs. 14, 15).

Most of the well-known integrable equations in two independent variables, such as the KdV and the sine-Gordon equations, admit a Lax pair of the form

$$\mathrm{d}\Psi=\Omega\Psi, \tag{34}$$

where Ψ belongs in $\mathbf{SL}(2,\mathbb{R})$ and Ω is a $\mathfrak{sl}(2,\mathbb{R})$-valued one form. The associated equation arises from the integrability condition $\mathrm{d}\Omega=\Omega\wedge\Omega$ and the particular way the independent and dependent variables enter into Ω. Using the Iwasawa decomposition of $\mathbf{SL}(2,\mathbb{R})$ for Ψ, one may construct a Bäcklund transformation associated with the given PDE.[15]

The system $\Sigma=0$ also admits a Lax pair of the form (34). It can be derived algorithmically from the Lax pair (10) of the ASDYM equations by applying the invariance conditions[16]

$$\mathcal{L}_X\Psi=\mathcal{L}_Y\Psi=0. \tag{35}$$

These conditions imply that Ψ depends only on the invariant coordinates u, v and the spectral parameter ζ. Taking into account (16) and putting $A=B=0$, we find that Eqs. (10) reduce to

$$\Psi_{,u}=\frac{1}{f(u)-\lambda}\tilde{B}\Psi, \tag{36a}$$

$$\Psi_{,v}=\frac{1}{g(v)-\lambda}\tilde{A}\Psi, \tag{36b}$$

where we have set $\lambda=-\zeta^{-1}$. It is now easily verified that the integrability condition $\Psi_{,uv}=\Psi_{,vu}$ leads to $\Sigma=0$. This means that Eqs. (36) constitute a Lax pair for the system $\Sigma=0$.

At this point it is worth noting that using (18) and (26), the Lax pair (36) may be written in an invariant form as

$$(\sigma-\lambda-\rho*)\mathrm{d}\Psi=\mathrm{d}K\,\Psi, \tag{37a}$$

where

$$(\sigma-\lambda-\rho*)^{-1}=\frac{(\sigma-\lambda+\rho*)}{(\sigma-\lambda)^2-\rho^2}. \tag{37b}$$

We point out that the linear system (37), or equivalently (36), includes the Lax pair used by Hauser and Ernst in solving the initial value problem for colliding plane gravitational waves.[17]

We are now ready to construct a Bäcklund transformation of the system $\Sigma=0$, using the Lax pair (36). To this end, we generalize the technique employed in the case where $\Psi\in\mathbf{SL}(2,\mathbb{R})$ by considering the Gauss decomposition of $\mathbf{GL}(2,\mathbb{C})$.[18] It allows us to write the spectral potential Ψ in the form

$$\Psi=L^{-1}T, \tag{38}$$

where T is an upper triangular matrix and L is a lower triangular one of the form

$$L=\begin{pmatrix}1 & 0\\ -\tilde{U} & 1\end{pmatrix}. \tag{39}$$

Substituting (38) into the Lax pair (36) we obtain the following linear system for the matrix function T:

$$T_{,u}T^{-1}=L_{,u}L^{-1}+\frac{1}{f(u)-\lambda}L\tilde{B}L^{-1}, \tag{40a}$$

$$T_{,v}T^{-1}=L_{,v}L^{-1}+\frac{1}{g(v)-\lambda}L\tilde{A}L^{-1}. \tag{40b}$$

The lower left elements of the the above system lead to the following Riccati system for the function $\tilde{U}$:

$$\tilde{U}_{,u}=\frac{P-\tilde{U}}{f(u)-\lambda}\big(n(u)-(P-\tilde{U})U_{,u}\big), \tag{41a}$$

$$\tilde{U}_{,v}=\frac{R-\tilde{U}}{g(v)-\lambda}(m(v)-(R-\tilde{U})U_{,v}). \tag{41b}$$

The integrability condition of (41) is satisfied if the system $\Sigma=0$ holds. In other words, (41) defines a Bäcklund map for the system $\Sigma=0$. Using such a map, one may construct an auto-Bäcklund transformation for the system under consideration in a manner presented in Ref. 14.

More specifically, solving Eqs. (41) for the derivatives of U we obtain

$$U_{,u}=\frac{1}{P-\tilde{U}}\left(n(u)-\frac{f(u)-\lambda}{P-\tilde{U}}\tilde{U}_{,u}\right), \tag{42a}$$

$$U_{,v}=\frac{1}{R-\tilde{U}}\left(m(v)-\frac{g(v)-\lambda}{R-\tilde{U}}\tilde{U}_{,v}\right). \tag{42b}$$

When Eqs. (32a), (32b) and (42) are satisfied, the integrability condition $U_{,uv}=U_{,vu}$ implies that $\tilde{U}$ satisfies the following equation:

$$(f(u)-g(v))\tilde{U}_{,uv}=n(u)\tilde{U}_{,v}-m(v)\tilde{U}_{,u}+2\left(\frac{f(u)-\lambda}{\tilde{U}-P}-\frac{g(v)-\lambda}{\tilde{U}-R}\right)\tilde{U}_{,u}\tilde{U}_{,v}. \tag{43}$$

If not stated otherwise, we will assume that $P\neq R$ in what follows. The particular case $P=R$ will be considered separately in the next section.

Equation (43) takes the form

$$(f(u)-g(v))\tilde{U}_{,uv}=n(u)\tilde{U}_{,v}-m(v)\tilde{U}_{,u}-2(\tilde{P}-\tilde{R})\tilde{U}_{,u}\tilde{U}_{,v}, \tag{44}$$

by introducing

$$\tilde{P}=W-\frac{f(u)-\lambda}{\tilde{U}-P}, \quad \tilde{R}=W-\frac{g(v)-\lambda}{\tilde{U}-R}, \tag{45}$$

where $W=W(u,v)$ is an auxiliary function. Equation (44) is none other than Eq. (32c) with (U,P,R) replaced by $(\tilde{U},\tilde{P},\tilde{R})$. It suffices to determine W in such a way that the functions $(\tilde{U},\tilde{P},\tilde{R})$, defined by (41) and (45), satisfy $\Sigma(u,v,\tilde{U},\tilde{P},\tilde{R})=0$ whenever $\Sigma(u,v,U,P,R)=0$ holds. This requirement and the differential consequences of (45) result in

$$W=U, \tag{46}$$

up to a nonsignificant constant of integration. In conclusion, we have established the following.

Proposition 1: The algebro-differential system,

$$\tilde{U}_{,u}=\frac{P-\tilde{U}}{f(u)-\lambda}(n(u)-(P-\tilde{U})U_{,u}), \tag{47a}$$

$$\tilde{U}_{,v}=\frac{R-\tilde{U}}{g(v)-\lambda}(m(v)-(R-\tilde{U})U_{,v}), \tag{47b}$$

$$(\tilde{U}-P)(U-\tilde{P})=f(u)-\lambda, \tag{48}$$

$$(\tilde{U}-R)(U-\tilde{R})=g(v)-\lambda, \tag{49}$$

$$\begin{tikzcd}
 & (P_1,R_1,U_1) \arrow[dr, "\lambda_2"] & \\
(P_0,R_0,U_0) \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] & & (P_3,R_3,U_3) \\
 & (P_2,R_2,U_2) \arrow[ur, "\lambda_1"'] &
\end{tikzcd}$$

FIG. 1. Bianchi commuting diagram.

constitutes an auto-Bäcklund transformation for the system $\Sigma=0$.

By purely algebraic procedures Proposition 1 leads to the following permutability theorem.

Permutability theorem: *Let* (U_i,P_i,R_i), $i=1,2$, *be a solution of the system (32a)–(32c), generated by means of the Bäcklund transformation (47)–(49) from a known solution* (U_0,P_0,R_0) *via the Bäcklund parameters* λ_1 *and* λ_2, *respectively. Then there exists a new solution* (U_3,P_3,R_3) *which is given by*

$$(U_3-U_0)(U_2-U_1)=\lambda_2-\lambda_1, \tag{50a}$$

$$(P_3-P_0)(P_2-P_1)=\lambda_2-\lambda_1, \tag{50b}$$

$$(R_3-R_0)(R_2-R_1)=\lambda_2-\lambda_1, \tag{50c}$$

where (U_3,P_3,R_3) *is constructed according to* Fig. 1.

IV. NEW AND OLD EQUATIONS RELATED TO THE SYSTEM Σ

Many physically interesting integrable equations arise from the system $\Sigma=0$ by imposing further algebraic constraints on the independent and the dependent variables. This is illustrated in the following sections in terms of a new family of fourth order equations, which represents a generalization of the RPDE introduced by Nijhoff *et al.*[9] recently, the RPDE itself and the well-known Euler–Poisson–Darboux equation. The Ernst–Weyl equation is also contained in the system $\Sigma=0$ and is presented thoroughly in the next section.

A. The generalized RPDE

The system $\Sigma=0$ leads to a fourth order PDE for the function U solely in the following manner. First, one may solve Eq. (32c) for the difference $P-R$ to obtain

$$P-R=\mathcal{A}[U]:=\frac{1}{2}\left(2\rho\frac{U_{,uv}}{U_{,u}U_{,v}}+\frac{n(u)}{U_{,u}}-\frac{m(v)}{U_{,v}}\right), \tag{51}$$

where ρ is given by (26). Taking the partial derivative of both sides of (51) with respect to u and using Eq. (32b), we arrive at an expression for $P_{,u}$ involving the derivatives of U only. The latter expression and (32a) form a linear first order system for P which takes the following form:

$$P_{,u}=-\frac{\mathcal{A}}{2\rho}(n(u)-\mathcal{A}U_{,u})+\partial_u\mathcal{A},$$

$$P_{,v}=-\frac{\mathcal{A}}{2\rho}(m(v)+\mathcal{A}U_{,v}).$$

The compatibility condition $P_{,uv}=P_{,vu}$ leads to the equation

$$\partial_u\left(\partial_v\mathcal{A}+\frac{\mathcal{A}^2}{\rho}U_{,v}+m(v)\frac{\mathcal{A}}{\rho}\right)+\partial_v\left(\partial_u\mathcal{A}+\frac{\mathcal{A}^2}{\rho}U_{,u}-n(u)\frac{\mathcal{A}}{\rho}\right)=0. \tag{52}$$

This equation, when written out explicitly, reads as

$$\begin{aligned}U_{,uuvv}={}&U_{,uuv}\left(-\frac{\rho_{,v}}{\rho}+\frac{U_{,vv}}{U_{,v}}+\frac{U_{,uv}}{U_{,u}}\right)+U_{,uvv}\left(-\frac{\rho_{,u}}{\rho}+\frac{U_{,uu}}{U_{,u}}+\frac{U_{,uv}}{U_{,v}}\right)-U_{,uu}U_{,vv}\frac{U_{,uv}}{U_{,u}U_{,v}}\\&+U_{,uu}\left(\frac{n(u)^2}{4\rho^2}\frac{U_{,v}^2}{U_{,u}^2}+\frac{\rho_{,v}}{\rho}\frac{U_{,uv}}{U_{,u}}-\frac{U_{,uv}^2}{U_{,u}^2}\right)+U_{,vv}\left(\frac{m(v)^2}{4\rho^2}\frac{U_{,u}^2}{U_{,v}^2}+\frac{\rho_{,u}}{\rho}\frac{U_{,uv}}{U_{,v}}-\frac{U_{,uv}^2}{U_{,v}^2}\right)\\&-\frac{n(u)^2}{8\rho^3}\frac{U_{,v}}{U_{,u}}(\rho_{,v}U_{,u}-\rho_{,u}U_{,v}+2\rho U_{,uv})+\frac{m(v)^2}{8\rho^3}\frac{U_{,u}}{U_{,v}}(\rho_{,v}U_{,u}-\rho_{,u}U_{,v}-2\rho U_{,uv})\\&+\frac{1}{2\rho}U_{,uv}^2\left(\frac{\rho_{,u}}{U_{,u}}+\frac{\rho_{,v}}{U_{,v}}\right)-\frac{n(u)n'(u)}{4\rho^2}\frac{U_{,v}^2}{U_{,u}}-\frac{m(v)m'(v)}{4\rho^2}\frac{U_{,u}^2}{U_{,v}}.\end{aligned} \tag{53}$$

This is a new integrable equation which generalizes the RPDE, as it will be clear from the following section. It will be referred to as the generalized RPDE (GRPDE). In order to clarify its relation to the system $\Sigma=0$, we first introduce a shorthand notation whereby Eq. (53) is written in the form

$$\mathcal{R}(u,v,U;m(v),n(u))=0. \tag{54}$$

In a manner to be explained shortly, one is led to similar equations for the functions P, R, starting from the system $\Sigma=0$. More specifically, these functions satisfy the fourth order equations

$$\mathcal{R}(u,v,P;m(v),n(u)-2\rho_{,u})=0, \tag{55}$$

and

$$\mathcal{R}(u,v,R;m(v)+2\rho_{,v},n(u))=0, \tag{56}$$

respectively. Thus, it becomes clear that the system $\Sigma=0$ represents an involution of a triad of GRPDEs. The members of this triad differ only by specific changes in the parameter functions $m(v)$ and $n(u)$.

Returning to the derivation of Eqs. (55), (56), let us first note that one may eliminate the derivatives of U from the system $\Sigma=0$ by solving (32a) and (32b) for $U_{,v}$ and $U_{,u}$, respectively. Using compatibility conditions and (32c), one then ends up with the following system for P, R:

$$P_{,uv}=\frac{2}{P-R}P_{,u}P_{,v}+\frac{m(v)}{2\rho}P_{,u}+\frac{n(u)-2\rho_{,u}}{2\rho}P_{,v}, \tag{57a}$$

$$R_{,uv}=\frac{2}{R-P}R_{,u}R_{,v}-\frac{m(v)+2\rho_{,v}}{2\rho}R_{,u}-\frac{n(u)}{2\rho}R_{,v}. \tag{57b}$$

It is now a matter of straightforward, but lengthy, calculations to decouple the above system and arrive to Eqs. (55), (56) for P and R, respectively.

At this point, it is worth noting that Eqs. (57) may be combined to yield[19]

$$(\rho SS_{,u})_{,v}+(\rho SS_{,v})_{,u}+\tau_{,u}S_{,v}+\tau_{,v}S_{,u}=0, \tag{58}$$

where

$$\tau:=\sigma+\beta, \tag{59a}$$

and S is a 2×2 matrix defined by

$$S=\frac{1}{R-P}\begin{pmatrix} P+R & -2 \\ 2PR & -(P+R)\end{pmatrix}. \tag{59b}$$

The matrix equation (58) represents a complexified hyperbolic version of the stationary Loewner–Konopelchenko–Rogers (LKR) system[20] proposed by Schief[21] as a 2×2 real matrix representation of the Ernst–Weyl equation in stationary, axisymmetric spacetimes. It should be pointed out that LKR systems have proven to be a repository of mathematically and physically interesting integrable equations in $2+1$-dimensions, including a $2+1$-dimensional Ernst-type equation, introduced by Schief recently.[22]

B. The RPDE

By choosing

$$f(u)=u, \quad g(v)=v \tag{60a}$$

and

$$m(v)=m, \quad n(u)=n, \tag{60b}$$

where n, m are complex parameters, Eq. (53) becomes

$$\begin{aligned} U_{,uuvv}=&U_{,uuv}\left(\frac{1}{u-v}+\frac{U_{,vv}}{U_{,v}}+\frac{U_{,uv}}{U_{,u}}\right)+U_{,uvv}\left(\frac{1}{v-u}+\frac{U_{,uu}}{U_{,u}}+\frac{U_{,uv}}{U_{,v}}\right)-U_{,uu}U_{,vv}\frac{U_{,uv}}{U_{,u}U_{,v}}\\ &+U_{,uu}\left(\frac{n^2}{(u-v)^2}\frac{U_{,v}^2}{U_{,u}^2}-\frac{1}{u-v}\frac{U_{,uv}}{U_{,u}}-\frac{U_{,uv}^2}{U_{,u}^2}\right)+U_{,vv}\left(\frac{m^2}{(u-v)^2}\frac{U_{,u}^2}{U_{,v}^2}+\frac{1}{u-v}\frac{U_{,uv}}{U_{,v}}-\frac{U_{,uv}^2}{U_{,v}^2}\right)\\ &+\frac{n^2}{2(u-v)^3}\frac{U_{,v}}{U_{,u}}(U_{,u}+U_{,v}+2(v-u)U_{,uv})-\frac{m^2}{2(u-v)^3}\frac{U_{,u}}{U_{,v}}(U_{,u}+U_{,v}+2(u-v)U_{,uv})\\ &+\frac{1}{2(u-v)}U_{,uv}^2\left(\frac{1}{U_{,u}}-\frac{1}{U_{,v}}\right), \end{aligned} \tag{61}$$

which is the RPDE introduced by Nijhoff *et al.*[9] recently. Its importance stems from the fact that it is a generating equation for the whole hierarchy of the KdV equation. We presented several aspects of the integrability of the RPDE in Ref. 23.

C. The EPD equation

By choosing

$$P=R, \tag{62}$$

and referring to Eqs. (32a), (32b), we find that

$$P=c, \quad c \text{ constant}. \tag{63}$$

Without loss of generality we set $c=0$. This can be achieved by performing a gauge transformation on $\tilde{A}$, $\tilde{B}$ by $g=\left(\begin{smallmatrix}1 & 0\\ -c & 1\end{smallmatrix}\right)$, whereupon one obtains equivalent Higgs fields with $P=R=0$. Within this setting, the system $\Sigma=0$ reduces to a single linear, second order PDE for U,

$$U_{,uv}=\frac{1}{f(u)-g(v)}(n(u)U_{,v}-m(v)U_{,u}), \tag{64}$$

which is known as the *Euler–Poisson–Darboux* (EPD) equation. From this point of view, the EPD equation may be considered as a linearization of the system $\Sigma=0$.

The substitution $\psi=\tilde{U}^{-1}$ transforms the Riccati equations (41) into the linear system,

$$\psi_{,u}=\frac{n(u)}{f(u)-\lambda}\psi+\frac{U_{,u}}{f(u)-\lambda}, \tag{65a}$$

$$\psi_{,v}=\frac{m(v)}{g(v)-\lambda}\psi+\frac{U_{,v}}{g(v)-\lambda}. \tag{65b}$$

It can be easily verified that the integrability condition $U_{,uv}=U_{,vu}$ and the system (65) lead to

$$\psi_{,uv}=\frac{1}{f(u)-g(v)}(n(u)\psi_{,v}-m(v)\psi_{,u}). \tag{66}$$

A comparison of this with Eq. (64) shows that the system (65) represents an auto-Bäcklund transformation for the EPD equation. The linearity of the same system in ψ indicates that it is also a Lax pair for the EPD equation.

Restricting $U(u,v),m(v),n(u)$ to be real functions of their arguments and choosing

$$n(u)=-\tfrac{1}{2}f'(u), \quad m(v)=-\tfrac{1}{2}g'(v),$$

Eq. (64) becomes

$$2U_{,uv}+\frac{\rho_{,u}}{\rho}U_{,v}+\frac{\rho_{,v}}{\rho}U_{,u}=0, \quad \rho=\frac{1}{2}(g(v)-f(u)). \tag{67}$$

This particular form of the EPD equation governs the collision of two plane gravitational waves with *collinear polarization*[24] in the context of Einstein's General Relativity. The reduced Lax pair (65) becomes the Lax pair used in Ref. 25 for solving the corresponding initial value problem.

V. REDUCTION TO THE ERNST–WEYL EQUATION

One of the most extensively studied problems in General Relativity is the collision of two plane gravitational waves. The Ernst–Weyl equation describes the collision of neutrino waves accompanied by gravitational waves.[10] The latter equation, which reduces to the famous Ernst equation when the neutrino fields vanish everywhere, arises naturally by applying appropriate reality conditions to the system $\Sigma=0$. Our reduction scheme unifies many aspects of the integrability of the Ernst equation like the Hauser–Ernst Lax pair, Neugebauer–Kramer involution, and the Harrison Bäcklund transformation and gives analogous generalizations for the Ernst–Weyl equation.

A. The reality conditions

The system $\Sigma=0$ may be written in the following invariant form:

$$\mathrm{d}(P+R)-*\mathrm{d}(P-R)=\frac{P-R}{\rho}((P-R)*\mathrm{d}U-\mathrm{d}\beta), \tag{68a}$$

$$\rho\mathrm{d}*\mathrm{d}U-\mathrm{d}\beta\wedge\mathrm{d}U=(P-R)\mathrm{d}U\wedge*\mathrm{d}U, \tag{68b}$$

where

$$\rho=\tfrac{1}{2}(g(v)-f(u)) \quad \text{and} \quad \mathrm{d}\beta=n(u)\mathrm{d}u+m(v)\mathrm{d}v. \tag{69}$$

Here, we have used the Poincaré lemma to express the one-form $n(u)\mathrm{d}u+m(v)\mathrm{d}v$ as the exterior derivative of a scalar complex function β.

Let us now impose the reality condition,

$$P=-R^{\star},\tag{70}$$

on (68) with $\star$ denoting complex conjugation. Furthermore, in order to make contact with the notation employed in General Relativity,[26] let us introduce the complex potentials $E,\mathcal{E}$ by

$$\mathcal{E}=\mathcal{F}+\mathrm{i}\chi=P,\quad E=F+\mathrm{i}\omega=U,\tag{71}$$

where $\mathrm{i}=\sqrt{-1}$ and $\mathcal{F}$, F and χ, ω are the real and imaginary parts, respectively, of $\mathcal{E}$, E. In the same fashion, we split the one-form $\mathrm{d}\beta$ into real and imaginary part by setting

$$\mathrm{d}\beta=\mathrm{d}\delta+\mathrm{i}\mathrm{d}a,\tag{72}$$

where a, δ are real solutions of the wave equation. Inserting these into Eq. (68a), the system separates into a real and an imaginary part:

$$\mathrm{d}\mathcal{F}=\frac{\mathcal{F}}{\rho}*\mathrm{d}\delta-\frac{2\mathcal{F}^2}{\rho}\mathrm{d}F\tag{73a}$$

and

$$\mathrm{d}\chi=\frac{2\mathcal{F}^2}{\rho}*\mathrm{d}\omega-\frac{\mathcal{F}}{\rho}\mathrm{d}a,\tag{73b}$$

respectively. Equation (68b) on the other hand becomes

$$\rho\mathrm{d}*\mathrm{d}E-\mathrm{d}\beta\wedge\mathrm{d}E=2\mathcal{F}\mathrm{d}E\wedge*\mathrm{d}E.\tag{74}$$

The integrability condition $\mathrm{d}^2\mathcal{F}=0$ of Eq. (73a) yields

$$2\mathcal{F}(\mathrm{d}\rho-*\mathrm{d}\delta)\wedge\mathrm{d}F=\mathrm{d}\rho\wedge*\mathrm{d}\delta,\tag{75}$$

which is satisfied for general real functions $\mathcal{F},F$ of (u,v) if

$$\mathrm{d}\delta=*\mathrm{d}\rho=-\mathrm{d}\sigma.\tag{76}$$

Consequently,

$$\mathrm{d}\beta=*\mathrm{d}\rho+\mathrm{i}\mathrm{d}a.\tag{77}$$

Then, Eq. (73a) can be integrated to give

$$\mathcal{F}F=\frac{\rho}{2},\tag{78}$$

by setting the integration constant equal to zero. The integrability condition of Eq. (73b) is satisfied when Eqs. (74), (77) and (78) hold. In view of (77) and (78), Eq. (74) becomes

$$\mathcal{R}e(E)(\mathrm{d}(\rho*\mathrm{d}E)-\mathrm{i}\,\mathrm{d}a\wedge\mathrm{d}E)=\rho\,\mathrm{d}E\wedge*\mathrm{d}E.\tag{79}$$

Equivalently we may apply the reality condition (70) to the system (57). This condition is compatible with the system for general function $\mathcal{E}$ of (u,v) when (77) holds. In this case, the system (57) reduces to the following single second order PDE for $\mathcal{E}$:

$$\mathcal{R}e(\mathcal{E})(\mathrm{d}(\rho * \mathrm{d}\mathcal{E}) + \mathrm{i} * \mathrm{d}a \wedge \mathrm{d}\mathcal{E}) = \rho\, \mathrm{d}\mathcal{E} \wedge * \mathrm{d}\mathcal{E}. \tag{80}$$

Hence, the above considerations lead to the following.

Proposition 2: The conditions

$$P = -R^{\star} \tag{81a}$$

and

$$m(v) = -\rho_{,v} + \mathrm{i}\ a_{,v}\,, \quad n(u) = \rho_{,u} + \mathrm{i}\ a_{,u} \tag{81b}$$

reduce the system (68) and equivalently the system (57) to

$$\mathcal{R}e(E)(\mathrm{d}(\rho * \mathrm{d}E) - \mathrm{i}\ \mathrm{d}a \wedge \mathrm{d}E) = \rho\, \mathrm{d}E \wedge * \mathrm{d}E, \tag{82}$$

$$\mathcal{R}e(\mathcal{E})(\mathrm{d}(\rho * \mathrm{d}\mathcal{E}) + \mathrm{i} * \mathrm{d}a \wedge \mathrm{d}\mathcal{E}) = \rho\, \mathrm{d}\mathcal{E} \wedge * \mathrm{d}\mathcal{E}, \tag{83}$$

respectively, where E *and* $\mathcal{E}$ *are related through the involution*

$$\mathcal{F}F = \frac{\rho}{2}, \quad \mathrm{d}\,\chi = \frac{2\mathcal{F}^2}{\rho} * \mathrm{d}\,\omega - \frac{\mathcal{F}}{\rho}\mathrm{d}a. \tag{84}$$

- When $\mathrm{d}a = 0$, Eq. (82) becomes the Ernst equation for colliding plane pure gravitational waves in a flat background. The map given by (84) which connects two solutions $(E,\mathcal{E})$ of the Ernst equation, is known as the *Neugebauer–Kramer involution*.[27]
- When $\mathrm{d}a \neq 0$, Eq. (82) is the Ernst–Weyl equation for colliding plane neutrino waves accompanied by gravitational waves. The form of the function a is specified by the initial profile of the neutrino waves on the null hypersurfaces $u=0$, $v=0$. After solving Eq. (82) for E, the neutrino fields and the metric components can be found by quadrature. It should be mentioned here that, in the present case, the spacetime metric is not of the block diagonal form and the two Killing vectors characterizing the plane symmetry of the spacetime are not surface-forming. The map given by (84), connecting solutions of the two different Ernst–Weyl equations (82) and (83), may be viewed as a *generalization of the Neugebauer–Kramer involution*.

The considerations in Sec. IV indicate that the solutions $(E,\mathcal{E})$ of the Ernst–Weyl equations (82), (83) also satisfy

$$\mathcal{R}(u,v,E; -\rho_{,v} + \mathrm{i}\ a_{,v}\,, \rho_{,u} + \mathrm{i}\ a_{,u}) = 0 \tag{85a}$$

and

$$\mathcal{R}(u,v,\mathcal{E}; -\rho_{,v} + \mathrm{i}\ a_{,v}\,, -\rho_{,u} + \mathrm{i}\ a_{,u}) = 0, \tag{85b}$$

respectively. Thus, the solution space of the Ernst–Weyl equation is imbedded into the solution space of Eq. (53) for the specific choices of $m(v), n(u)$ given by (81b).

B. Reduction of the auto-Bäcklund transformation

Finding exact solutions of Einstein's field equations is quite a difficult task. This is mainly due to the nonlinearity of the field equations. Another reason is that the solutions should respect various types of boundary (side) conditions. In stationary axisymmetric spacetimes for example, the corresponding metric should be asymptotically flat. For colliding wave solutions, on the other hand, the metric must satisfy appropriate junction conditions across the wavefront surfaces. Therefore, solution generating techniques, such as the inverse scattering transform and Bäcklund transformations, become invaluable methods in finding exact solutions or determining global properties of the field equations (for a survey of results see Ref. 28).

In this section we show how the auto-Bäcklund transformation (47)–(49) of the system $\Sigma=0$ can be reduced to a Bäcklund transformation for the Ernst–Weyl equation. Moreover, we establish the relation of this reduction to the well-known Harrison Bäcklund transformation[29] for the Ernst equation.

To this end, let us first denote by $\mathcal{D}$ the solution space of the system $\Sigma=0$ and by $\mathcal{D}_E\subset\mathcal{D}$ the corresponding space of the Ernst–Weyl equation. Then, the auto-Bäcklund transformation B, defined by (47)–(49), may be viewed as a symmetry transformation in $\mathcal{D}$. In order to construct the reduced Bäcklund transformation for the Ernst–Weyl equation we require $B(\mathcal{D}_E)\subset\mathcal{D}_E$. This requirement implies that the new functions should satisfy the system $\Sigma=0$ *and* conditions (81). Following the notation of the previous section, we shall denote in what follows $\tilde{P}$ by $\tilde{\mathcal{E}}$ and $\tilde{U}$ by $\tilde{E}$.

Under the above conditions the auto-Bäcklund transformation (47)–(49) takes the following form:

$$\tilde{E}_{,u}=\frac{\mathcal{E}-\tilde{E}}{f(u)-\lambda}(\rho_{,u}+\mathrm{i}\ a_{,u}-(\mathcal{E}-\tilde{E})E_{,u}), \tag{86a}$$

$$\tilde{E}_{,v}=\frac{\mathcal{E}^\star+\tilde{E}}{g(v)-\lambda}(\rho_{,v}-\mathrm{i}\ a_{,v}-(\mathcal{E}^\star+\tilde{E})E_{,v}), \tag{86b}$$

$$(\tilde{E}-\mathcal{E})(E-\tilde{\mathcal{E}})=f(u)-\lambda, \tag{87a}$$

$$(\tilde{E}+\mathcal{E}^\star)(E+\tilde{\mathcal{E}}^\star)=g(v)-\lambda. \tag{87b}$$

One can now easily verify the following fact. Starting with a pair of potentials $(E,\mathcal{E})$ related as in (84), then the system (86), (87) delivers a new pair of potentials $(\tilde{E},\tilde{\mathcal{E}})$ which satisfy the same relation.

In order to establish the connection of the above Bäcklund transformation to the one proposed by Harrison, let us first note that in terms of $(E,\mathcal{E})$ the first of Eqs. (84) reads as

$$(E+E^\star)(\mathcal{E}+\mathcal{E}^\star)=2\rho. \tag{88}$$

Then, combining (87a), (87b) and (88) one finds that

$$\left|\frac{\tilde{\mathcal{E}}+E^\star}{\tilde{\mathcal{E}}-E}\right|^2=\gamma, \tag{89}$$

where $|\,|$ stands for the modulus of a complex number and γ is defined by

$$\gamma:=\frac{g(v)-\lambda}{f(u)-\lambda}. \tag{90}$$

Introducing the function α via

$$\alpha:=\frac{\tilde{\mathcal{E}}+E^\star}{\tilde{\mathcal{E}}-E}, \tag{91}$$

we can write Eq. (89) as

$$\alpha\,\alpha^{\star}=\gamma, \tag{92}$$

and Eqs. (87), (91) as

$$\tilde{E}=\frac{\gamma-\alpha}{\gamma-1}\mathcal{E}+\frac{1-\alpha}{\gamma-1}\mathcal{E}^{\star}, \tag{93a}$$

$$\tilde{\mathcal{E}}=\frac{\alpha}{\alpha-1}E+\frac{1}{\alpha-1}E^{\star}, \tag{93b}$$

respectively. Finally, Eqs. (86) take the following form:

$$\alpha_{,u}=\alpha(\alpha-1)\frac{E_{,u}}{E+E^{\star}}+(\alpha-\gamma)\frac{E^{\star}_{,u}}{E+E^{\star}}+\frac{\alpha}{2}(\gamma-1)\frac{\rho_{,u}-\mathrm{i}\,a_{,u}}{\rho}, \tag{94a}$$

$$\alpha_{,v}=(\alpha-1)\frac{E^{\star}_{,v}}{E+E^{\star}}+\frac{\alpha}{\gamma}(\alpha-\gamma)\frac{E_{,v}}{E+E^{\star}}+\frac{\alpha}{2\gamma}(\gamma-1)\frac{\rho_{,v}-\mathrm{i}\,a_{,v}}{\rho}. \tag{94b}$$

The converse also holds. More specifically, let us suppose that a pair of potentials $(E,\mathcal{E})$ satisfying Eqs. (82), (84) is given. Then, the members of the Ricatti system (94) for the auxiliary function $\alpha(u,v)$ are compatible and can be integrated to yield a solution which satisfies the condition (92). Moreover, Eqs. (93) deliver a new pair of potentials $(\tilde{E},\tilde{\mathcal{E}})$ which *also* satisfy the generalized version of the Neugebauer–Kramer involution. The system (92)–(94) constitutes a generalization of the *single Harrison transformation* for the Ernst equation, to which it reduces when $\mathrm{d}a=0$. We would like to point out that the above novel construction is purely algebraic and yields *explicit* expressions for the new potentials. As far as we are aware only implicit expressions for the new potentials occur in the literature. Let it also be noted that if one considers each member of the pair $(E,\mathcal{E})$ separately, then the relations (92)–(94b) must be viewed as an auto- or hetero-Bäcklund transformation, depending on whether $\mathrm{d}a$ equals 0 or not, respectively. This follows from the fact that E and $\mathcal{E}$ satisfy Eqs. (82) and (83), respectively, which are identical or different depending on whether $\mathrm{d}a$ vanishes or not.

Returning to the considerations of the paragraph preceding the last one, we note that one has to apply the transformation (92)–(94) twice in order to decouple the new potentials. This can be achieved by using the superposition principle (50) in the form resulting by applying the reality conditions. This form is given by

$$E_3=E_0+\frac{\lambda_2-\lambda_1}{E_2-E_1}, \tag{95a}$$

$$\mathcal{E}_3=\mathcal{E}_0+\frac{\lambda_2-\lambda_1}{\mathcal{E}_2-\mathcal{E}_1}. \tag{95b}$$

The potentials $E_i,\mathcal{E}_i$, $i=1,2$, appearing in these relations are determined by

$$E_i=\frac{\gamma_i-\alpha_i}{\gamma_i-1}\mathcal{E}_0+\frac{1-\alpha_i}{\gamma_i-1}\mathcal{E}^{\star}_0,$$

$$\mathcal{E}_i=\frac{\alpha_i}{\alpha_i-1}E_0+\frac{1}{\alpha_i-1}E^{\star}_0.$$

$$\begin{tikzcd}
 & (E_1,\mathcal{E}_1) \arrow[dr, "(\alpha_{12},\gamma_2)"] & \\
(E_0,\mathcal{E}_0) \arrow[ur, "(\alpha_1,\gamma_1)"] \arrow[dr, "(\alpha_2,\gamma_2)"'] & & (E_3,\mathcal{E}_3) \\
 & (E_2,\mathcal{E}_2) \arrow[ur, "(\alpha_{21},\gamma_1)"'] &
\end{tikzcd}$$

FIG. 2. Commuting diagram for the Harrison transformation.

Here, the γ_i's are defined by (92) with λ replaced by λ_i and the α_i's are obtained from the system (94) by replacing γ by γ_i. Using these expressions, Eqs. (95) reads as

$$E_3=E_0+\frac{(\gamma_2-\gamma_1)(E_0+E_0^\star)}{\gamma_1-\gamma_2-(1-\gamma_2)\alpha_1+(1-\gamma_1)\alpha_2}, \tag{96a}$$

$$\mathcal{E}_3=\mathcal{E}_0-\frac{\gamma_2-\gamma_1}{\alpha_2-\alpha_1}\frac{(1-\alpha_1)(1-\alpha_2)}{(1-\gamma_1)(1-\gamma_2)}(\mathcal{E}_0+\mathcal{E}_0^\star). \tag{96b}$$

Equations (96) represent the double Harrison transformation and are valid for both cases, $\mathrm{d}a=0$ and $\mathrm{d}a\neq 0$.

In the Bianchi commuting diagram given in Fig. 2 one can focus on the auxiliary functions α_i's instead of the potentials $(E_i,\mathcal{E}_i)$. Then the superposition principle is expressed by the following relations among the α's:

$$\frac{\alpha_{12}}{\alpha_{21}}=\frac{\alpha_2}{\alpha_1}, \tag{97}$$

where

$$\alpha_{12}=\frac{1}{\alpha_1}\frac{\alpha_1\alpha_2(\gamma_2-\gamma_1)+\alpha_1\gamma_2(\gamma_1-1)+\alpha_2\gamma_1(1-\gamma_2)}{\gamma_1-\gamma_2+\alpha_1(\gamma_2-1)+\alpha_2(1-\gamma_1)}. \tag{98}$$

VI. REDUCTIONS TO THE PAINLEVÉ TRANSCENDENTS

Similarity reductions of the Ernst equation to Painlevé transcendents have been of particular interest for a long time. More recently Schief[21] has shown that the Ernst–Weyl equation for stationary axially symmetric spacetimes [the elliptic analog of Eq. (82)] admits similarity reductions to Painlevé III, V, and VI. His reduction procedure to Painlevé III and V is based on solving the system of ordinary differential equations (ODEs) resulting from the application of the invariance conditions. For the reduction to Painlevé VI, however, the author resorts to a different approach, based on a matrix formulation of the Ernst–Weyl equation, motivated by the Loewner system. In this section, we present a reduction of the hyperbolic Ernst–Weyl equation (82) by a method which exploits the relation of the latter equation to the RPDE and the straightforward manner in which the symmetry group of the RPDE leads to Painlevé's transcendents.

As we have shown,[23] the RPDE admits straightforward similarity reductions to Painlevé III, V, and VI in full form. This was done by considering the invariant solutions under the optimal system of one-dimensional subalgebras of the Lie point symmetries of the RPDE. The link between the RPDE and the Ernst–Weyl equation is the system $\Sigma=0$. Specifically, in Sec. IV, it was shown that the RPDE may be given in involutive form by system (32) with the choices

$$f(u)=u, \quad g(v)=v, \quad m(v)=m, \quad n(u)=n. \tag{99}$$

Within the above setting, the Ernst–Weyl equation arises from the system $\Sigma=0$ by requiring that

$$P=-R^{\star}, \quad m=-\tfrac{1}{2}+\mathrm{i}a_1, \quad n=-\tfrac{1}{2}+\mathrm{i}a_2, \tag{100}$$

where a_1, a_2 are real constants.

On the basis of the above connection, we shall search for invariant solutions of the Ernst–Weyl equation that follow from imposing conditions (100) on corresponding invariant solutions of the RPDE. As we shall immediately show, this can best be achieved by first prolonging the symmetries of the RPDE to symmetries of the system $\Sigma=0$. For illustration purposes, we shall restrict our presentation to invariant solutions that are related to the full Painlevé V and VI cases, only. To make the presentation self-contained, we first summarize briefly the Lie point symmetries of the RPDE.

The Lie point symmetry group of the RPDE consists of transformations leaving the dependent variables unaffected (base transformations) and transformations acting on the dependent variables only (vertical transformations). The base transformations act on $(u,v)\in\mathbb{R}^2$ by

$$(u,v)\to(\lambda u+\epsilon,\lambda v+\epsilon), \quad \lambda\in\mathbb{R}^+,\ \epsilon\in\mathbb{R}, \tag{101}$$

and the generators are given by the vector fields

$$\partial_u+\partial_v, \quad u\partial_u+v\partial_v. \tag{102}$$

The group of vertical transformations is the most general group of transformations acting locally effectively on a one-dimensional complex manifold and is given by

$$U\to\frac{aU+b}{cU+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathbf{SL}(2,\mathbb{C}). \tag{103}$$

The corresponding generators are the vector fields

$$\partial_U, \quad U\partial_U, \quad U^2\partial_U. \tag{104}$$

A. Reduction to Painlevé V

The group invariant solutions of the RPDE under the symmetry generator,

$$\partial_u+\partial_v+2\mu U\partial_U, \tag{105}$$

have the form

$$U(u,v)=S(y)\exp(\mu(u+v)), \quad y=u-v. \tag{106}$$

The prolongation of the vector field (105) in the P and R directions which yields a symmetry of the system $\Sigma=0$ reads as

$$\partial_u+\partial_v+2\mu U\partial_U-2\mu(P\partial_P+R\partial_R). \tag{107}$$

This implies that the invariant form of the functions P and R is

$$P(u,v)=p(y)\exp(-\mu(u+v)), \quad R(u,v)=r(y)\exp(-\mu(u+v)), \tag{108}$$

respectively. Inserting (106), (108) into the system $\Sigma=0$ we get the following system of ODEs:

$$(S'-\mu S)(p-r)^2-m(p-r)-y(p'+\mu p)=0, \tag{109a}$$

$$(S'+\mu S)(p-r)^2-n(p-r)+y(r'-\mu r)=0, \tag{109b}$$

$$yS''+2(p-r)S'^2-(m+n)S'-2\mu^2(p-r)S^2-\mu(m-n+\mu y)S=0. \tag{109c}$$

By a simple differentiation and elimination process applied to system (109), one ends up with a fourth order ODE for the function S, which can be integrated once leading to

$$\begin{aligned} y^2S(S'^2-\mu^2S^2)S'''-y^2SS'S''^2+y(-yS'^3+SS'^2+3\mu^2yS^2S'-\mu^2S^3)S''-yS'^4+\mu(m^2-n^2 \\ +\mu y)S^2S'^2+\mu^2[2(m^2+n^2)-\mu^2y^2]S^3S'+\mu^3(m^2-n^2)S^4-ly(S'^2-\mu^2S^2)^2=0, \end{aligned} \tag{110}$$

where l is the constant of integration. For later use, we note that, using system (109), the above first integral can be written in the following remarkably simple form:

$$2y\frac{(pr)'}{(p-r)^2}+(m-n)\frac{p+r}{p-r}+1+l=0. \tag{111}$$

Setting

$$\frac{S'(y)}{S(y)}=\mu\frac{1+G(y)}{1-G(y)}, \tag{112}$$

Eq. (110) becomes

$$\mathcal{P}_V\left(G,y;\frac{n^2}{2},-\frac{m^2}{2},2\mu l,-2\mu^2\right)=0. \tag{113}$$

Here $\mathcal{P}_V(G,y;\alpha,\beta,\gamma,\delta)=0$, with α, β, γ and δ arbitrary complex parameters, stands for the full Painlevé V equation, i.e.,

$$-G''+\left(\frac{1}{2G}+\frac{1}{G-1}\right)G'^2-\frac{1}{y}G'+\alpha\frac{G(G-1)^2}{y^2}+\beta\frac{(G-1)^2}{y^2G}+\gamma\frac{G}{y}+\delta\frac{G(G+1)}{G-1}=0. \tag{114}$$

Having determined S in the manner described above, one can return to system (109) to find the following explicit expressions for the functions p, r:

$$p(y)=\frac{yG'+nG^2-(l+2n+1-2\mu y)G+l+n+1}{4\mu S(1-G)}, \tag{115a}$$

$$r(y)=\frac{yG'-(l+m+1)G^2+(l+2m+1+2\mu y)G-m}{4\mu SG(1-G)}. \tag{115b}$$

At this point it is worth noting that using Eqs. (112), (113) and (115), one can easily verify that the functions W, V defined by

$$\frac{p'(y)}{p(y)}=\mu\frac{W(y)+1}{W(y)-1}, \quad \frac{r'(y)}{r(y)}=\mu\frac{V(y)+1}{V(y)-1}, \tag{116}$$

satisfy

$$\mathcal{P}_V\left(W,y;\frac{(n+1)^2}{2},-\frac{m^2}{2},2\mu(l+1),-2\mu^2\right)=0, \tag{117a}$$

$$\mathcal{P}_V\left(V,y;\frac{n^2}{2},-\frac{(m+1)^2}{2},2\mu(l+1),-2\mu^2\right)=0, \tag{117b}$$

respectively. Thus, one may consider Eqs. (115) as Bäcklund transformations among the Painlevé V equations (113), (117).

Restricting our considerations to solutions of the Ernst–Weyl equation, we need to take conditions (100) into account. Under these conditions, Eqs. (108) imply that

$$p=-r^\star, \quad \mu=\mu^\star, \tag{118a}$$

while (111) yields

$$l=l^\star. \tag{118b}$$

Hence, the corresponding similarity solutions of the Ernst–Weyl equation (82) are determined by the potential

$$E=S(y)\exp(\mu(u+v)), \quad y=u-v, \tag{119}$$

where S is given by Eq. (112). In the latter, G represents a solution of the Painlevé equation (113), with

$$n=-\tfrac{1}{2}+\mathrm{i}a_1, \quad m=-\tfrac{1}{2}+\mathrm{i}a_2, \quad l=l^\star, \quad \mu=\mu^\star. \tag{120}$$

The analogous similarity solutions of the Ernst–Weyl equation (83) are specified by the potential

$$\mathcal{E}=p(y)\exp(-\mu(u+v)), \quad y=u-v, \tag{121}$$

where p is determined by the first of Eqs. (116). In the latter, W stands for a solution of the Painlevé equation (117a), with n, m, l, and μ satisfying the conditions (120) above.

B. Reduction to Painlevé VI

In a similar manner one may construct solutions of the Ernst–Weyl equation from the Painlevé VI transcendents, i.e., solutions of the ODE,

$$-G''+\frac{1}{2}\left(\frac{1}{G}+\frac{1}{G-1}+\frac{1}{G-y}\right)G'^2-\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{G-y}\right)G' \\ +\frac{G(G-1)(G-y)}{y^2(y-1)^2}\left(\alpha+\beta\frac{y}{G^2}+\gamma\frac{y-1}{(G-1)^2}+\delta\frac{y(y-1)}{(G-y)^2}\right)=0, \tag{122}$$

where α, β, γ, and δ arbitrary complex parameters, which will be denoted by $\mathcal{P}_{VI}(G,y;\alpha,\beta,\gamma,\delta)=0$ in the following.

The solutions of the RPDE which are invariant under the symmetry generated by the vector field

$$u\partial_u+v\partial_v+2\mu U\partial_U, \tag{123}$$

have the form

$$U(u,v)=S(y)(uv)^\mu, \quad y=u/v. \tag{124}$$

The prolongation of the vector field (123) in the P, R directions leading to a symmetry of system $\Sigma=0$ reads as

$$u\partial_u + v\partial_v + 2\mu U\partial_U + (1-2\mu)(P\partial_P + R\partial_R). \tag{125}$$

This implies that the invariant form of the functions P and R is

$$P(u,v) = p(y)(uv)^{-\mu+1/2}, \quad R(u,v) = r(y)(uv)^{-\mu+1/2}, \tag{126}$$

respectively. Inserting (124), (126) into the system $\Sigma = 0$ we obtain the following system of ODEs:

$$y^{1/2}(\mu S - yS')(p-r)^2 + (m + (\mu - \tfrac{1}{2})(y-1))p - mr + (y-1)yp' = 0, \tag{127a}$$

$$y^{1/2}(\mu S + yS')(p-r)^2 + (ny - (\mu - \tfrac{1}{2})(y-1))r - nyp + (y-1)yr' = 0, \tag{127b}$$

$$(1-y)y^2S'' + 2y^{1/2}(p-r)(\mu^2S^2 - y^2S'^2) + y(m + (n-1)y + 1)S' + \mu(m - ny + \mu(y-1))S = 0. \tag{127c}$$

Through a lengthy but straightforward process of differentiation and elimination one ends up with a fourth order ODE for the function S. The latter may be integrated once leading to a third order equation which is omitted because of its length. We note, however, that this first integral can be written in the following simple form by using system (127):

$$(1-2\mu)\frac{y^2-1}{(p-r)^2}(pr)' - 2y\frac{(y-1)^2}{(p-r)^2}p'r' + \frac{(2\mu-1)^2}{2(p-r)^2}\left(p^2 + r^2 - \frac{1+y^2}{y}pr\right)$$
$$+ (m-n)(1-2\mu)\frac{p+r}{p-r} + \frac{1}{2}(m-n)^2 = l, \quad l \text{ constant}. \tag{128}$$

Substituting

$$\frac{S'(y)}{S(y)} = \frac{\mu}{y}\frac{y+G(y)}{y-G(y)} \tag{129}$$

into the omitted third order ODE for S, we find that G satisfies the full Painlevé equation,

$$\mathcal{P}_{VI}\left(G,y;\frac{n^2}{2},-\frac{m^2}{2},l,\frac{1-4\mu^2}{2}\right) = 0. \tag{130}$$

Once S is determined by solving the differential equations (129), (130), one can find the functions p, r algebraically using system (127). The explicit expressions for these functions are given by

$$p(y) = \frac{y^2(y-1)^2G'^2 + \Gamma G^2(G-1)G' + A_i(y,m,n)G^i}{8\,\mu\,(2\mu-1)\,y^{1/2}(y-G)\,(G-1)\,G\,S}, \tag{131a}$$

$$r(y) = \frac{y^2(y-1)^2G'^2 + \Gamma yG(G-1)G' + B_i(y,m,n)G^i}{8\,\mu\,(2\mu-1)\,y^{1/2}(y-G)\,(G-1)\,G\,S}, \tag{131b}$$

where summation over the repeated index $i = 0,\ldots,4$ is understood and the coefficients A_i, B_i and Γ are given by

$$A_0(y,m,n) = -m^2y^2,$$

$$A_1(y,m,n) = y[(m^2 - 2l + (n-2\mu+1)^2)y + 2m^2],$$

$$A_2(y,m,n) = -(n-2\mu+1)^2y^2 - 2\big(m^2-2l+(n-2\mu+1)^2\big)y + (1-2\mu)^2 - m^2,$$

$$A_3(y,m,n) = 2(n-2\mu+1)^2y + m^2 + 2n^2 - 2l - (n+2\mu-1)^2,$$

$$A_4(y,m,n) = -n(n-4\mu+2),$$

$$B_i(y,m,n) = y^2 A_{4-i}(y^{-1},n,m), \quad i=0,\ldots,4,$$

$$\Gamma = 2(2\mu-1)y(y-1).$$

As in the previous case, one may use Eqs. (129), (130) and (131) to verify that the functions W, V defined by

$$\frac{p'(y)}{p(y)} = \frac{1-2\mu}{2y}\,\frac{y+W(y)}{y-W(y)}, \quad \frac{r'(y)}{r(y)} = \frac{1-2\mu}{2y}\,\frac{y+V(y)}{y-V(y)}, \tag{132}$$

satisfy

$$\mathcal{P}_{VI}\left(W,y;\frac{(n+1)^2}{2},-\frac{m^2}{2},l,2\mu(1-\mu)\right) = 0, \tag{133a}$$

$$\mathcal{P}_{VI}\left(V,y;\frac{n^2}{2},-\frac{(m+1)^2}{2},l,2\mu(1-\mu)\right) = 0, \tag{133b}$$

respectively. Hence, one may view Eqs. (131) as defining Bäcklund transformations among the Painlevé VI equations (130) and (133).

Taking into account conditions (100), one is restricted to solutions of the Ernst–Weyl equation. Equations (126) imply that

$$p = -r^\star, \quad \mu = \mu^\star, \tag{134}$$

while Eq. (128) gives

$$l = l^\star. \tag{135}$$

Hence the invariant solutions of the Ernst–Weyl equation (82) are of the form

$$E = S(y)(uv)^\mu, \quad y = u/v, \tag{136}$$

where S is given by integrating (129) and G satisfies (130) with

$$n = -\tfrac{1}{2} + \mathrm{i}a_1, \quad m = -\tfrac{1}{2} + \mathrm{i}a_2, \quad l = l^\star, \quad \mu = \mu^\star. \tag{137}$$

Last, the similarity solutions of the Ernst–Weyl equation (83) have the form

$$\mathcal{E} = p(y)(uv)^{-\mu+1/2}, \quad y = u/v, \tag{138}$$

where p is given by integrating the first equation of (132) and W satisfies (133a) with n, m, l, and μ given by (137).

VII. PERSPECTIVES

We presented a two-dimensional reduction of the ASDYM equations which leads to a novel system of integrable equations. This system incorporates the well-known Ernst–Weyl equation, as well as a significant generalization of the fourth order hyperbolic equation proposed by Nijhoff *et al.*[9] recently as the generating PDE for the KdV hierarchy. We have also constructed a Lax pair and an auto-Bäcklund transformation for the above system and similarity solutions based on the Painlevé V and VI transcendents. Our reduction scheme unifies many aspects of integrability of the Ernst equation, like the Ernst–Hauser deformation problem, the Neugebauer–Kramer involution, the single and double-Harrison Bäcklund transformations, and gives analogous generalizations to the Ernst–Weyl equation. All these aspects follow algorithmically from the new system and the associated Lax pair, by imposing purely algebraic and compatibility conditions. Moreover, our reduction scheme allows for an easier construction of solutions to the Ernst–Weyl equation related to Painlevé transcendents by Lie group techniques. Using the reduction scheme presented in this paper and higher-dimensional gauge groups than $\mathbf{GL}(2,\mathbb{C})$, one should obtain integrable systems which incorporate the Ernst–Maxwell–Weyl equations, or even more general integrable equations describing the interaction of the gravitational field with other sources.[30,31]

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