A family of Non-Monotonic Inference Systems based on Conditional Logics

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Abstract

A large number of common sense assertions such as prototypical properties, obligation, possibility, non-monotonic rules, can be expressed using conditional logics. They are precisely under the lights because of their great representation power for common sense notions. But representation is only the half of the work: reasoning is also needed. Furthermore, non-monotonic reasoning necessites both powerful representation and powerful reasoning. For this last point it is well known that the deduction relations of conditional logics are powerless[1].

In this article we propose to use conditional logics to represent defeasible rules and we present different ways, based on semantics, to increase their deductive power. This leads to different non-monotonic systems, the less powerful being equivalent to Pearl's system Z. A tableau like theorem proving method is proposed to implement all of these formalisms.

1 Introduction.

The family of formal systems named conditional logics has been introduced at the end of the sixties with the aim of giving a formal account of linguistic structures of the form *if it where the case that ... then it would be the case that ...* ([Lew73], [Sta68], [Nut80], ...). More recently, these systems have been used to deal with a main problem of artificial intelligence: non-monotonic reasoning. Indeed, even if these systems are monotonic, the non-monotonic conditional operator they introduce makes them powerful not only to study non-monotonic deduction ([ACS92], [Bel91], [CL92], [KS91], [Nej91], ...) but also to represent possibility notions, default rules, prototypical properties ([Che75], [Del87], [Del88], [FndCH91], [Lam92b], [Lew73], [Nej91], [Nut80], [Nut84], ...). In this article we will focus on this last point. A conditional will be used to represent rules such as '*Normally, if it is 12am, then the sun shines*' (eclipses are exceptions) as proposed by Delgrande ([Del87]).

The non-monotonic property of the conditional operator allows to represent defeasible rules in the language. But, because of this property, '*Normally, if it is 12am and I am not awaken, then the sun shines*' can not be deduced from '*Normally if it is 12am, then the sun shines*'. Indeed, there is no logical evidence that '*I am not awaken*' has no influence on the sun. Common sense proceeds in a different way: as far as there is no evidence that '*I am not awaken*' has any influence on the sun, the previous sentence is deduced. To resume, the conditional logics allow us to represent defeasible rules in the language, but not to reason with them as common sense would do.

This last constatation seems to show that conditional logics are not the good solution to deal with common sense reasoning. But, we prefer to say that they represent, half of the work. Using their semantics to represent defeasible rules enable to focus on the problem of the deduction, and try to improve this point.

Some methods have been already proposed to obtain a more natural (non-monotonic) deduction. Delgrande [Del88] has done one of the most interesting proposal (even if it is syntactical and very difficult to compute).

In this article, a practical semantical approach to deal with the same problem is presented. In order to decide if a conclusion Q is a non-monotonic consequence of a theory T (noted $T \mid\!\sim Q$) we will not consider all the models of T (as done in monotonic deduction), but only some of them. Model checking proceeds in the same way but considering one model only. [GP91], [Leh89] and [Pea90] also propose to select one particular model.

Choosing the models involves some problems of different natures:

- Theoretical ones.
  - What is the 'good form' for these models?
- Practical ones.

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[1] In logical formalism there is a duality between the representation and the deduction power. This can be explained semantically. On the one hand, the more interpretations there are, the most powerful is the representation power. On the other hand, the less interpretations there are, the greater is the deduction relation.

- How to build such models?
- How to evaluate if a formula is true in all these models?

The rest of this paper is organized as follows. In the next section, we present some brief recalls on the conditionals semantics. Section 3 briefly recalls a theorem proving method for conditional logics based on semantics. In section 4, this method is used to characterize some interesting models to be used in a non-monotonic way. Finally section 5 presents some complexity results.

2 Conditional logics.

The language of conditional logics is defined begining by a set $\mathcal{P}$ of propositional variables. Classical connectives ($\wedge$, $\vee$, $\neg$, $\rightarrow$) are defined, and another conditional operator ($\Rightarrow$) is introduced: if α and β are formulas, so is $\alpha \Rightarrow \beta$.

The semantics based on selection functions [Che75] is certainly the most general one. But here we will use a generalization of the sphere semantics proposed by Lewis ([Lew73]). It allows to deal with a large class of conditional logics including the most usual ones.

Definition 1 *Conditional structure.*

A conditional **structure** is a triple $M = \langle \mathcal{W}, F, m \rangle$ where

- $\mathcal{W}$ is a non-empty set of possible worlds.
- F is a function which associates to each possible world w a structure: $\langle \mathcal{W}_w, R_w \rangle$ such that:
  - $\mathcal{W}_w \subseteq \mathcal{W}$.
  - $R_w \subseteq \mathcal{W}_w \times \mathcal{W}_w$.
- m : $\mathcal{W} \times \mathcal{P} \mapsto$ {True, False}.

Intuitively, given w, $\mathcal{W}_w$ is the set of possible worlds which can be 'imagined' in w, and R_w is an accessibility relation. A further condition on the accessibility relations is that there no infinite descending chain[2]. This condition corresponds to the limit assumption in [Sta80] and [Lew73] and the smoothness assumption in [KLM90].

A conditional **interpretation** is a couple (M,w) where M is a structure $\langle \mathcal{W}, F, m \rangle$ and w belongs to $\mathcal{W}$. The satisfiability relation is then defined as follows:

Definition 2 *Satisfiability relation.*

Let p be a propositional variable, and let α, β be formulas.

$M,w \models p$ iff $m(w, p) = $ True.

$M,w \models \neg\alpha$ iff not $M,w \models \alpha$.

$M,w \models \alpha \wedge \beta$ iff $M,w \models \alpha$ and $M,w \models \beta$.

$M,w \models \alpha \Rightarrow \beta$ iff for all R_w-minimal worlds w' in $\mathcal{W}_w \cap ||\alpha||^M$, $M,w' \models \beta$.

$||\alpha||^M$ is the set of worlds in M satisfying α. Given a set $\mathcal{S}$ and a relation R, a world w' is said R-minimal in $\mathcal{S}$ iff for all worlds w" in $\mathcal{S}$ if w'Rw" then w"Rw'.

When the accessibility relations are reflexive, transitive and connected[3] (we can then speak about them as total pre-orders), the connection with Lewis sphere semantics is obvious, and we get the system V [Lew73]. Only assuming reflexivity, transitivity and the fact that the world w is one of the minimal ones in $\mathcal{W}_w$ according to R_w, we obtain the system named WC (in [Nut80]) or PO (in [KS91]).

For the sake of simplicity, nested conditionals are not considered in the following. Furthermore, since the intuitive meaning of such formulas is not clear (see [Lew73], [Nut80], [Del87]) and as far as our purpose is not to propose some new reading for them, it seems meaningless to speak of them here (see [Bou92] and [Wey92] for some proposals on this subject). The technical consequence of this restriction is that to evaluate formulas in an interpretation (M,w), we only have to consider w and its associated structure F(w). No matter how F is defined for the other worlds.

3 Theorem proving in conditional logics.

In [Lam92b] and [Lam92a] a method for theorem proving in conditional logics has been presented. The main principles of the system are recalled for the system V.

The method is based on semantics. To prove if α is (or is not) a theorem, an attempt to build a model for $\neg\alpha$ is done. Two cases: 1 - a model is found, then α is not a theorem or 2 - no model can be found and then α is a theorem.

A decomposition of $\neg\alpha$ according to the classical connectives can be used to simplify a little this problem:

- $\neg(\alpha\wedge\beta)$ is rewritten $\neg\alpha\vee\neg\beta$.
- $\neg(\alpha\vee\beta)$ is rewritten $\neg\alpha\wedge\neg\beta$.
- $\{X, \beta\wedge\chi, Y\}$ gives $\{X, \beta, \chi, Y\}$.
- $\{X, \beta\vee\chi, Y\}$ gives $\{X, \beta, Y\}$ and $\{X, \chi, Y\}$.

This gives a set of sets of classical and conditional formulas. Finding a model for $\neg\alpha$ is equivalent finding a model for one of these sets. This last problem is simpler since only conjuctions (sets) of classical literals and conditionals have to be considered.

As far as total pre-orders are considered, the structure associated to a possible world can be viewed as a set of ordered clusters (semantics of modal logic S4.3). Furthermore, since there is no infinite descending chain, a minimal cluster exists. The main principle is then to compute the structure cluster by cluster beginning by the smallest one. The following considerations will help us:

[2]That is there is no infinite set $\{w_1, w_2, \ldots, w_i, \ldots\}$ such that $w_i \neq w_j$ if $i \neq j$ and $w_i R w_j$ if $i \leq j$.

[3]Connected condition: if w_1 and w_2 belongs to $\mathcal{W}_w$ then $w_2 R_w w_3$ or $w_3 R_w w_2$

Let $\mathcal{S}$ be the set of formulas under consideration.

- Let us assume that the positive conditional $\alpha \Rightarrow \beta$ belongs to $\mathcal{S}$.
  A structure satisfies this conditional iff each minimal world satisfying α satisfies β also. So two possibilities: 1 - there is one world in the smallest cluster satisfying α. A simple reasoning leads us to conclude that all the worlds of the smallest cluster satisfying α are minimal in $||\alpha||$ and then satisfy β. 2 - there is no world satisfying α in the smallest cluster. In both cases, each world of the smallest cluster satisfies $\alpha \rightarrow \beta$.
- Let us assume that the negative conditional $\neg(\alpha \Rightarrow \beta)$ belongs to $\mathcal{S}$.
  According to the definition of $\Rightarrow$, either the smallest cluster contains no world satisfying α or it contains at least one world satisfying $\alpha \wedge \neg\beta$.

Let us consider an example. $\mathcal{S} = \{(\alpha \Rightarrow \beta), (\beta \Rightarrow \chi), (\alpha \Rightarrow \neg\chi), \neg(\alpha \Rightarrow \neg\alpha), \alpha\}$. The previous considerations may be shematized as in figure 1.

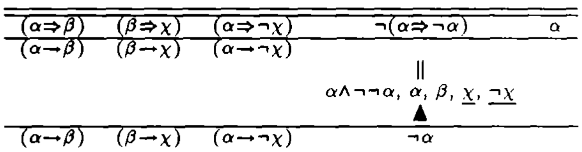

Figure 1: Tableau like presentation.

Under a double line we rewrite all the formulas of the set. According to the first consideration, all the classical entailments written under the first simple line must be satisfied in every world of the smallest cluster. Then according to the second consideration, as far as $\alpha \wedge \neg\neg\alpha$ $(\alpha \Rightarrow \neg\alpha)$ is not consistent with this set of material implications, $\neg\alpha$ must also be satisfied in all the worlds of the smallest cluster. The formulas which enable to detect an inconsistency are underlined and the symbol ▲ signals this inconsitency.

Definition 3 *Core, state core, captured conditionals.*
The **core** of a cluster is the set of formulas which are true in all the worlds of the cluster.
A **state core** is a consistent set ST such that ST = Core ∪ $\{\alpha\}$, α being the hypothesis of a considered conditional.
A conditional is said to be **captured** in a cluster if there is one world satisfying its hypothesis in this cluster.

In a level of a tableau, the core is the set of formulas appearing under the last simple line, and for each considered conditional captured at this level, there is a state core. For example in figure 1, the core of the smallest cluster is: $\{(\alpha \rightarrow \beta), (\beta \rightarrow \chi), (\alpha \rightarrow \neg\chi), \neg\alpha\}$. The conditional $(\beta \Rightarrow \chi)$ is the only one to be captured and there is only one state core: $\{(\alpha \rightarrow \beta), (\beta \rightarrow \chi), (\alpha \rightarrow \neg\chi), \neg\alpha\} \cup \{\beta\}$.

The same kind of computation is done for the just upper cluster considering only conditionals which are not captured yet.

When no more conditional can be captured, the procedure stops. Two cases: 1 - there is no conditional under the last double line, (as in figure 2) or 2 - the same set of conditionals appears under the two last double lines.

No model can be found when: 1 - a classical inconsistency is detected among considered formulas or 2 - a negative conditional cannot be captured.

Applying completely this method to the previous example leads to figure 2.

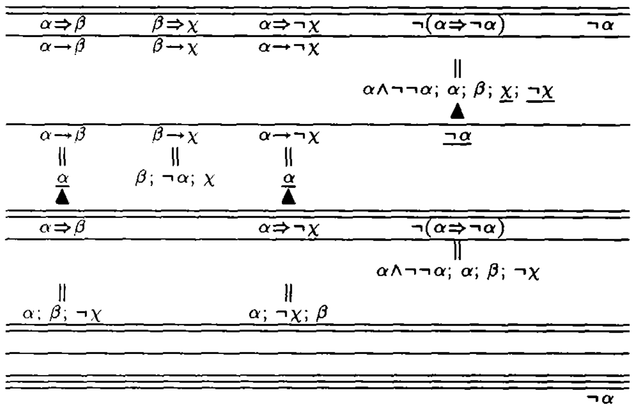

Figure 2: A tableau like presentation for theorem proving in system V.

The triple line corresponds to the treatment of the classical formulas which consists in verifying that they are consistent. Note that, this test can as well be performed at the beginning of the procedure.

According to this tableau, we can conclude that models satisfying $((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \chi) \wedge (\alpha \Rightarrow \neg\chi) \wedge \neg(\alpha \Rightarrow \neg\alpha) \rightarrow \neg\alpha$ can be found. Figure 3 represents one of the simplest: it just contains an actual world and one world for each state core.

Theorem 1 *Soundness/Completeness.*
Let us note $\vdash_{\text{Proc}}\alpha$ the fact that for any set obtained from $\neg\alpha$, the previous method does not give any model.
$\vdash_{\text{Proc}}\alpha$ if and only if $\vdash_{\text{V}}\alpha$

The proof can be found in [Lam92b] or in [Lam92a].

Definition 4 *T-interpretation.*
Given a tableau T (as in figure 2) obtained from a finite theory Γ via the previous method, a T-interpretation of Γ is a V-interpretation $\langle \mathcal{W}, \text{F}, \text{m}\rangle$,w such that:

- $F(v) = \langle \mathcal{W}_v, R_v \rangle$ for all v in $\mathcal{W}$[4].
- $\mathcal{W}_v = \{v\}$ for all v in $\mathcal{W}$ such that $v \neq w$.
- F(w) contains exactly as many clusters as there are levels (double lines) in the tableau.
- For each cluster of F(w), each world satisfies the core of its level.
- For each particular level, there is a world satisfying each state core.
- w satisfies all the formulas written under the triple line.

The first level of the structure corresponds to the smallest cluster, the following level corresponds to the just upper cluster, and so on.

Theorem 2 *T-interpretations are models of* Γ.
Let Γ be a finite consistent set of formulas.
A T-interpretation of Γ exists and if M,w is such an interpretation then $M,w \models \Gamma$.

The proof is not difficult according to the construction of the tableau and the definition of a T-model.

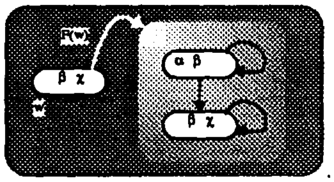

Figure 3: Extracted counter-model.

Note that, figure 3 represents a T-interpretation of the previous example.

4 Non-monotonic proving based on theorem proving.

It seems reasonable to say that we use our "intuition" in common-sense reasoning. This notion can be approximate as a 'modelisation' of the reality. This is the basic idea of the model checking approach: considering a finite theory Γ (formulas describing a phenomena), a particular model of it is built. Then when a question arises, this model is used to answer.

In the previous section we have briefly recalled a method for theorem proving in conditionals systems which provides a way to compute models. In the following, we will propose to use the models obtained via this method as suggested in the model checking approach.

In the following paragraphs, we present our method more formally, and we discuss some possible alternatives.

[4] Remember that nested conditionals are not considered. This point and the following one would be different if they were.

4.1 General principle

Assuming that Γ is the finite consistent theory, the above method gives some way to obtain tableaux (as in figure 2) from it. We will consider the T-models defined according to the tableaux and satisfying one meta assumption called **normality assumption on formulas** (abreviated by 'NF'):

Definition 5 *V.NF-interpretations.*
Given a tableau T obtained from a finite theory Γ via the previous method, a **V.NF-interpretation** of Γ is a T-interpretation $\langle \mathcal{W}, F, m \rangle$,w such that, if a formula appears in one cluster[5] then it appears in every smaller cluster, the core of which is consistent with this formula.

The first condition ensures that the considered models satisfy the initial theory. The last assumption just means that if a formula is considered possible (appears in the structure), it must also be considered in cases which are the most normal ones for it. This may be summarized by "a formula must not be considered more exceptional than the theory tells!".

Definition 6 *Non-monotonic consequence.*
A formula α is a non-monotonic consequence of a finite theory Γ noted $\Gamma \mid\!\sim_{V.NF} \alpha$[6] iff it is true in all the V.NF-models of Γ.

The main difference between the model checking approach and ours is that a class of models and not only one model is considered here. This avoids some problem on the difference between positive and negative formulas. Indeed, if only one interpretation is considered the deduction relation has the following property: $T \mid\!\sim Q$ or $T \mid\!\sim \neg Q$ which seems in general not intuitive to us.

A very interesting point is that we do not need to build these interpretations to know if a formula is true in all of them. Indeed, the tableau can be used for this purpose:

Theorem 3 *Link between V.NF-models and the tableau.*
Let Γ be a conjuction of conditionals and classical formulas.
α is true in all V.NF-models of Γ iff:

- If α is a classical formula.
  α is logically entailed by the set of formulas appearing under the triple line.
- If α is a positive conditional of the form $(\varphi \Rightarrow \psi)$.
  1 - φ is not consistent with the core of any level, or
  2 - the core of the smallest level which is consistent with φ logically entails $\varphi \rightarrow \psi$.

[5] We say that a formula appears in one cluster when there is a world in this cluster satisfying it.

[6] In $\mid\!\sim_{V.NF}$ the subscript V means that the underlying logical system is V, and NF means that we reason under the Normality assumption on Formulas.

- If α is a negative conditional of the form $\neg(\varphi \Rightarrow \psi)$.
  In the smallest level consistent with φ (if there isn't any, the conditional is not satisfied) there is at least one state core which logically entails $\varphi \wedge \neg \psi$.

The case of classical formulas is not difficult.

Let us consider a positive conditional. 1 - Its hypothesis is not consitent with the core of any level of the tableau iff there is no world satisfying its hypothesis in the structure. The conditional is then satisfied in all V.NF-models. 2 - Because of the Normality assumption on the Formulas, the condition 2 is enough to prove that, if φ appears in the structure then any world of the smallest cluster containing a world satisfying φ satisfies $\varphi \rightarrow \psi$ too.

Let us consider negative conditionals. The last condition is enough to prove that for all V.NF-interpretations: 1 - since there is a state core satisfying φ, φ appears at least one time. Furthermore, there is at least one world satisfying $\varphi \wedge \neg \psi$ in the smallest cluster containing at least one world satisfying φ. This is exactly the satisfiability condition for negative conditionals.

In [Lam92b], the positive fragment[7] of this approach has been proved to correspond to the Pearl's system Z [Pea90] which is based on a probabilistic interpretation of the conditional similar to Adams' system [Ada75], and also to the approach presented by Lehmann in [Leh89].

The only difference between what has been presented in [Lam92b] and what is presented here concerns the treatment of negative formulas. Indeed, in [Lam92b] only one interpretation was considered: a T.NF-interpretation satisfying the additional assumption: as many worlds as possible belong to the structure associated to the actual world. This leads to the problem presented above.

This system ($\mid\!\sim_{\mathrm{V.NF}}$) shares some important limitations with Pearl's systenm Z (see [Pea90] ans [GP91]). This leads us to consider some other solutions. In [GP91] Goldszmidt and Pearl have proposed another system named Z^+. The main principle is to ask the knowledge base designer to add some meta-informations: numbers associated to the conditionals. They express some priority (order) and guide the system to build a better model. But in this approach, the system lies and the user works: the quality of the result mainly depends on the quality of the order given by the designer.

In the following, we explore some ways to make the system more active so that the knowledge base designer may have more rest!

[7]Negative conditionals are not considered.

$\alpha \Rightarrow \beta$	$\beta \Rightarrow \chi$	$\neg(\alpha \Rightarrow \delta)$
$\alpha \rightarrow \beta$	$\beta \rightarrow \chi$	
‖	‖	‖
α ; β; χ	β ; χ	α ; β; χ (captured) There is no state core such that (state$\vdash \alpha \wedge \neg \delta$)

The conclusion is that $((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \chi)) \not\mid\!\sim_{\mathrm{V.NF}} \neg(\alpha \Rightarrow \delta)$. According to the approach presented in [Lam92b] this would be deduced.

To prove $((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \chi)) \mid\!\sim_{\mathrm{V.NF}} (\alpha \Rightarrow \chi)$, considering the left part of the tableau, we have to check that $\alpha \rightarrow \chi$ is a monotonic consequence of the smallest core which is consistent with α.

Figure 4: A non-monotonic tableau (NF).

4.2 Some variations of this approach.

The constraints presented above do not completely specify the model. Indeed, changing the way the tableau is computed may modify:

- $\mathcal{W}_{\mathrm{w}}$: what does the structure contain.
- R_{w}: how it is structured.

In the following, some possibilities are investigated.

4.2.1 Distinguishing omissions. If we analyse how the tableau is obtained, it is not difficult to see that a considered conditional is forgoten as soon as it is captured. Forgetting conditionals allows to capture some other considered conditionals which are not captured yet. But droping away all the captured ones is too strong.

Definition 7 *Interesting omissions.*
Let us consider the set of conditionals $\mathcal{C}$ appearing at a level i of a tableau.
Let us call $\mathcal{CC}$ the set of captured conditionals at this level.
A **possible omission** $\mathcal{PO}$ is a set such that $\mathcal{PO} \subseteq (\mathcal{CC})$ and $\{\neg(\alpha \Rightarrow \beta) \mid \neg(\alpha \Rightarrow \beta) \in (\mathcal{CC})\} \subseteq \mathcal{PO}$.
An **interesting omission** $\mathcal{IO}$ is a possible omission such that forgetting it enables to capture in the next level some conditionals uncaptured at this level.
A **maximal interesting omission** ($\mathcal{MIO}$) is an interesting omission such that none of its subsets (except itself) is an interesting omission.

Note that it is not difficult to write a simple algorithm to compute the maximal interesting omissions from the previous definition. It may be the case that, at some particular level, different maximal interesting omissions are found. Then we choose to forget all of them at the same time. This solution has the advantage to give only one tableau.

Definition 8 *Non-monotonic deduction.*

$\Gamma\mid\!\sim_{V.NF.MIO}\alpha$ iff α is satisfied in all the V.NF.MIO-interpretations of Γ (i.e. the T.NF-interpretations obtained from the MIO-tableau).

Applied to the theory $\Gamma = \{\alpha\Rightarrow\beta,\ \alpha\Rightarrow\chi,\ \beta\Rightarrow\delta,\ \beta\Rightarrow\neg\psi,\ \chi\Rightarrow\psi,\ \delta\Rightarrow\neg\varphi,\ \psi\Rightarrow\varphi\}$ this approach is still too weak: it does not allow to deduce $\alpha\Rightarrow\delta$ which seems an intuitive conclusion (see [THT97] page 478).

In this case, different maximal interesting omissions ($\{\beta\Rightarrow\delta,\ \beta\Rightarrow\neg\psi\}$ and $\{\delta\Rightarrow\neg\varphi,\ \beta\Rightarrow\neg\psi\}$ in the given example) solve the same conflict. Forgetting only some (but not all) of them is then sufficient. But which one to forget and which one to keep?

Definition 9 *Specificity of sets of conditionals.*
Let Γ_1 and Γ_2 be two finite sets of conditionals. Γ_1 is said more **specific** that Γ_2 at a level i of a tableau iff for all conditionals $\alpha_2\Rightarrow\beta_2$ in Γ_2 there is one conditional $\alpha_1\Rightarrow\beta_1$ in Γ_1 such that $core_i\vdash\alpha_1\rightarrow\alpha_2$ ($core_i$ is the core of the level i).

Only the less specific maximal interesting omissions will be forgotten.

Definition 10 *Non-monotonic deduction.*
$\Gamma\mid\!\sim_{V.NF.MIO.MS}\alpha$
iff α is satisfied in all T.NF.MIO.MS-models of Γ.

Theorem 4 *Soundness/Completeness with V.*
Let Γ be a finite set of formulas.
$\Gamma\mid\!\sim_{V.NF.MIO.MS}\bot$ iff $\Gamma\vdash_V\bot$.

This means that the change of strategy to build the tableau is coherent with the underlying logic: this method can still be used as a theorem prover.

4.2.2 Assuming the actual world as normal as possible.

Classical formulas are not mentioned in the previous section. Indeed, in logic V, there is no link between conditional and classical formulas so that it is not so interesting to consider them. Changing the underlying conditional logics is a solution. We can consider NP, proposed by Delgrande (the actual world belongs to the structure), or VW proposed by Lewis [Lew73] and Nute [Nut80] (the actual world belongs to the smallest cluster). Here, we propose to apply some default assumptions on the actual world:

- As far as it is possible, a world belongs to its associated structure.
- If a world belongs to its associated structure then there is no world in a smaller cluster satisfying the classical formulas appearing under the triple line.

Note that no inconsistency problem arises if the world does not belong to its associated structure. This is the difference from adding an axiom to the underlyinglogic.

Because of the normality assumption on formulas, the last condition corresponds to a normality assumption on the actual world presented in [Del88].

In order to capture this intuition, classical formulas of the considered theory (let us call $\mathcal{C}$ this subset) are considered as the hypothesis of a conditional ($\mathcal{C}\Rightarrow\top$) which is forgotten as far as it is captured. Note that this conditional does not take any role in the computation of the core. The triple line appears at the level where this conditional is captured. All the classical formulas ($\mathcal{C}$) and the core of this level are under it. If the conditional ($\mathcal{C}\Rightarrow\top$) is not captured, then the triple line appears at the end of the tableau with only $\mathcal{S}$ under it and the actual world does not bellongs to its associated structure.

All the T.NF-models sastisfying these assumptions are called T.NF.NAW-models.

Definition 11 *Non-monotonic deduction.*
$\Gamma\mid\!\sim_{V.NF.NAW}\alpha$ iff α is satisfied in all the T.NF.NAW-models of Γ.

Theorem 5 *Soundness/Completeness with V.*
Let Γ be a finiteset of formulas.
$\Gamma\mid\!\sim_{V.NF.NAW}\bot$ iff $\Gamma\vdash_V\bot$.

4.2.3 An example.

It is interesting to note that this assumption on the actual world can be combined with any of the previous approaches to give a new non-monotonic deduction (for example $\mid\!\sim_{V.NF.MIo.MS.NAW}$).

Let us consider the set $\Gamma=\{\alpha\Rightarrow\beta,\ \alpha\Rightarrow\chi,\ \beta\Rightarrow\neg\varphi,\ \beta\Rightarrow\delta,\ \chi\Rightarrow\omega,\ \chi\Rightarrow\varphi,\ \delta\Rightarrow\neg\psi,\ \psi\Rightarrow\varphi,\ \chi,\ \neg\delta\}$.

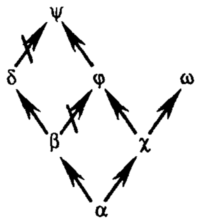

Figure 5: Conditional theory.

Because of space, it is not possible to draw the tableau here. At the first level, the captured conditionals are $\{\beta\Rightarrow\neg\varphi,\ \beta\Rightarrow\delta,\ \chi\Rightarrow\omega,\ \chi\Rightarrow\varphi,\ \delta\Rightarrow\neg\psi,\ \varphi\Rightarrow\psi\}$. The classical formulas of Γ are consistent with the core of this first level so that the conditional $((\chi\wedge\neg\delta)\Rightarrow\top)$ is captured here. The triple line corresponding to the actual world appears just after this level and all classical formulas of Γ and the core of this level appear under it. The maximal interesting omissions are $\{\chi\Rightarrow\varphi\}$, $\{\beta\Rightarrow\neg\varphi,\ \varphi\Rightarrow\psi\}$, $\{\beta\Rightarrow\neg\varphi,\ \beta\Rightarrow\delta\}$ and $\{\beta\Rightarrow\neg\varphi,\ \delta\Rightarrow\neg\psi\}$, but as far as $\{\beta\Rightarrow\neg\varphi,\ \beta\Rightarrow\delta\}$ is more specific that $\{\beta\Rightarrow\neg\varphi,\ \delta\Rightarrow\neg\psi\}$, the forgotten conditionals are $\{\beta\Rightarrow\neg\varphi,\ \chi\Rightarrow\varphi,\ \delta\Rightarrow\neg\psi,\ \varphi\Rightarrow\psi\}$. The set of conditional considered for the next level are: $\{\alpha\Rightarrow\beta,\ \alpha\Rightarrow\chi,\ \beta\Rightarrow\delta,\ \chi\Rightarrow\omega\}$ where all of them are captured.

It is then not so difficult to check that $\alpha\Rightarrow\delta$ and $\beta\Rightarrow\neg\psi$ ψ, and $\neg\beta$ are deductible.

5 Complexity.

5.1 Complexity of the monotonic theorem prover.

Theorem 6 *Complexity of the theorem prover in the worst case.*

Let us consider a theory Γ of flat conditionals and classical formulas.

Let us call N the number of conditionals.

Let us call n the number of literals in Γ.

Let us call Sat(k) the complexity of propositional satisfiability for a formula containing k literals.

- The complexity of the monotonic theorem prover is $O(3^{n/3}\ N^3\ Sat(n))$.
- If Γ is a conjunction of conditionals and classical literals, then the complexity is $O(N^3\ Sat(n))$.
- If Γ is a conjunction of positive conditionals and classical literals, then the complexity is $O(N^2\ Sat(n))$.

5.2 Complexity of the non-monotonic deduction method.

The complexity of the non-monotonic inference relations presented in this paper must be divided into two parts: 1 - the complexity of building the tableaux, and 2 - the complexity of checking in a tableau.

Let us begin by this last point. Let us call n' the number of literals in the question. In the worst case the complexity of checking the truth of a conditional (positive or negative one) is $O(Log_2(N)\ Sat(n+n'))$. For classical formulas, the complexity is $O(Sat(n+n'))$.

As far as we only have to check in a tableau, the method is efficient. But, the tableaux must also be built. For the simplest non-monotonic deduction which has been presented in this article ($\mid\!\sim_{V.NF}$), the complexity of this last problem is identical to the theorem prover one. For the other non-monotonic systems, it is exponential.

Once more, this result seems in contradiction with the argument "as far as only some cases are considered, non-monotonic deduction must be at least as efficient as logical deduction which considers all possible cases" which has been used to criticize non-monotonic formalisms complexity. But an important point is missed: how are these cases chosen? Indeed, this problem is really difficult: hours but also days, months, years, and sometimes the whole life are not always enough to build a good (human) intuition of a phenomena!

According to this last comment, the results presented here seems more reasonable: the system needs time to build the tableaux, but using them, it can very quickly jump to conclusions.

Computing tableaux can also be viewed as a compilation where the resulting code is the tableau itself. Then answering to a question corresponds to a run with particular data (the question). This analogy with program compiling makes clear that tableaux need to be built only when the theory Γ is modified.

6 Conclusion.

All the non-monotonic formalisms presented in this paper are based on conditional logic V and on the theorem proving method presented in [Lam92b]. Basically, the strategy is modified to improve the deduction power.

Some interesting points are:

- A powerful language.

Two levels of non-monotonicity (language $\Rightarrow$ and meta-language $\mid\!\sim$) increase the accuracy of the formalism. For example $\Gamma\mid\!\sim\alpha\Rightarrow\beta$and $\Gamma\wedge\alpha\mid\!\sim\beta$ do not express the same thing.

- A powerful representation.

The full representation power of the underlying conditional logic is used. This point is not shared by all non-monotonic formalisms: sometimes consistent theories do not have any 'stable expansion' or more generally are not 'non-monotonically consistent'.

- An interesting flexibility.

Defeasible reasoning, may make mistakes and give some undesirable conclusion. When this happens adding the negation of this conclusion to the theory seems to be the most natural way ($\Gamma\cup\{\neg\alpha\}\not\mid\!\sim_{...}\alpha$). This solution is possible because the language allows negative conditionals. Furthermore, as far as $\Gamma\cup\{\neg\alpha\}$ is non-monotonically consistent iff it is logically consistent, the theory Γ has to be reconsidered only in case of logical inconsistency.

- A powerful deduction.

As far as the underlying conditional logic and the non-monotonic properties are flexible, a large class of deductions may be captured. Furthermore, the most powerful deductions presented in this paper (as $\mid\!\sim_{V.NF.MIO.MS.NAW}$) seem to be useful even in non toy examples.

The general principle of the approach presented in this article can be used with some other underlying logics (as modal logics) and with some other theorem proving methods as far as they provide a way to select models. This can be a very general and powerful approach to non-monotonic proving. Indeed, it can be linked to the preference approach presented by Shoham, where, in order to know if $\alpha\mid\!\sim\beta$, only the most preferred interpretations satisfying α are considered. The approach presented here is related in the following way: from α, tableaux characterizing a set of interpretations (the most preferred ones) are built. Then, they are used to determine if α is true in all these interpretations.

As in the system Z^+, numbers may be associated to conditionals in order to help the system, but some new mechanisms must be added [Lam92a].

An important remark is that in this approach two logically equivalent formulas may different in a non-monotonic point of view. For example, using the deduction $\vdash_{\text{T.NF.MIO.MS.NAW}}$, $\{\alpha \Rightarrow \beta,\ \beta \Rightarrow (\chi \wedge \delta),\ \alpha,\ \text{notcl}\chi\} \not\vdash \delta$ but $\{\alpha \Rightarrow \beta,\ \beta \Rightarrow \chi,\ \beta \Rightarrow \delta,\ \alpha,\ \text{notcl}\chi\} \vdash \delta$.

It is interesting to note that even the strongest non-monotonic deduction presented here ($\vdash_{\text{V.NF.MIO.MS.NAW}}$) is still weaker that the approach presented by Delgrande in [Del88].

In some future work, we will also explore other conditions which may be applied to the tableau in order to obtain stronger deductions.

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