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A Family of Irreducible Representations of the Witt Lie Algebra with Infinite-Dimensional Weight Spaces

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Abstract. We define a 4-parameter family of generically irreducible and inequivalent representations of the Witt Lie algebra on which the infinitesimal rotation operator acts semisimply with infinite-dimensional eigenspaces. They are deformations of the (generically indecomposable) representations on spaces of polynomial differential operators between two spaces of tensor densities on S^1, which are constructed by composing each such differential operator with the action of a rotation by a fixed angle.

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1. Introduction

Let $\mathcal{W}$ be the Witt Lie algebra, the space of Laurent polynomial vector fields on the circle S^1, and let $\mathcal{V}$ be its universal central extension, the Virasoro Lie algebra. Representations of $\mathcal{W}$ and $\mathcal{V}$ are important in many areas of Mathematics and Physics, and have been studied since the 1970s [GoOl]. However, essentially only the Harish-Chandra representations have been considered, on which the infinitesimal rotation $z(\mathrm{d}/\mathrm{d}z)$ acts semisimply with finite-dimensional eigenspaces (or weight spaces) [Ka1, FF1, FF2]. One exception is Kirillov's paper [Ki], in which a class of irreducible representations of $\mathcal{W}$ with infinite-dimensional weight spaces is defined (see the final remark of Section 5, and also the related paper of Ushirobira [Us]).

In this paper we introduce a 4-parameter family of irreducible representations of $\mathcal{W}$, which act in spaces of operators between two spaces of tensor densities on S^1, on which $z(\mathrm{d}/\mathrm{d}z)$ acts semisimply with countably infinite-dimensional weight spaces. In order to be more precise, denote by $A(a, \gamma)$ the space of (multivalued) tensor densities of the form $\mathrm{d}z^{\gamma} z^{a-\gamma} p(z)$, where a and γ are in $\mathbb{C}$, and $p(z)$ is a polynomial on S^1. These spaces were introduced by Feigin and Fuchs [FF2], and they carry a natural action of $\mathcal{W}$ which permits the realization of all irreducible Harish-Chandra representations of $\mathcal{V}$ that are not of highest or lowest weight [Ka2],

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[Ma], [MP1], [CP]. For any additional complex numbers b and p, let $E_0(a, b, \gamma, p)$ be the span of the 'polynomial' differential operators from $A(a, \gamma)$ to $A(a+b, \gamma+p)$, i.e., those mapping $(\mathrm{d}z)^\gamma z^{a-\gamma} p(z)$ to $(\mathrm{d}z)^{\gamma+p} z^{\mu-p} (\mathrm{d}/\mathrm{d}z)^k z^{a-\gamma} p(z)$, for some μ in $b+\mathbb{Z}$ and k in $\mathbb{N}$ (the quote marks indicate the fact that $z^{\mu-p}$ may be multivalued; this particular operator is of weight μ for all k). The natural adjoint action of $\mathcal{W}$ on $E_0(a, b, \gamma, p)$ is in general indecomposable, but not irreducible, and is itself composed of infinitely many tensor density representations. The study of these representations goes back to the nineteenth century, and is still active [CMZ], [BOv], [GaOv], [Mat].

We shall define a 1-parameter deformation $E_h(a, b, \gamma, p)$ of $E_0(a, b, \gamma, p)$, and our main result is that for generic values of the parameters, the representations of $\mathcal{W}$ on $E_h(a, b, \gamma, p)$ are irreducible, and if a is fixed, inequivalent. In the case that h is purely imaginary, the deformation $E_h(a, b, \gamma, p)$ has the following geometric definition, which has a holomorphic analog for arbitrary h: if ϕ_h is the rotation of S^1 given as multiplication by e^{-h}, then:

$$E_h(a, b, \gamma, p) = \left\{ T \circ \phi_h^*\colon T \in E_0(a, b, \gamma, p) \right\},$$

where ϕ_h^* is the natural action of ϕ_h on tensor densities. This is invariant under the adjoint action of $\mathcal{W}$, because conjugation by ϕ_h^* preserves polynomial differential operators.

We remark that the choice of ϕ_h is natural: if ϕ is an arbitrary diffeomorphism of S^1, we have the more general space $E_\phi(\gamma, p)$, spanned by the maps $T \circ \phi^*$ such that T is an arbitrary differential operator from tensor densities of degree γ to those of degree $\gamma + p$. It is still invariant under the adjoint action of $\mathcal{W}$, but when ϕ is not a rotation, it has no natural subspace that preserves the polynomial tensor densities, and $z(\mathrm{d}/\mathrm{d}z)$ does not act upon it semisimply.

This paper is organized as follows. In Section 2 we collect our notation and definitions, in Section 3 we prove the generic irreducibility of the deformations $E_h(a, b, \gamma, p)$ (let us remark that our proof uses difference operators, and is reminiscent of q-calculus), and in Section 4 we prove their generic inequivalence. In the final section we make some remarks: in particular, we relate $E_h(0, 0, 0, 0)$ to the q-Weyl algebra (considered, for example, by Kassel [Kas]), and note that $E_0(0, 0, 0, 0)$ is the Weyl algebra with z^{-1} adjoined, whose deformation $E_h(0, 0, 0, 0)$ was previously considered by Pinczon [Pi] (with only positive powers of z present), in connection with his theory of non-commutative deformations of associative algebras.

2. Definitions

In this section we establish our notation, give definitions of $A(a, \gamma)$ and $E_h(a, b, \gamma, p)$ more suited to calculation than those in the introduction, and exhibit some of the intertwining maps and subrepresentations of the $E_h(a, b, \gamma, p)$ at certain special values of the parameters.

2.1. TENSOR DENSITY REPRESENTATIONS

Let $\{e_n : n \in \mathbb{Z}\}$ be the standard basis of the Witt Lie algebra $\mathcal{W}$, the space of polynomial vector fields on the circle: $e_n = z^{n+1}(\mathrm{d}/\mathrm{d}z)$, so $[e_n, e_m] = (m-n)e_{n+m}$. Let $\mathfrak{a}_n$ be the subalgebra $\mathrm{Span}\{e_0, e_{\pm n}\}$, which is isomorphic to $\mathfrak{sl}(2)$, and recall that the Casimir operator Q_n of $\mathfrak{a}_n$ is defined by $n^2 Q_n = e_0^2 + ne_0 - e_{-n}e_n$. Note that e_0 is the infinitesimal rotation $z(\mathrm{d}/\mathrm{d}z)$, and $\mathfrak{a}_1$ is the projective subalgebra of linear fractional transformations.

In any representation V of $\mathcal{W}$, the eigenvalues of the action of e_0 are called the weights of V, and the eigenspace of eigenvalue μ is called its μ-weight space, written V_μ. The irreducible Harish-Chandra representations are classified as follows: those with a highest or a lowest weight are irreducible quotients of Verma modules, and those of uniformly bounded weight space dimension may be realized in the spaces $A(a, \gamma)$ of tensor densities. We now rephrase the definition of the $A(a, \gamma)$: let M be the 1-form $z^{-1}dz$ on S^1, and for $\lambda \in \mathbb{C}$, think of z^λ as a function on S^1, multivalued if λ is not an integer. For any complex scalars a and γ, define $A(a, \gamma)$ to be $\mathrm{Span}\{M^\gamma z^\lambda : \lambda \in a + \mathbb{Z}\}$. Then the natural action of $\mathcal{W}$ on tensor densities gives:

$$e_n(M^\gamma z^\lambda) = (\lambda + n\gamma)M^\gamma z^{\lambda+n}.$$

Of course, we may always assume that $0 \leqslant \mathrm{Re}(a) < 1$. These are all irreducible, except for $A(0, 0)$ and $A(0, 1)$: the 0-weight space of $A(0, 0)$ is invariant, and thus is the trivial representation; the corresponding quotient is called $\tilde{A}$, and is irreducible; and $A(0, 1)$ is the dual of $A(0, 0)$. If $A(a, \gamma)$ and $A(a', \gamma')$ are different, then they are equivalent if and only if $a = a' \neq 0$ and $\{\gamma, \gamma'\} = \{0, 1\}$, and so the following list classifies the irreducible bounded representations: the trivial representation, $\tilde{A}$, the $A(a, 0)$ with a not 0, and the $A(a, \gamma)$ with γ not 0 or 1. We mention that the (restricted) dual of $A(a, \gamma)$ is equivalent to $A(-a, 1-\gamma)$.

It will be convenient to define $\mathcal{A}$ to be the algebraic direct sum of all the $A(a, \gamma)$:

$$\mathcal{A} \;=\; \bigoplus_{\substack{\gamma \in \mathbb{C} \\ 0 \leqslant \mathrm{Re}(a) < 1}} A(a, \gamma) \;=\; \mathrm{Span}\left\{M^\gamma z^\lambda : \gamma, \lambda \in \mathbb{C}\right\}.$$

Of course $\mathcal{A}$ is a representation of $\mathcal{W}$, and we shall write $\tilde{e}_n$ for the operator on $\mathcal{A}$ by which e_n acts.

Let ad denote the adjoint action of $\mathrm{End}(\mathcal{A})$ on itself: if τ and T are operators on $\mathcal{A}$, then $\mathrm{ad}(\tau)T = \tau T - T\tau$. In particular, we denote by σ the adjoint action of $\mathcal{W}$ on $\mathrm{End}(\mathcal{A})$:

$$\sigma(e_n) = \mathrm{ad}(\tilde{e}_n).$$

We will also have occasion to use the following subspace of $\mathcal{A}$:

$$A(\gamma) \;=\; \bigoplus_{0 \leqslant \mathrm{Re}(a) < 1} A(a, \gamma) \;=\; \mathrm{Span}\left\{M^\gamma z^\lambda : \lambda \in \mathbb{C}\right\}.$$

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We shall write $r_{a,\gamma}$ and r_γ for restriction of operators on $\mathcal{A}$ to $A(a,\gamma)$ and $A(\gamma)$, respectively, and r_a^γ for restriction of maps with domain $A(\gamma)$ to $A(a,\gamma)$. Of course, $r_{a,\gamma} = r_a^\gamma \circ r_\gamma$. Note that the adjoint action σ of $\mathcal{W}$ is also defined on the spaces $\mathrm{Hom}[A(a,\gamma), A(a+b,\gamma+p)]$ and $\mathrm{Hom}[A(\gamma), A(\gamma+p)]$.

2.2. AN ALGEBRA OF OPERATORS ON TENSOR DENSITIES

Here we define a subalgebra $\mathcal{E}$ of $\mathrm{End}(\mathcal{A})$, which contains $\mathcal{W}$. For p and μ in $\mathbb{C}$, and $f\colon \mathbb{C}\to\mathbb{C}$ any set-theoretic function, let M^p, z^μ, $f(\tilde{e}_0)$, and Γ be the following operators on $\mathcal{A}$:

$$M^p(M^\gamma z^\lambda) = M^{p+\gamma}z^\lambda, \qquad z^\mu(M^\gamma z^\lambda) = M^\gamma z^{\mu+\lambda},$$
$$f(\tilde{e}_0)(M^\gamma z^\lambda) = f(\lambda)M^\gamma z^\lambda, \qquad \Gamma(M^\gamma z^\lambda) = \gamma M^\gamma z^\lambda.$$

Then we have the following commutation relations:

$$[z^\mu, M^p] = 0, \qquad [\tilde{e}_n, M^p] = npM^pz^n, \qquad f(\tilde{e}_0)z^\mu = z^\mu f(\tilde{e}_0+\mu),$$
$$[\Gamma, z^\mu] = 0, \qquad [\Gamma, \tilde{e}_n] = 0, \qquad \Gamma M^p = M^p(\Gamma + p),$$

where $f(\tilde{e}_0+\mu)$ denotes the operator mapping $M^\gamma z^\lambda$ to $f(\lambda+\mu)M^\gamma z^\lambda$.

We define the subspace $\mathcal{E}$ of $\mathrm{End}(\mathcal{A})$ by

$$\mathcal{E} = \mathrm{Span}\left\{M^pz^\mu f(\tilde{e}_0)\Gamma^j : p,\mu\in\mathbb{C},\ j\in\mathbb{N},\ f\colon\mathbb{C}\to\mathbb{C}\ \text{ arbitrary}\right\}.$$

In light of the above commutation relations, it is a subalgebra of $\mathrm{End}(\mathcal{A})$. The operator $\tilde{e}_n$ is $z^n(\tilde{e}_0+n\Gamma)$, which is contained in $\mathcal{E}$, and so $\mathcal{E}$ is invariant under the action σ of $\mathcal{W}$. To compute this action, recall that $\sigma(e_n)$ is the derivation $\mathrm{ad}(\tilde{e}_n)$, and check that:

$$\sigma(e_n)M^p = npM^pz^n, \qquad \sigma(e_n)z^\mu = \mu z^{\mu+n},$$
$$\sigma(e_n)f(\tilde{e}_0) = z^n(\tilde{e}_0+n\Gamma)\Big[f(\tilde{e}_0) - f(\tilde{e}_0+n)\Big], \qquad \sigma(e_n)\Gamma = 0.$$

Hence,

$$\begin{aligned}&\sigma(e_n)\Big[M^pz^\mu f(\tilde{e}_0)\Gamma^j\Big]\\ &\quad = M^pz^{\mu+n}\Big[(\mu+np+n\Gamma+\tilde{e}_0)f(\tilde{e}_0) - (n\Gamma+\tilde{e}_0)f(\tilde{e}_0+n)\Big]\Gamma^j.\end{aligned} \tag{1}$$

The μ-weight space $\mathcal{E}_\mu$ is the span of the $M^pz^\mu f(\tilde{e}_0)\Gamma^j$ with μ fixed, and $\mathcal{E}$ is the algebraic direct sum of its weight spaces.

2.3. THE REPRESENTATION $E_h(b, \gamma, p)$

In this section we give two equivalent definitions of the representation considered in this paper: $E_h(b, \gamma, p)$, a space of maps from $A(\gamma)$ to $A(\gamma + p)$ or, equivalently, $E_h(a, b, \gamma, p)$, a space of maps from $A(a, \gamma)$ to $A(a + b, \gamma + p)$, for any a.

Fix h, a, b, γ, and p in $\mathbb{C}$, recall the restriction maps r_γ and r_a^γ from Section 2.1, and define a map $T_\mu^k(h, \gamma, p)$ from $A(\gamma)$ to $A(\gamma + p)$ by:

$$T_\mu^k(h, \gamma, p) = r_\gamma\Big[M^p z^\mu \exp(h\tilde{e}_0)\tilde{e}_0^k\Big].$$

Now define:

$$E_h(b, \gamma, p) = \text{Span}\left\{T_\mu^k(h, \gamma, p) \colon \mu \in b + \mathbb{Z},\ \ k \in \mathbb{N}\right\},$$

and

$$E_h(a, b, \gamma, p) = r_a^\gamma\Big[E_h(b, \gamma, p)\Big].$$

Throughout this paper, we will write E_h for $E_h(b, \gamma, p)$, and T_μ^k for $T_\mu^k(h, \gamma, p)$, whenever the parameters are fixed by the context.

It is not hard to check that the maps T_μ^k are linearly independent, and so we may define the *degree* of T_μ^k to be k. This gives a filtration of E_h by degree:

$$E_h^0 \subset E_h^1 \subset E_h^2 \subset \cdots \subset E_h.$$

It will be useful, because $\mathcal{W}$ acts on it by operators of degree 1. The basic properties of $E_h(b, \gamma, p)$ and $E_h(a, b, \gamma, p)$ are summarized by the following lemma, whose proof follows from Equation (1).

LEMMA 2.1. *The μ-weight space $E_h(b, \gamma, p)_\mu$ has basis $\{T_\mu^k \colon k \in \mathbb{N}\}$, and so it is infinite-dimensional. Although the filtration $E_h^k(b, \gamma, p)$ is not invariant under $\mathcal{W}$, it does have weight spaces: the μ-weight space $E_h^k(b, \gamma, p)_\mu$ is the $(k+1)$-dimensional span of the T_μ^j with $0 \leqslant j \leqslant k$.*

The adjoint action of $\mathcal{W}$ on $T_\mu^k(h, \gamma, p)$ is:

$$\begin{aligned}\sigma(e_n)T_\mu^k = T_{\mu+n}^{k+1}(1 - \mathrm{e}^{nh}) + T_{\mu+n}^k n\Big[\frac{\mu}{n} + p + \gamma - (\gamma + k)\mathrm{e}^{nh}\Big] - \\ - \mathrm{e}^{nh}\sum_{j=0}^{k-1} T_{\mu+n}^j n^{k-j+1}\left[\binom{k}{j-1} + \gamma\binom{k}{j}\right],\end{aligned} \tag{2}$$

where we use the convention that $\binom{k}{j} = 0$ for $j < 0$. Therefore, $\sigma(e_n)$ maps $E_h^k(b, \gamma, p)_\mu$ to $E_h^{k+1}(b, \gamma, p)_{\mu+n}$, and in particular, $E_h(b, \gamma, p)$ is invariant under σ.

The space $E_h(a, b, \gamma, p)$ is also invariant under σ, and the restriction map r_a^γ gives a $\mathcal{W}$-isomorphism from $E_h(b, \gamma, p)$ to $E_h(a, b, \gamma, p)$, so the representation structure of $E_h(a, b, \gamma, p)$ is independent of a.

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2.4. INTERTWINING MAPS AND SUBREPRESENTATIONS

In this section we exhibit some subrepresentations of the $E_h(b, \gamma, p)$, and some intertwining maps between them. These will allow us to restrict the parameters h, b, γ, and p so as to eliminate most equivalences, and to explain the reducibility of our representations at certain special values of the parameters.

EQUIVALENCES. First, $E_h(b, \gamma, p)$ and $E_h(b+1, \gamma, p)$ are equal, and so henceforth we assume $0 \leqslant \mathrm{Re}(b) < 1$. Here we will define a fundamental domain $\mathcal{H}$ for the action of the equivalences we will define on the parameter h. This will permit us to assume that h is in $\mathcal{H}$ for the rest of the paper.

For $n \in \mathbb{Z}$, let $\rho_n : \mathcal{E} \to \mathcal{E}$ (see Section 2.2) be right multiplication by $\exp(2i\pi n\tilde{e}_0)$. It is clear from Equation (2) that ρ_n defines an equivalence from E_h to $E_{h+2i\pi n}$, mapping $T^k_\mu(h, \gamma, p)$ to $T^k_\mu(h + 2i\pi n, \gamma, p)$. However, one should note that $E_h(b, \gamma, p)$ and $E_{h+2i\pi}(b, \gamma, p)$ are not the same space, as $\mathrm{e}^{2i\pi\tilde{e}_0}$ maps z^λ to $\mathrm{e}^{2i\pi\lambda} z^\lambda$. This corresponds to the fact that z^λ is multivalued unless λ is in $\mathbb{Z}$.

There is also an equivalence β from $E_h(b, \gamma, p)$ to $E_{-h}(b, 1-p-\gamma, p)$, defined by:

$$\beta\Big[T^k_\mu(h, \gamma, p)\Big] = r_{1-p-\gamma}\Big[\mathrm{e}^{-h\tilde{e}_0}(-\tilde{e}_0)^k z^\mu M^p\Big]$$
$$= (-1)^k \mathrm{e}^{-h\mu} r_{1-p-\gamma}\Big[M^p z^\mu (\tilde{e}_0 + \mu)^k \mathrm{e}^{-h\tilde{e}_0}\Big].$$

It satisfies $\beta^2 = 1$, and may be understood as conjugation of differential operators, or algebraically as the adjoint map corresponding to the following nondegenerate $\mathcal{W}$-invariant form on $\mathcal{A}$, which explicitly identifies the dual of $A(a, \gamma)$ with $A(-a, 1-\gamma)$:

$$B(M^\gamma z^\lambda, M^{\gamma'} z^{\lambda'}) = \delta(\gamma + \gamma' - 1)\delta(\lambda + \lambda').$$

The two equivalences ρ_1 and β generate a group of equivalences, which is isomorphic to the semidirect product $\mathbb{Z}_2 \times_s \mathbb{Z}$. Its action on the parameter h is faithful, and it is easy to see that the set

$$\mathcal{H} = \Big\{h \in \mathbb{C}\colon 0 \leqslant \mathrm{Im}(h) \leqslant \pi,\ \ \mathrm{Re}(h) \geqslant 0 \text{ if } \ \mathrm{Im}(h) = 0 \text{ or } \pi\Big\}$$

is a fundamental domain for this action. Therefore, up to equivalence we may, and henceforth do, assume that h is in $\mathcal{H}$.

INVOLUTIONS. The only points of $\mathcal{H}$ that have a non-trivial stabilizer under the action of $\mathbb{Z}_2 \times_s \mathbb{Z}$ are 0 and $i\pi$. Therefore there are equivalences which do not change these values of h, and when $p = 1 - 2\gamma$ they turn out to be non-trivial involutions.

The Case $h = i\pi$. One checks that $\beta \circ \rho_{-1} = \mathrm{e}^{2i\pi b}\rho_1 \circ \beta$, and so

$$\alpha \ = \ \mathrm{e}^{i\pi b}\rho_1 \circ \beta\colon\ E_{i\pi}(b, \gamma, p) \to E_{i\pi}(b, 1-p-\gamma, p)$$

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is an equivalence such that $\alpha^2 = 1$. Hence it is an involution of $E_{i\pi}(b, \gamma, 1-2\gamma)$, and we define $E_{i\pi,\pm}(b, \gamma, 1-2\gamma)$ to be its ± 1-eigenspaces. Then there is a $\mathcal{W}$-splitting:

$$E_{i\pi}(b, \gamma, 1-2\gamma) = E_{i\pi,+}(b, \gamma, 1-2\gamma) \oplus E_{i\pi,-}(b, \gamma, 1-2\gamma),$$

and we can write explicit bases for the summands as follows. Define:

$$S^k_\mu(h, \gamma, p) = r_\gamma\Big[M^p z^{\mu/2} \tilde{e}_0^k \mathrm{e}^{h\tilde{e}_0} z^{\mu/2}\Big] = \mathrm{e}^{h\mu/2} r_\gamma\Big[M^p z^\mu (\tilde{e}_0 + \mu/2)^k \mathrm{e}^{h\tilde{e}_0}\Big];$$

then:

$$\beta\Big[S^k_\mu(h, \gamma, p)\Big] = (-1)^k S^k_\mu(-h, \gamma, p),$$
$$\rho_n\Big[S^k_\mu(h, \gamma, p)\Big] = \mathrm{e}^{-i\pi\mu n} S^k_\mu(h + 2i\pi n, \gamma, p),$$

and so $S^k_\mu(i\pi, \gamma, p)$ is an eigenvector of α of eigenvalue $(-1)^{k+\mu-b}$. Therefore the $S^k_\mu(i\pi, \gamma, 1-2\gamma)$ with $k + \mu - b$ even, and odd, form bases of $E_{i\pi,+}(b, \gamma, 1-2\gamma)$, and $E_{i\pi,-}(b, \gamma, 1-2\gamma)$, respectively. This can be pictured as follows: if we think of the S^k_μ as points on a half-infinite lattice indexed by μ and k, and color the lattice as a chess board, then the decomposition into $E_{i\pi,\pm}$ is along colors.

The Case $h = 0$. The other special value of h is 0. Here β is an equivalence from $E_0(b, \gamma, p)$ to $E_0(b, 1-p-\gamma, p)$, and hence an involution of $E_0(b, \gamma, 1-2\gamma)$. One checks as above that its $+1$ and -1-eigenspaces are the spans of the maps $S^k_\mu(0, \gamma, 1-2\gamma)$ with k even and odd, respectively, and so here the picture of the decomposition is along stripes of alternating color.

QUOTIENT AND SUBREPRESENTATIONS. The cases that γ is 0 or $1-p$ are special: $E_h(b, 0, p)$ contains a copy of $E_h(b, 1, p-1)$, and $E_h(b, 1-p, p)$ contains a copy of $E_h(b, 1-p, p-1)$. In both cases, the quotient is equivalent to $A(b, p)$, and one case may be obtained from the other via the equivalence β. These facts are consequences of the equivalence $A(a, 0) \cong A(a, 1)$ for non-zero a; the details are as follows.

The Case $\gamma = 0$. It is immediate from Equation (2) that the subspace

$$E^{>0}_h(b, 0, p) = \mathrm{Span}\Big\{T^k_\mu(h, 0, p) \colon \mu \in b + \mathbb{Z},\ k > 0\Big\}$$

of $E_h(b, 0, p)$ is $\mathcal{W}$-invariant, and

$$E_h(b, 0, p)/E^{>0}_h(b, 0, p) \cong A(b, p),$$

by the map $T^0_\mu \mapsto M^p z^\mu$. This can be understood in terms of the isomorphism $A(a, 0) \cong A(a, 1)$ for $a \neq 0$, which in our language is the map $r_0(M\tilde{e}_0)\colon A(0) \to A(1)$. More precisely, the right multiplication operator $R[r_0(M\tilde{e}_0)]$ is an injective $\mathcal{W}$-map from $E_h(b, 1, p-1)$ to $E_h(b, 0, p)$, whose image is $E^{>0}_h(b, 0, p)$. In this context, let us simply write $M\tilde{e}_0$ instead of $r_0(M\tilde{e}_0)$.

The Case $\gamma = 1 - p$. Changing the sign of h and then applying β in the last paragraph, we see that the left multiplication operator $L(M\tilde{e}_0)$ is an injective $\mathcal{W}$-map from $E_h(b, 1-p, p-1)$ to $E_h(b, 1-p, p)$. Its image is $\beta[E_{-h}^{>0}(b, 0, p)]$, and since β is an equivalence, the associated quotient is again a copy of $A(b, p)$.

The Case $\gamma = 0$, $p = 1$. This is the only simultaneous occurrence of the two preceding cases: $L(M\tilde{e}_0)$ injects $E_h(b, 1, -1)$ into $E_h(b, 1, 0)$, which is injected into $E_h(b, 0, 1)$ by $R(M\tilde{e}_0)$. In the remark on cohomology in Section 5, we prove that the resulting quotient is trivial:

$$E_h(b, 0, 1)\Big/\Big(M\tilde{e}_0\Big[E_h(b, 1, -1)\Big]M\tilde{e}_0\Big) \cong A(b, 1) \oplus A(b, 0).$$

The Case $h = i\pi$, $\gamma = 0$, $p = 1$, $b \neq 0$. If $E_{i\pi}(b, \gamma, p)$ admits both the involution α and a quotient $A(b, p)$ in either of the above ways, then necessarily $\gamma = 0$ and $p = 1$. In this case, we define:

$$E_{i\pi,\pm}^{>0}(b, 0, 1) = E_{i\pi,\pm}(b, 0, 1) \cap E_{i\pi}^{>0}(b, 0, 1).$$

Suppressing the obvious parameters, one checks that when $b \neq 0$, the weight spaces of $E_{i\pi,+}^{>0}$, or $E_{i\pi,-}^{>0}$, are of codimension 1 in the weight spaces of $E_{i\pi,+}$, or $E_{i\pi,-}^{>0}$, and that they have bases given by $S_\mu^k - (\mu/2)^2 S_\mu^{k-2}$, where $k \geqslant 2$, and $k + \mu - b$ is even, or odd, respectively. Both $E_{i\pi,+}$ and $E_{i\pi,-}$ have full image in the quotient $E_{i\pi}/E_{i\pi}^{>0} \cong A(b, 1)$, as S_μ^k and $(\mu/2)^k T_\mu^0$ have the same image under projection. Similar reasoning gives:

$$E_{i\pi,\pm}/E_{i\pi,\pm}^{>0} \cong A(b, 1), \qquad E_{i\pi,\pm}^{>0} = M\tilde{e}_0\Big[E_{i\pi,\pm}(b, 1, -1)\Big]M\tilde{e}_0.$$

The Case $h = i\pi$, $\gamma = 0$, $p = 1$, $b = 0$. When furthermore $b = 0$, the only additional change in the picture is that the 0-weight space $E_{i\pi,-}(0, 0, 1)_0$ is contained in $E_{i\pi}^{>0}$, and so:

$$E_{i\pi,-}/E_{i\pi,-}^{>0} \cong E_{i\pi}^{>0}\Big/\Big(E_{i\pi,+}^{>0} \oplus E_{i\pi,-}^{>0}\Big) \cong \tilde{A}.$$

Here $E_{i\pi,-}^{>0}(0, 0, 1)$ contains $M\tilde{e}_0[E_{i\pi,-}(0, 1, -1)]M\tilde{e}_0$ properly, and the quotient is a trivial representation.

The Case $h = 0$. It will be useful to collect some elementary facts about $E_0(b, \gamma, p)$. It follows from Equation (2) that the filtration E_0^k is $\mathcal{W}$-invariant, and the subquotient E_0^k/E_0^{k-1} is equivalent to $A(b, p-k)$, by the map $T_\mu^k \mapsto M^{p-k} z^\mu$. We have seen that β is an equivalence from $E_0(b, \gamma, p)$ to $E_0(b, 1-p-\gamma, p)$, and hence an involution of $E_0(b, \gamma, 1-2\gamma)$.

When $\gamma = 0$, we have the $\mathcal{W}$-splitting:

$$E_0(b, 0, p) = E_0^0(b, 0, p) \oplus E_0^{>0}(b, 0, p).$$

For future reference, note that $E_0^0(b, 0, p)$ is equivalent to $A(b, p)$, which is equivalent

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to the quotient $E_h(b,0,p)/E_h^{>0}(b,0,p)$, for any h. Therefore there exists an intertwining map from $E_h(b,0,p)$ to $E_0(b,0,p)$. Applying β gives a dual statement at $\gamma = 1-p$. As we saw above, $E_0^{>0}(b,0,p)$ is equivalent to $E_0(b,1,p-1)$, and in $E_0(b,0,1)$ we have all of the subrepresentations of this paragraph and the last, so it splits as the direct sum of $A(b,1)$, $A(b,0)$, and the images of the ± 1-eigenspaces of β acting on $E_0(b,1,-1)$.

3. Irreducibility

In the first theorem of this section we prove that the representations $E_h(b,\gamma,p)$ are irreducible, provided that h is not contained in a certain finite set $\mathcal{F}$, and neither of the roots of a certain quadratic, whose coefficients depend on γ and p, lie in $\mathbb{Z}^+$. The idea of the proof is to use the Casimir operators Q_n to construct operators $N_{-1}(n)$ of weight 0, that lower the degree of elements of E_h by 1. Since Q_n itself raises the degree by 2, we must use a linear combination of distinct Q_n that cancels the higher degree terms.

The other two theorems treat irreducibility when $h = 2i\pi m'/m$ for $m > 2$, and when $h = i\pi$. All three theorems impose certain conditions on γ and p, which for example rule out the cases that γ is 0 or $1-p$, where E_h is known to reduce (see Section 2.4). We begin by computing the action of Q_n on T_μ^k, modulo E_h^{k-2}. For convenience, we use the hyperbolic version of the classical function versine: $\operatorname{versh} z = 1 - \cosh z$.

LEMMA 3.1. *Define*

$$A_0(k) = \gamma(\gamma + p - 1) - \tfrac{1}{2}k(k+1-2p),$$
$$A_{-1}(k) = \gamma(\gamma + p - 1) - \tfrac{1}{6}(k-1)(k+1-3p).$$

Then modulo E_h^{k-2}, we have the formula:

$$\begin{aligned} &-n^2\sigma(Q_n)T_\mu^k \\ &\quad\equiv 2T_\mu^{k+2}\operatorname{versh} nh + \\ &\qquad + 2T_\mu^{k+1}\Big[(p-k-1)n\sinh nh + \mu \operatorname{versh} nh\Big] - \\ &\qquad - 2T_\mu^{k}\Big[\tfrac{1}{2}(p-k)(p-k-1)n^2 + A_0(k)n^2\operatorname{versh} nh + \mu(k+\gamma)n\sinh nh\Big] + \\ &\qquad + 2T_\mu^{k-1}\Big[kA_{-1}(k)n^3\sinh nh - \tfrac{1}{2}\mu k(2\gamma+k-1)n^2\cosh nh\Big]. \end{aligned} \tag{3}$$

Proof. One way to prove this is simply to write down the definitions of Q_n, σ, and T_μ^k, and calculate. This is essentially what we will do, but with a few organizational tricks to bring us down to the case that $\mu = 0$. The first step is to calculate $-n^2\sigma(Q_n)T_0^k$. Since e_0 annihilates vectors of weight 0, this is $\sigma(e_{-n}e_n)T_0^k$. Use

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Equation (2) to write down $\sigma(e_n)T_0^k$ modulo E_h^{k-3}, and apply $\sigma(e_{-n})$ to it to prove the lemma for $\mu = 0$; this requires a long computation, which can be done with a computer.

Now let us write L and R for the left and right actions of End($\mathcal{A}$) on itself: if τ and T are operators on $\mathcal{A}$, then $L(\tau)T = \tau T$ and $R(\tau)T = T\tau$. The next step is to prove that as operators on $\mathcal{E}$,

$$[n^2\sigma(Q_n), L(z^\mu)] = \mu L(z^\mu)\Big[R(\tilde{e}_{-n})\,\mathrm{ad}(z^n) + R(\tilde{e}_n)\,\mathrm{ad}(z^{-n})\Big]. \tag{4}$$

For this, recall that $\sigma(e_n) = L(\tilde{e}_n) - R(\tilde{e}_n)$, which leads to $[\sigma(e_n), L(z^\mu)] = \mu L(z^{\mu+n})$. Some computation gives:

$$[n^2\sigma(Q_n), L(z^\mu)] = \mu\Big[n + 2\,\mathrm{ad}(\tilde{e}_0) - \mathrm{ad}(\tilde{e}_{-n})L(z^n) - L(z^{-n})\,\mathrm{ad}(\tilde{e}_n)\Big].$$

Now replace $\mathrm{ad}(\tilde{e}_{-n})L(z^n)$ by $L(z^n)[\mathrm{ad}(\tilde{e}_{-n}) + nL(z^{-n})]$, replace each 'ad' by '$L - R$,' and expand: the terms involving only L's will all cancel. Then move the $L(z^{\pm n})$ terms to the right of the $R(\tilde{e}_{\pm n})$ terms, and replace each 'L' by 'ad $+R$.' Expanding again cancels all the terms involving only R's, proving Equation (4).

Finally, apply the right hand side of Equation (4) to T_0^k and calculate modulo E_h^{k-2}:

$$\begin{aligned}&[n^2\sigma(Q_n), L(z^\mu)]T_0^k\\ &\quad\equiv 2\mu\Big[-T_\mu^{k+1}\,\mathrm{versh}\,nh + T_\mu^k(k+\gamma)n\sinh nh + \tfrac{1}{2}T_\mu^{k-1}k(2\gamma+k-1)n^2\cosh nh\Big].\end{aligned}$$

Since $L(z^\mu)T_0^k = T_\mu^k$, combining this with the first paragraph of the proof gives Equation (3). □

Next, we use difference operators to define linear combinations $N_2(n)$, $N_1(n)$, $N_0(n)$, and $N_{-1}(n)$ of the operators $\sigma(Q_n)$, such that $N_j(n)T_\mu^k$ is of degree $k+j$. For any function f defined on some subset of $\mathbb{N}\times\mathbb{C}$, let

$$\Delta f(n)(h) = f(n+1,h) - f(n,h),$$

and define functions:

$$\begin{aligned}
F(n,h) &= n\sinh nh/\,\mathrm{versh}\,nh,\\
G(n) &= \Delta\frac{\Delta n^2}{\Delta F}(n),\\
H(n) &= \Delta\frac{\Delta(n^2/\,\mathrm{versh}\,nh)}{\Delta F}(n),\\
I(n) &= \Delta\frac{\Delta(n^2F)}{\Delta F}(n),\\
J(n) &= \Delta\frac{\Delta(I/G)}{\Delta(H/G)}(n).
\end{aligned}$$

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Now define operators:

$$\begin{aligned}
N_2(n) &= -n^2\sigma(Q_n)/2\,\mathrm{versh}\, nh,\\
N_1(n) &= \frac{\Delta N_2}{\Delta F}(n),\\
N_0(n) &= \frac{\Delta N_1}{G}(n),\\
\hat{N}_0(n) &= \frac{\Delta N_0}{\Delta(H/G)}(n),\\
N_{-1}(n) &= \Delta \hat{N}_0(n).
\end{aligned}$$

The action of these operators on T^k_μ, modulo E^{k-2}_h, follows from Lemma 3.1:

$$\begin{aligned}
N_1(n)T^k_\mu \equiv{}& T^{k+1}_\mu(p-k-1)\\
&- T^k_\mu\Big[\frac{1}{2}(p-k)(p-k-1)\frac{\Delta(n^2/\,\mathrm{versh}\, nh)}{\Delta F} + A_0(k)\frac{\Delta n^2}{\Delta F} + \mu(k+\gamma)\Big]\\
&+ T^{k-1}_\mu\Big[kA_{-1}(k)\frac{\Delta(n^2F)}{\Delta F} - \frac{1}{2}\mu k(2\gamma+k-1)\Big(\frac{\Delta(n^2/\,\mathrm{versh}\, nh)}{\Delta F} - \frac{\Delta n^2}{\Delta F}\Big)\Big],\\
N_0(n)T^k_\mu \equiv{}& -T^k_\mu\Big[\frac{1}{2}(p-k)(p-k-1)\frac{H}{G} + A_0(k)\Big]\\
&+ T^{k-1}_\mu\Big[kA_{-1}(k)\frac{I}{G} - \frac{1}{2}\mu k(2\gamma+k-1)\Big(\frac{H}{G}-1\Big)\Big],\\
\hat{N}_0(n)T^k_\mu \equiv{}& -\frac{1}{2}T^k_\mu(p-k)(p-k-1)\\
&+ T^{k-1}_\mu\Big[kA_{-1}(k)\frac{\Delta(I/G)}{\Delta(H/G)} - \frac{1}{2}\mu k(2\gamma+k-1)\Big],\\
N_{-1}(n)T^k_\mu \equiv{}& T^{k-1}_\mu kA_{-1}(k)J(n).
\end{aligned}$$

DEFINITION. For $n \in \mathbb{Z}^+$, let $\mathcal{F}_n$ be the set of $h \in \mathbb{C}$ such that any of the following conditions is satisfied:

(1) $\mathrm{versh}\, n'h = 0$ for some n' in $\{n, \ldots, n+4\}$.
(2) $\Delta F(n')(h) = 0$ for some n' in $\{n, \ldots, n+3\}$.
(3) $G(n')(h) = 0$ for some n' in $\{n, n+1, n+2\}$.
(4) $\Delta(\frac{H}{G})(n')(h) = 0$ for some n' in $\{n, n+1\}$.
(5) $J(n)(h) = 0$.

Note that each of these conditions makes sense only if h does not satisfy any of the preceding ones. For example, $G(n)$, $G(n+1)$, and $G(n+2)$ are only defined if h does not satisfy (1) or (2). For the next lemma, recall the set $\mathcal{H}$ from Section 2.4.

LEMMA 3.2. *First, if $h \notin \mathcal{F}_n$ then $N_{-1}(n)$ is a well-defined non-zero operator. Second, for each n the set $\mathcal{F}_n$ is a finite number of additive cosets of $2i\pi\mathbb{Z}$ in $\mathbb{C}$, and so we have the finite set: $\mathcal{F} = \left(\cap_1^\infty \mathcal{F}_n\right) \cap \mathcal{H}$.*

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Proof. First, it follows from $h \notin \mathcal{F}_n$ that N_2 is defined at $n, \ldots, n+4$, that N_1 is defined at $n, \ldots, n+3$, that N_0 is defined at $n, n+1, n+2$, that $\hat{N}_0$ is defined at $n, n+1$, and that $N_{-1}(n)$ is defined and non-zero.

Second, let $q = e^h$. We show that for each n, the functions $\Delta F(n)$, $G(n)$, $\Delta(H/G)(n)$, and $J(n)$ become well-defined rational functions of q that are not identically zero. It is clear that each of the functions in question is either rational in q or undefined for all q, and so it will suffice to prove that if $|q|$ is a large enough positive real number, all of them are defined and $J(n) \neq 0$. Given two functions $t(n, q)$ and $T(n, q)$, let us say that t is of Order(T), or $t = \mathrm{O}(T)$, if there are constants C and q_0 independent of n and q such that $|t(n, q)| \leqslant CT(n, q)$ for all $n \in \mathbb{Z}^+$ and all $|q| \geqslant |q_0|$ (for our purposes, it would be enough for C and q_0 to be independent of q).

Now we easily prove that:

$$F(n)(h) = -n\frac{1+q^{-n}}{1-q^{-n}} = -n(1+\mathrm{O}q^{-n}),$$
$$\Delta F(n)(h) = -1 + \mathrm{O}(nq^{-n}),$$
$$G(n)(h) = \Delta\frac{\Delta n^2}{\Delta F} = \Delta(2n+1)(-1+\mathrm{O}nq^{-n}) = -2 + \mathrm{O}(n^2q^{-n}).$$

Thus the functions arising in the first three conditions are defined, and not identically zero. The same is true of the function $\Delta(H/G)$ arising in the fourth condition, because:

$$H(n)(h) = \Delta\frac{\Delta(n^2/\operatorname{versh} nh)}{\Delta F} = \Delta\Big[-2n^2q^{-n}(1+\mathrm{O}q^{-1})\Big] = 2n^2q^{-n}(1+\mathrm{O}q^{-1}),$$
$$\Delta\left(\frac{H}{G}\right)(n)(h) = \Delta\Big[-n^2q^{-n}(1+\mathrm{O}q^{-1})\Big] = n^2q^{-n}(1+\mathrm{O}q^{-1}).$$

Finally, we find:

$$I(n)(h) = \Delta\frac{\Delta(n^2F)}{\Delta F} = 6n + 6 + \mathrm{O}(n^3q^{-n}),$$
$$\Delta\left(\frac{I}{G}\right)(n)(h) = \Big[\Delta(-3n-3+\mathrm{O}(n^3q^{-n})\Big] = -3 + \mathrm{O}(n^3q^{-n}),$$
$$J(n)(h) = \Delta\frac{\Delta(I/G)}{\Delta(H/G)} = \Delta\Big[-3n^{-2}q^{n}(1+\mathrm{O}q^{-1})\Big] = -3(n+1)^{-2}q^{n+1}(1+\mathrm{O}q^{-1}).$$

Therefore $J(n)$ is not identically zero in q for any n.

To prove the second point of the lemma, it suffices to remark that $\mathcal{F}_n$ is the inverse image under the exponential map of the finite set of poles and zeroes of the following functions of q: $\operatorname{versh} n'h$ for $n' = n, \ldots, n+4$, ΔF at $n, \ldots, n+3$, G at $n, n+1, n+2$, $\Delta(H/G)$ at $n, n+1$, and $J(n)$. In fact, the set of poles at each step is included in the set of zeroes and poles of the previous step. □

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THEOREM 3.3. *If $h \notin \mathcal{F}$ and the roots of the quadratic $A_{-1}(k)$ are not positive integers, then $\mathcal{W}$ acts irreducibly on $E_h(b, \gamma, p)$.*

Proof. Recall from Section 2.4 that we are assuming $h \in \mathcal{H}$, and so $h \notin \mathcal{F}$ implies $h \notin \mathcal{F}_n$ for some n. By Lemma 3.2 and our assumption on A_{-1}, the operator $N_{-1}(n)$ maps elements of weight μ and degree k to elements of weight μ and degree exactly $k-1$. Suppose that W is a non-zero $\mathcal{W}$-subrepresentation of E_h; then for some μ and k, it contains an element R^k_μ of weight μ and degree k. Then $R^j_\mu = N_{-1}(n)^{k-j} R^k_\mu$ has degree exactly j, and so since $N_{-1}(n)$ is the image of an element of the universal enveloping algebra $\mathfrak{U}(\mathcal{W})$ under σ, W contains the span of $\{R^0_\mu, \dots, R^k_\mu\}$, which is all of $E^k_{h,\mu}$. Finally, versh $nh \neq 0$ by the first condition above, and so $h \notin 2i\pi\mathbb{Z}/n$. Therefore by Equation (2), $\sigma(e_{\pm 1})$ maps R^k_μ to an element of weight $\mu \pm 1$ and degree $k+1$, and so W contains elements of arbitrarily high degree in every weight. The theorem follows. □

It seems likely that $2i\pi m'/m \notin \mathcal{F}$ for $m \geqslant 6$, and this could perhaps be verified by a computer. However, we will prove the irreducibility of $E_{2i\pi m'/m}$ for $m \geqslant 3$ by a different argument, using the fact that the vector fields e_{mn} for $n \in \mathbb{Z}$ act on it as they do on E_0. Our proof will require different conditions on γ and p. First, we need a lemma describing the action of Q_{mn}. It is essentially Lemma 3.1 in the case that $h = 0$, extended to terms of degree $\geqslant k-2$. Its proof is much easier than that of Lemma 3.1, and we leave it to the reader.

LEMMA 3.4. *Suppose that* $\mathrm{e}^{mh} = 1$, *and define:*

$$B_{-2}(k) = \gamma(\gamma + p - 1) - \tfrac{1}{12}(k-2)(k+1-4p).$$

Then modulo E^{k-3}_h, we have the formula:

$$\sigma(Q_{mn})T^k_\mu \equiv T^k_\mu(p-k)(p-k-1) + T^{k-1}_\mu \mu k(2\gamma + k - 1) - \\ - 2T^{k-2}_\mu m^2 n^2 \binom{k}{2} B_{-2}(k). \tag{5}$$

THEOREM 3.5. *Let e^h be a primitive* mth *root of unity with $m > 2$. If the roots of the quadratic $B_{-2}(k)$ defined in Lemma 3.4 are not integers greater than* 1, *then $E_h(b, \gamma, p)$ is irreducible.*

Proof. We adapt the proof of Theorem 3.3. Let $M_{-2}(n) = -\sigma(\Delta Q_{mn})/2m^2(2n+1)$. Then modulo degree $k-3$, Lemma 3.4 gives:

$$M_{-2}(n)T^k_\mu \equiv T^{k-2}_\mu \binom{k}{2} B_{-2}(k).$$

Therefore, under our assumptions $M_{-2}(n)$ lowers the degree of any element of E_h by exactly 2, and so the proof of Theorem 3.3 goes through once we show that any non-zero subrepresentation of E_h contains elements of arbitrarily high degree *and arbitrary parity* in each weight. This follows from the fact that $e_{\pm 1}$ and e_2

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all raise the degree by exactly 1, because $m > 2$. For example, to raise the degree by 3 without changing the weight, apply $e_{-1}^2 e_2$. □

We remark that when $e^{2h} \neq 1$, the operator $N_1(1)$ from Theorem 3.3 is defined, and it increases the degree by 1 except in degree $p-1$. This can be used to prove Theorem 3.5 in most of the cases that B_{-2} does not have two consecutive integer roots greater than 1, along the line of the proof of Theorem 3.7.

It remains to consider $E_{i\pi}(b, \gamma, p)$, which we know decomposes when $p = 1 - 2\gamma$. Here $e_{\pm 2}$ does not raise the degree, and so we begin by constructing an operator that changes the parity of the degree without changing the weight when p is not 0 or $1 - 2\gamma$. The proof is a little longer than that of Lemma 3.4, but it is a direct calculation so we omit it.

LEMMA 3.6. *Let* $P_2(n) = \sigma(e_{2-4n} e_{2n-1}^2)/4$, *and let* $P_1(n) = \Delta^2 P_2(n)/16$. *Then modulo degree k:*

$$P_2(n) T_\mu^k \equiv T_\mu^{k+2}\Big[\mu + 2(2n-1)(k+3-p)\Big] + \\ + 2T_\mu^{k+1}\Big[\mu^2 + \mu(2n-1)(3k+5+\gamma-p) + 2(2n-1)^2 p(1-p-2\gamma)\Big], \tag{6}$$

and so $P_1(n) T_\mu^k \equiv T_\mu^{k+1} p(1-p-2\gamma)$.

THEOREM 3.7. *First, suppose that* γ *is not* 0, $(1-p)/2$, *or* $1-p$, *and* $p \neq 0$. *If* $B_{-2}(k)$ *does not have two consecutive integer roots greater than* 1, *then* $E_{i\pi}(b, \gamma, p)$ *is irreducible.*

Second, suppose that $p = 1 - 2\gamma$. *If* B_{-2} *has no positive even roots, then* $E_{i\pi,+}(b, \gamma, 1-2\gamma)$ *is irreducible, and if it has no odd roots greater than* 1, *then* $E_{i\pi,-}(b, \gamma, 1-2\gamma)$ *is irreducible.*

Proof. Let us begin by remarking that by Section 2.4, we know that $E_{i\pi}(b, \gamma, p)$ reduces when γ is 0, $1-p$, or $(1-p)/2$: when $\gamma = 0$ we have the subrepresentation $E_{i\pi}^{>0}(b, 0, p) \cong E_{i\pi}(b, 1, p-1)$, when $\gamma = 1 - p$ we have the subrepresentation $\beta[E_{-i\pi}^{>0}(b, 0, p)]$, and when $\gamma = (1-p)/2$ we have the involution α. However, we do not know what happens in general when $p = 0$.

Now the first statement is proven just as Theorem 3.5 was, except that when $B_{-2}(k) = 0$, we get down from degree k to degree $k-2$ by going up to degree $k+1$ with P_1, then down to $k-3$ with M_{-2}^2, and then up to $k-2$ with P_1. Note that under our assumptions, $B_{-2}(2) \neq 0$, and so this always works. Use Lemma 3.6 to prove that as long as p is not 0 or $1-2\gamma$, $E_{i\pi}(b, \gamma, p)$ contains elements of arbitrarily high degree and arbitrary parity in all weights. To prove the second statement, it is enough to use $e_{\pm 1}$ to increase the degree and M_{-2} to reduce it. □

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4. Equivalences

In this section we will prove that there are no equivalences between two different irreducible representations from Section 3, unless both of them have $h = i\pi$; in this case, the only possibility is the intertwining map α, defined in Section 2.4. Recall from that section that we are restricting the parameters so that $0 \leqslant \mathrm{Re}(b) < 1$, and h lies in the fundamental domain $\mathcal{H}$ of the action of the group of equivalences generated by ρ_1 and β on h.

Throughout this section, we consider two representations $F = E_h(b, \gamma, p)$ and $F' = E_{h'}(b', \gamma', p')$ of $\mathcal{W}$, and we write σ and σ', respectively, for the two actions of $\mathcal{W}$. We begin by stating an elementary lemma; to prove it, use n-asymptotics to come down to the case that $C = 0$, and then use the identity $\operatorname{versh} 2z / \operatorname{versh} z = 2(1 + \cosh z)$.

LEMMA 4.1. *Suppose that h and h' are in $\mathcal{H}$, and A, B, and C are constants such that $A \neq 0$ and $A \operatorname{versh} nh' = B \operatorname{versh} nh + Cn \sinh nh$ for all $n \in \mathbb{N}$. Then either $h = h'$, $A = B$, and $C = 0$; or $h = h' = i\pi$ and $A = B$; or $h = h' = 0$; or $h' = B = C = 0$.*

The essential part of the next proposition is that if F and F' are equivalent, then $h = h'$. In order to simultaneously handle equivalence questions regarding the subrepresentations $E_{i\pi,\pm}$ occurring at $h = i\pi$ (and any hitherto undiscovered subrepresentations occurring at the special values of the parameters not covered by the results of Section 3), we have written it to apply to arbitrary subrepresentations.

PROPOSITION 4.2. *Let $V \subset F$ and $V' \subset F'$ be $\mathcal{W}$-subrepresentations, and suppose that $\tau\colon V \to V'$ is a non-zero intertwining map. First, $b = b'$, and τ maps the weight space V_μ into V'_μ. Second, h' is either h or 0. Third, if τ is an equivalence, then $h = h'$.*

Proof. We introduce some notation. If V is any subrepresentation of F, it is not hard to see that we can pick elements R^k_μ of V_μ, such that R^k_μ is either zero, or of degree k and congruent to T^k_μ modulo degree $k - 1$, which have the property that the set of non-zero R^k_μ form a basis of V. Note that if $h \neq 0$ and $R^k_\mu \neq 0$, then R^{k+2}_μ cannot be zero by Equation (3), as $\sigma(Q_1)R^k_\mu$ is an element of V_μ of degree $k + 2$.

Suppose in addition that V' is a subrepresentation of F', and let R'^k_μ be a similar basis of V'. Then if $\tau\colon V \to V'$ is an equivalence, whenever $\tau(R^k_\mu) \neq 0$ we can define scalars $J_k(\mu)$ and $\tau_{k,j}(\mu)$ by the equation:

$$\tau(R^k_\mu) = \sum_{j=0}^{J_k(\mu)} R'^j_\mu \tau_{k,j}(\mu),$$

where $R'^{J_k(\mu)}_\mu$ and $\tau_{k,J_k(\mu)}(\mu)$ are not zero, and $\tau_{k,j}(\mu) = 0$ whenever $R'^j_\mu = 0$. We will usually suppress the μ-dependence of these scalars.

The first statement of the proposition holds because τ commutes with e_0. For the second, we retain the notation above, and begin with the case $h \neq 0$. Whenever

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R^k_μ is non-zero, Equation (3) gives:

$$-\tfrac{1}{2}n^2\sigma(Q_n)(R^k_\mu) \equiv R^{k+2}_\mu \operatorname{versh} nh + R^{k+1}_\mu\Big[(p-k-1)n\sinh nh + C_k \operatorname{versh} nh\Big] \tag{7}$$

modulo degree k, for some constant C_k. Choose K minimal such that $\tau(R^K_\mu) \neq 0$, and consider the equality $\sigma'(Q_n)\tau(R^K_\mu) = \tau\sigma(Q_n)(R^K_\mu)$ modulo degree $J_K + 1$. We have:

$$\begin{aligned} &R'^{J_K+2}_\mu \tau_{K,J_K} \operatorname{versh} nh' \\ &\quad\equiv -\tfrac{1}{2}n^2\sigma'(Q_n)\tau(R^K_\mu) \equiv -\tfrac{1}{2}n^2\tau\sigma(Q_n)(R^K_\mu) \\ &\quad\equiv \sum_j R'^j_\mu\Big(\tau_{K+2,j}\operatorname{versh} nh + \tau_{K+1,j}\Big[(p-k-1)n\sinh nh + C_K \operatorname{versh} nh\Big]\Big), \end{aligned} \tag{8}$$

as τ maps terms of degree $\leqslant K$ to terms of degree $\leqslant J_K$. The sum over j is necessarily of degree at most J_K+2, and an application of Lemma 4.1 to the degree J_K+2 term of the equation gives $h' = h$ or 0.

Now if $h=0$, $\sigma(Q_n)$ does not increase degree, and so Equation (7) becomes $\sigma(Q_n)R^k_\mu \equiv 0$ modulo degree k, which leads to $\tau\sigma(Q_n)(R^K_\mu) \equiv 0$ modulo degree J_K+1. Therefore Equation (8) becomes $R'^{J_K+2}_\mu\tau_{K,J_K}\operatorname{versh} nh' \equiv 0$, which gives $h' = 0$. Note that the case $h'=0$ does occur for all values of h, for example when $\gamma = \gamma' = 0$, as we saw Section 2.4.

To prove the third statement, it suffices to apply the second to τ^{-1}. □

In light of Proposition 4.2, we may assume henceforth that $h = h'$ and $b = b'$. Now we prove the main results of this section.

THEOREM 4.3. *Suppose that* $F = E_h(b,\gamma,p)$ *and* $F' = E_h(b,\gamma',p')$ *both satisfy the conditions of Theorem 3.3. If* $\tau\colon F \to F'$ *is an equivalence, then* $F = F'$.

Proof. Let us establish some notation. Define $N'_j(n)$ and $\hat{N}'_0(n)$ to be the operators on F' analogous to $N_j(n)$ and $\hat{N}_0(n)$. Note that since $h = h'$, we have $N'_j(n)\tau = \tau N_j(n)$ for all j. Fix n_0 so that h is not in the set $\mathcal{F}_{n_0}$ of Section 3, and during this proof, write simply N_j for $N_j(n_0)$, and N'_j for $N'_j(n_0)$, and so on.

Given any map θ from F to F', there are unique maps $\theta^{(i)}\colon F \to F'$ and scalars $\theta^{(i)}(k,\mu)$, defined by:

$$\theta = \sum_{\mathbb{Z}} \theta^{(i)}, \qquad \theta^{(i)}(T^k_\mu) = \theta^{(i)}(k,\mu)T^{k+i}_\mu.$$

We say that $\theta^{(i)}$ is the *degree i part of* θ, and we usually suppress the μ-dependence of the scalars. This notation is cumbersome, but it expedites the proof that $\gamma = \gamma'$. Note that if $k+i<0$, then $\theta^{(i)}(k) = 0$. If $\theta = \sum_{i\leqslant I}\theta^{(i)}$ and $\theta^{(I)} \neq 0$, we say that the map θ is of *degree I*, and we will also use this terminology for endomorphisms of F and F'; for example, N_j and N'_j are of degree j.

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We begin by proving that $p = p'$. We have the following facts: $\tau N_{-1}^k = N_{-1}'^k \tau$, the kernel of N_{-1}^k is F^{k-1}, the kernel of $N_{-1}'^k$ is F'^{k-1}, and τ is invertible. These lead to $\tau(F^k) = F'^k$ for all k, i.e., τ is of degree 0. Therefore, the degree 0 part of the equation $\hat{N}_0'\tau = \tau\hat{N}_0$ is:

$$\hat{N}_0'^{(0)}\tau^{(0)} = \tau^{(0)}\hat{N}_0^{(0)}.$$

Since $\hat{N}_0^{(0)}(k) = -\frac{1}{2}(p-k)(p-k-1)$, and similarly in the primed case, applying this equation to T_μ^k yields:

$$(p'-k)(p'-k-1)\tau^{(0)}(k) = (p-k)(p-k-1)\tau^{(0)}(k)$$

for all k. Since $\tau^{(0)}(k) \neq 0$, this gives $p' = p$.

Now we prove that γ' must be either γ or $1-p-\gamma$, and that $\tau^{(0)}(k)$ is independent of k. A analysis of the degree 0 part of $N_0'\tau = \tau N_0$ similar to that above yields $A_0'(k) = A_0(k)$ for all k, which proves that γ' is either γ or $1-p-\gamma$. Next, $p' = p$ gives:

$$N_1^{(1)}(k) = N_1'^{(1)}(k) = p-k-1,$$

and so the degree 1 part of $N_1'\tau = \tau N_1$, applied to T_μ^k, gives $\tau^{(0)}(k) = \tau^{(0)}(k+1)$, unless $k = p-1$. A similar argument, using the degree 2 part of $N_2'^{(2)}\tau = \tau N_2^{(2)}$, gives $\tau^{(0)}(k) = \tau^{(0)}(k+2)$, and so $\tau^{(0)}(k)$ is independent of k.

Finally, we prove that γ is γ'. To prove that it is not $1-p-\gamma$, it is not enough to consider only top degree parts as above. Applying the degree 1 part of $N_2'\tau = \tau N_2$ to T_μ^k gives:

$$N_2'^{(2)}(k-1)\tau^{(-1)}(k) + N_2'^{(1)}(k)\tau^{(0)}(k) = \tau^{(-1)}(k+2)N_2^{(2)}(k) + \tau^{(0)}(k+1)N_2^{(1)}(k),$$

where:

$$N_2^{(2)}(k) = N_2'^{(2)}(k) = 1, \qquad N_2^{(1)}(k) = N_2'^{(1)}(k) = (p-k-1)F(n) + \mu.$$

This leads to $\tau^{(-1)}(k+2) = \tau^{(-1)}(k)$, and at $k = 0$, to $\tau^{(-1)}(2) = 0$. Therefore:

$$\tau^{(-1)}(2k) = 0, \qquad \tau^{(-1)}(2k+1) = \tau^{(-1)}(1),$$

for all k. Finally,

$$N_1'^{(0)}(k) - N_1^{(0)}(k) = \mu(\gamma - \gamma'),$$

and so a similar analysis of the degree 0 part of $N_1'\tau = \tau N_1$, applied to T_μ^{2k}, gives:

$$\mu(\gamma - \gamma')\tau^{(0)}(0) = (p-2k-1)\tau^{(-1)}(1)$$

for all k. Therefore $\tau^{(-1)}(1) = 0$, and $\gamma' = \gamma$. □

THEOREM 4.4. *Suppose that e^h is a root of unity other than ± 1, and $F = E_h(b, \gamma, p)$ and $F' = E_h(b, \gamma', p')$ both satisfy the conditions of Theorem 3.5. If $\tau\colon F \to F'$ is an equivalence, then $F = F'$.*

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Proof. Define $M'_{-2}(n)$ to be the operator on F' analogous to $M_{-2}(n)$, and note that $M'_{-2}(n)\tau = \tau M_{-2}(n)$. We may fix and suppress n. Since $\ker M^k_{-2} = F^{2k-1}$, the equation $\tau M^k_{-2} = M'^k_{-2}\tau$ gives $\tau(F^{2k-1}) = F'^{2k-1}$. Similarly,

$$\ker\Big[M^{k+1}_{-2}N_1(1)\Big] \cap \ker\Big[M^{k+2}_{-2}N_1(1)N_2(1)\Big] = F^{2k}$$

(unless $2k = p-1$, it is enough to use $\ker M^{k+1}_{-2}N_1$), and so we get $\tau(F^{2k}) = F'^{2k}$. Using this, the fact that $p = p'$ follows from $\sigma'(Q_{mn})\tau = \tau\sigma(Q_{mn})$ and Lemma 3.4. We find that $\tau^{(0)}(k)$ is independent of k just as before, and from this, $\tau M_{-2} = M'_{-2}\tau$ gives $B_{-2} = B'_{-2}$, which implies that γ' is either γ or $1-p-\gamma$. Finally, the same argument used for Theorem 4.3 gives $\gamma = \gamma'$.

THEOREM 4.5. *Suppose that $h = i\pi$. First, if $F = E_{i\pi}(b,\gamma,p)$ and $F' = E_{i\pi}(b,\gamma',p')$ both satisfy the conditions of the first statement of Theorem 3.7, and $\tau\colon F \to F'$ is an equivalence, then either $F = F'$, or τ is a multiple of α and $F' = E_{i\pi}(b, 1-p-\gamma, p)$.*

Second, suppose that W is one of $E_{i\pi,\pm}(b,\gamma,1-2\gamma)$ and W' is one of $E_{i\pi,\pm}(b,\gamma',1-2\gamma')$, and that both satisfy the conditions of the second statement of Theorem 3.7. If $\tau\colon W \to W'$ is an equivalence, then $W = W'$.

Third, $E_{i\pi}(b,\gamma,p)$ is not equivalent to either of $E_{i\pi,\pm}(b,\gamma',1-2\gamma')$.

Proof. First, we find that $\tau(F^k) = F'^k$ just as in Theorem 4.4, except that we use P_1 instead of N_1. Then we get $p = p'$ from $\sigma'(Q_2)\tau = \tau\sigma(Q_2)$, and we get $\tau^{(0)}(k+2) = \tau^{(0)}(k)$ from $N'_2\tau = \tau N_2$. Applying these facts to $M'_{-2}\tau = \tau M_{-2}$ gives $B'_{-2} = B_{-2}$, which implies that γ' is either γ or $1-p-\gamma$. This proves the first statement: both cases are possible.

For the second statement, suppose first that W and W' have opposite 'signs,' say $W = E_{i\pi,+}$ and $W' = E_{i\pi,-}$. Then the equation $M'_{-2}\tau = \tau M_{-2}$ implies that τ is of degree 1 in weight μ for $\mu - b$ even, and of degree -1 in weight μ for $\mu - b$ odd. Applying the equation $\sigma'(Q_2)\tau = \tau\sigma(Q_2)$ to the case that $\mu - b$ is even gives $p' = p+1$, but applying it to the case that $\mu - b$ is odd gives $p' = p-1$, so W and W' are not equivalent. If W and W' have the same sign, these arguments show that τ is of degree 0 and $p = p'$, so $W = W'$. The third statement follows from a consideration of the sizes of the kernels of M^k_{-2} and M'^k_{-2} in each weight. □

5. Remarks

We conclude with a few remarks. First, in the works [CMZ, BOv, GaOv, Mat], the eigenspaces of the Casimir operators Q_n play a crucial role in the analysis of E_0, but for nh not in $2i\pi\mathbb{Z}$, they act on E_h without eigenvalues, because they raise the degree. However, the operators $\hat{N}_0(n)$ defined in Section 3 leave the degree invariant, and in fact they have the same eigenvalues on E_h that Q_n has on E_0. It would be interesting to find some analogy between Q_n and $\hat{N}_0(n)$, perhaps by explaining the significance of the eigenspaces of the latter.

COHOMOLOGY. Here we show that $E_h(0, 1, -1)$ has a non-trivial 1-cohomology group for all h, and we justify the claim that $E_h(b, 0, 1)$ has a quotient equivalent to $A(b, 1) \oplus A(b, 0)$, made in Section 2.4.

Recall from Section 2.4 that $E_h(b, 0, p)$ is an extension of $A(b, p)$ by $E_h(b, 1, p-1)$, and that this extension splits when $h = 0$. It does not split for any non-zero value h_0 of h, for if it did, we could construct a non-zero intertwining map from $E_0(b, 0, p)$ to $E_{h_0}(b, 0, p)$, with kernel equivalent to $E_0(b, 1, p-1)$, and image equivalent to $A(b, p)$, which is impossible by Proposition 4.2. Therefore the representation on the space of maps from $A(b, p)$ to $E_h(b, 1, p-1)$ has non-zero 1-cohomology for all non-zero h.

Since $A(0, 0)$ contains the trivial representation D_0, an extension of D_0 by $E_h(0, 1, -1)$ is contained in $E_h(0, 0, 0)$. It is clear from Equation 1 that when h is non-zero, $E_h(0, 0, 0)$ does not contain D_0 as a subrepresentation, and so this extension cannot be trivial. Therefore $E_h(0, 1, -1)$ has a non-trivial 1-cocycle δ_h. However, it is a subrepresentation of the representation on the space of maps from $A(1)$ to $A(0)$, and in [MP2] it is proven that up to coboundaries, all cocycles of this representation take values in $E_0(0, 1, -1)$. This means that δ_h must be cohomologous to an $E_0(0, 1, -1)$-valued 1-cocycle, by a coboundary taking values in the larger space of all maps from $A(1)$ to $A(0)$.

In fact, δ_h is trivial in the cohomology of this larger space. To see this, note that if we regard $E_h(0, 1, -1)$ as a subrepresentation of $E_h(0, 0, 0)$, then unwinding the definitions gives $\delta_h = \partial(\mathrm{e}^{h\tilde{e}_0})$. Since the injection is right multiplication by $M\tilde{e}_0$, transporting $\mathrm{e}^{h\tilde{e}_0}$ back to $E(0, 1, -1)$ gives $M^{-1}\mathrm{e}^{h\tilde{e}_0}/\tilde{e}_0$, and so $\delta_h = \partial[M^{-1}\mathrm{e}^{h\tilde{e}_0}/\tilde{e}_0]$ whenever $\tilde{e}_0^{-1}$ is defined, i.e. on the weight spaces of $A(1)$ of non-zero weight. A little more work gives a formula valid on all weight spaces:

$$\delta_h = \partial\Big[M^{-1}\frac{\mathrm{e}^{h\tilde{e}_0} - 1}{\tilde{e}_0}\Big].$$

Summarizing, we have the following proposition; its irreducibility statements are corollaries of Theorems 3.3, 3.5, and 3.7, and its cohomology statement follows from the above discussion.

PROPOSITION 5.1. *The representation* $E_h(0, 1, -1)$ *is irreducible if either* $h \notin \mathcal{F}$ *or* e^h *is a root of unity other than* ± 1. *The representation* $E_{i\pi}(0, 1, -1)$ *splits as the direct sum of the two representations* $E_{i\pi,\pm}(0, 1, -1)$*, both of which are irreducible. All of these irreducible representations have non-trivial 1-cohomology groups.*

Now recall our claim from the paragraph on the case that $\gamma = 0$ and $p = 1$ in Section 2.4: the quotient of $E_h(b, 0, 1)$ by $M\tilde{e}_0[E_h(b, 1, -1)]M\tilde{e}_0$ is equivalent to $A(b, 1) \oplus A(b, 0)$. To prove this, check that the images $\overline{T}^0_\mu$ and $\overline{T}^1_\mu$ of T^0_μ and T^1_μ are a basis of the quotient, and that $T^2_\mu + \mu T^1_\mu$ projects to zero in the quotient. Hence by Equation (2), the quotient action is:

$$\overline{\sigma}(e_n)\overline{T}^0_\mu = (n+\mu)\overline{T}^0_{\mu+n} + (1-\mathrm{e}^{hn})\overline{T}^1_{\mu+n},$$
$$\overline{\sigma}(e_n)\overline{T}^1_\mu = (n+\mu-n\mathrm{e}^{hn})\overline{T}^1_{\mu+n} + (1-\mathrm{e}^{hn})\overline{T}^2_{\mu+n} \;=\; \mu\mathrm{e}^{hn}\overline{T}^1_{\mu+n}.$$

Define a linear bijection from the quotient to $A(b,1)\oplus A(b,0)$ by $\overline{T}^0_\mu \mapsto Mz^\mu$ and $\mathrm{e}^{h\mu}\overline{T}^1_\mu \mapsto z^\mu$. This carries the quotient action $\overline{\sigma}$ to an explicit extension of $A(b,1)$ by $A(b,0)$, and one checks that the associated cocycle is $e_n \mapsto M^{-1}z^n \mathrm{e}^{-h\tilde{e}_0}(\mathrm{e}^{-nh}-1)$. Computation shows that this is in fact $-\delta_{-h}$, and so the extension is trivial.

THE ALGEBRA OF DIFFERENCE OPERATORS. Fix $q=\mathrm{e}^h$, and define $\mathcal{D}$ to be:

$$\mathcal{D} = E_0(0,0,0) = \mathbb{C}[z, z^{-1}, \tilde{e}_0] = \mathbb{C}\left[z, z^{-1}, \frac{\mathrm{d}}{\mathrm{d}z}\right],$$

the algebra of polynomial differential operators on S^1, i.e. the Weyl algebra with z^{-1} adjoined. In the paper [Kas] of Kassel, a q-analog $\mathcal{D}_q$ of $\mathcal{D}$ is defined:

$$\mathcal{D}_q = \mathrm{Span}\left\{z^\mu\tau_q^n \colon \mu\in\mathbb{Z},\ n\in\mathbb{N}\right\},$$

where in our notation, τ_q is $\mathrm{e}^{h\tilde{e}_0}$. Note that $\tau_q z = qz\tau_q$, and Jackson's q-differentiation operator is $\partial_q = z^{-1}(\tau_q-1)/(q-1)$, which satisfies the q-Weyl relation $\partial_q z - qz\partial_q = 1$. We remark that $\mathrm{e}^{h\tilde{e}_0}$ may be adjoined to $\mathcal{D}$ to form the algebra $\mathcal{D}[\mathrm{e}^{h\tilde{e}_0}]$, which may be written in various ways:

$$\mathcal{D}[\mathrm{e}^{h\tilde{e}_0}] = \mathbb{C}[z, z^{-1}, \tilde{e}_0, \mathrm{e}^{h\tilde{e}_0}] = \oplus_{n=0}^\infty E_{nh}(0,0,0) = \oplus_{k=0}^\infty \mathcal{D}_q\tilde{e}_0^k.$$

DEFORMATIONS OF ALGEBRAS. First, note that when b and p are zero, the domain and range spaces of tensor densities of $E_0(0,\gamma,0)$ are the same, and so it is an algebra on which $\mathcal{W}$ acts by derivations. However, it is nothing new: the representation structure of $E_0(0,\gamma,0)$ depends on γ, but its algebra structure does not; it is easily seen to be isomorphic as an algebra to $E_0(0,0,0)$. In this setting, $E_hE_{h'} = E_{h+h'}$, and so $\oplus_h E_h$ is an algebra on which $\mathcal{W}$ acts by derivations, and E_h is an E_0-bimodule.

Recently, Pinczon [Pi] introduced a non-commutative version of Gerstenhaber's theory of deformations of algebras [GS]: given any associative algebra L, and an endomorphism τ of L, there is a natural algebra structure on $L[[\theta]]$ such that $\theta l = \tau(l)\theta$ for all $l\in L$, and a τ-*deformation* of L is a deformation of this structure which preserves this equation. It is easy to check that $\mathcal{D}[\mathrm{e}^{h\tilde{e}_0}]$ is the trivial $\mathrm{Ad}(\tau_q)$-deformation of $\mathcal{D}$. Note that if m is an integer, $\mathrm{e}^{2i\pi\tilde{e}_0}$ is a central element of $\mathcal{D}[\mathrm{e}^{2i\pi\tilde{e}_0/m}]$, and so the quotient of this algebra by the two-sided ideal generated by $e^{2i\pi\tilde{e}_0} - 1$ is, as a vector space, $\oplus_{m'=0}^{m-1} E_{2i\pi m'/m}(0,0,0)$. The quotient algebra structure on this space is of course simply a truncated trivial $\mathrm{Ad}(\tau_q)$-deformation of $\mathcal{D}$.

In [Pi], Pinczon defines a certain bimodule W_1^q of the Weyl algebra $\mathbb{C}[z, (\mathrm{d}/\mathrm{d}z)]$, and in fact as a $\mathcal{D}$-bimodule, $E_h(0,0,0)$ is nothing but W_1^q with z^{-1} adjoined (the reader may easily deduce the definition of W_1^q from this fact). He computes the Hochschild cohomology of W_1^q for all q, and uses the result to prove that at $q = -1$, the Weyl algebra has a unique $\mathrm{Ad}(\tau_{-1})$-deformation, which is realized as the universal enveloping algebra of $\mathfrak{osp}(1,2)$.

Let us remark that the case $q = -1$ is also special in our setting: we saw in Section 2.4 that $E_h(0,0,0)$ contains a submodule equivalent to $E_h(0,1,-1)$, which by Proposition 5.1 is irreducible under the adjoint action of $\mathcal{W}$ for most h, but splits as $E_{i\pi,+}(0,1,-1) \oplus E_{i\pi,-}(0,1,-1)$ when $h = i\pi$. It would be interesting to relate these two phenomena.

OTHER REPRESENTATIONS WITH INFINITE-DIMENSIONAL WEIGHT SPACES. Chari and Pressley [CP] have remarked that one of Kirillov's results in [Ki] is that the representations of $\mathcal{W}$ in the symmetric and anti-symmetric half-densities on $S^1 \times S^1$ are both unitary, irreducible and have infinite-dimensional weight spaces. At the algebraic level, this translates to the statement that the symmetric and anti-symmetric parts of $A(0,\frac{1}{2}) \otimes A(0,\frac{1}{2})$ are irreducible. These representations belong to the family $A(a,\gamma) \otimes A(a',\gamma')$ and their subrepresentations, which by the following proposition are not the same as the representations we have constructed in this paper.

PROPOSITION 5.2. *Let V be either $A(a,\gamma) \otimes A(a',\gamma')$ for some a, a', γ, and γ', or in the case that $a = a'$ and $\gamma = \gamma'$, its symmetric or anti-symmetric part. Then V is not equivalent to $E_h(b,\gamma,p)$ for any h, b, γ, and p.*

Proof. First, it is easy to check that the actions of e_n and Q_n on $A(a,\gamma) \otimes A(a',\gamma')$ are:

$$
\begin{aligned}
e_n(M^\gamma z^\lambda \otimes M^{\gamma'} z^{\lambda'}) &= (\lambda + n\gamma)(M^\gamma z^{\lambda+n} \otimes M^{\gamma'} z^{\lambda'}) + (\lambda' + n\gamma')(M^\gamma z^\lambda \otimes M^{\gamma'} z^{\lambda'+n}),\\
Q_n(M^\gamma z^\lambda \otimes M^{\gamma'} z^{\lambda'}) &= \left[n^2(\gamma^2 - \gamma + \gamma'^2 - \gamma') + 2\lambda\lambda'\right](M^\gamma z^\lambda \otimes M^{\gamma'} z^{\lambda'}) + \\
&\quad - (\lambda + n\gamma)(\lambda' - n\gamma')(M^\gamma z^{\lambda+n} \otimes M^{\gamma'} z^{\lambda'-n}) - \\
&\quad - (\lambda - n\gamma)(\lambda' + n\gamma')(M^\gamma z^{\lambda-n} \otimes M^{\gamma'} z^{\lambda'+n}).
\end{aligned}
$$

It follows from these equations that V contains an element v with the property that

$$\mathrm{Span}\left\{Q_n(v) \colon n \in \mathbb{Z}^+\right\}$$

is infinite-dimensional. Indeed, $v = M^\gamma z^\lambda \otimes M^{\gamma'} z^{\lambda'}$ will do, provided that we choose $\lambda \neq 0$ if $\gamma = 0$, and similarly, $\lambda' \neq 0$ if $\gamma' = 0$.

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On the other hand, Q_n acts on E_h by an operator of degree 2 with respect to the filtration E_h^k, which has finite-dimensional weight spaces, and so for any T in E_h,

$$\mathrm{Span}\left\{\sigma(Q_n)T : n \in \mathbb{Z}^+\right\}$$

is finite-dimensional. Therefore V and E_h are not equivalent. □

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