A FAMILY OF SOLUTIONS OF THE MAGNETO-HYDROSTATIC PROBLEM IN A CONDUCTING ATMOSPHERE IN A GRAVITATIONAL FIELD

J. W. Dungey

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Summary

Quiescent solar prominences appear to be in static equilibrium and it is believed that magnetic fields play an essential role ; the electromagnetic force must balance the resultant of the pressure gradient and the force of gravity. The static equation for an isothermal atmosphere is derived and a simple family of two-dimensional solutions is obtained. This consists of a set of lines of force such that any magnetic field with these lines of force can be in static equilibrium. A model is obtained which resembles a filament and its associated coronal arches.

1. *Introduction*.—In recent years the belief has grown among solar physicists that magnetic fields play an essential role in the phenomena of solar prominences and Alfvén (1) has discussed the possibility that their important feature is an electrical discharge. The reasons for this belief may be summarized as follows:

(i) The trajectories of falling material have the general appearance of lines of force.

(ii) Prominences are strongly correlated with sunspots and a large proportion of prominences make their first appearance near sunspots.

(iii) The quiescent prominences appear to be almost in static equilibrium, in spite of apparent pressure variations which could not exist in static equilibrium, if the only force available to balance the pressure gradient were the force of gravity.

Because of (iii) and the difficulty of treating the dynamical problem quantitatively it is expected that insight will be gained by a study of models in static equilibrium, the additional force required to balance the pressure gradient being due to a magnetic field. Such a static equilibrium has already been considered by Menzel (2), who obtained solutions which resemble prominences. The electrodynamic side of the problem is briefly as follows. In a perfectly conducting gas the variation of the magnetic field is such that the magnetic flux linked by any closed curve moving with the gas is constant. In the solar atmosphere the conductivity is so large that the decay time of the magnetic field is much longer than the life of a prominence, but the Hall electric field must be considered; it can be shown that for any solution of the equations given in Section 3 the Hall field is irrotational and hence does not contribute to $\partial \mathbf{H}/\partial t$. The origin of the magnetic field supposed to be associated with a prominence is a separate question, but, remembering the high conductivity of the solar atmosphere, the observation (ii) suggests that this field is a remnant of a sunspot field.

The main purpose of this paper is to present a simple exact solution of the equation for static equilibrium, under certain simplifying conditions, which it is hoped will shed some light on the general problem. The solution takes the form of a set of lines of force having the property that, for any magnetic field having these lines of force, a distribution of the gas exists which will be in equilibrium with this field. The advantage of such a solution is that the magnetic field can be made to vanish outside any region bounded by a line of force and hence the problem of convergence at infinity is avoided. Before describing this solution the magnetic force will be discussed in a general way.

2. *The magnetic force.*—The magnetic force density is $\mathbf{j}\wedge\mathbf{H}/c$, where $\mathbf{H}$ is the magnetic field and the current density $\mathbf{j}$ is given for a static field by

$$4\pi\mathbf{j}=c \operatorname{curl} \mathbf{H}. \tag{1}$$

Hence

$$\mathbf{j}\wedge\mathbf{H}/c=(\mathbf{H}\,.\,\nabla)\mathbf{H}/4\pi-\nabla(H^2/8\pi), \tag{2}$$

where $H=|\mathbf{H}|$. The first term in (2) depends on the curvature of the lines of force and the second represents the " magnetic pressure ", $H^2/8\pi$, the total force being perpendicular to $\mathbf{H}$. If p is the gas pressure, the expression $(p+H^2/8\pi)$ may be termed the " total pressure " and horizontal variations of the total pressure can result only from the first term in (2). Also, since the magnetic force does not affect the variation of p along a line of force, the values of p at different points on the same line of force are proportional to $\exp[-\int dz/h]$, where z is the height, h the scale height and the integral is taken along the line of force. Measurements by van de Hulst (**3**) of the brightness in eclipse photographs of the corona suggest that the electron density in the polar plumes varies approximately in this way and the polar plumes are thought to be due to a magnetic field.

The material in and above sunspots also is probably in a state approximating to static equilibrium. The magnetic field probably causes a low gas pressure in a spot and one may then expect a low pressure in the atmosphere above a spot. In this case there must be a magnetic force in the atmosphere to balance the horizontal pressure gradient, and hence there must be currents in the atmosphere. For this reason the model in which the magnetic field over a bipolar spot group is taken to approximate to a dipole field is not justified, for it is based on the absence of currents in the atmosphere. The postulate that H^2/p is constant on a line of force, which is exactly true for the model described in this paper, probably gives a better approximation; H then decreases exponentially with height instead of following an inverse cube law.

The model described in this paper and Menzel's models are all two-dimensional, and it should be noted that they are unstable. Their stability may be discussed by considering movements of the gas in which the magnetic lines of force are regarded as moving with the gas. The magnetic energy $\int H^2 dV/8\pi$ contained in a tube of force increases with the length of the tube if the volume remains constant, so that the energy of a toroidal tube of force can be reduced by radial contraction accompanied by expansion in the direction of the axis of the toroid, as shown in Fig. 1 (*a*); the tube becomes a long thin cylinder as in Fig. 1 (*b*). A topological argument now shows that, if there are no linked lines of force, the magnetic energy can be reduced to zero. By splitting each tube into smaller tubes, separating these and contracting them, and by repeating the process indefinitely, each line of force can be shrunk to a point. Thus any magnetic field containing no linked lines of force is unstable, but it is hard to say whether this instability is important for the solar applications.

3. *The simplified model.*—We now consider the simplified model in which the temperature and composition of the gas and the gravitational field are all uniform. The equation to be solved is the static equation

$$\mathbf{j}\wedge\mathbf{H}/c+\mu\mathbf{g}-\nabla p=0, \tag{3}$$

where μ is the mass density and the gravitational field $\mathbf{g}$ is constant and equal to $-g$ in the z-direction. It is convenient to write

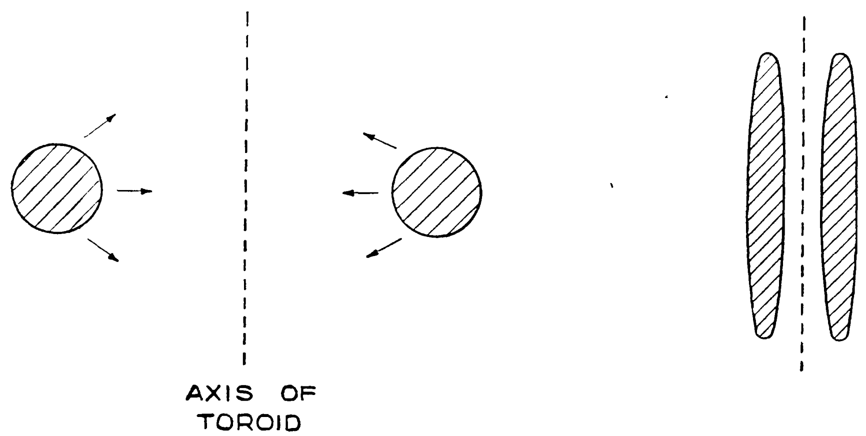
$$\mu g=\alpha p$$

with

$$\alpha=h^{-1}=mg/kT,$$

where m is the mean ionic mass and T is the temperature; h is then the scale height.

We now restrict the model to be two-dimensional: all variables are



(a) (b)

FIG. 1.

independent of the coordinate y and H_y is zero. We may then introduce a vector potential which has only a y-component, A, so that

$$\left.\begin{aligned} H_x&=-\partial A/\partial z,\\ H_z&=\ \ \partial A/\partial x.\end{aligned}\right\} \tag{4}$$

Equation (4) shows that A is constant on a line of force. Also (1) shows that $\mathbf{j}$ has only a y-component j given by

$$j=-\frac{c}{4\pi}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}\right). \tag{5}$$

The x- and z-components of equation (3) are

$$\frac{j}{c}\frac{\partial A}{\partial x}-\frac{\partial p}{\partial x}=0 \tag{6}$$

and

$$\frac{j}{c}\frac{\partial A}{\partial z}-\alpha p-\frac{\partial p}{\partial z}=0. \tag{7}$$

Introducing $q=p\,e^{\alpha z}$, (7) may be written

$$\frac{j}{c}\frac{\partial A}{\partial z}-e^{-\alpha z}\frac{\partial q}{\partial z}=0. \tag{8}$$

From (8) and (6) we obtain

$$\frac{\partial q/\partial x}{\partial q/\partial z}=\frac{\partial A/\partial x}{\partial A/\partial z}.$$

Hence q is a function of A, and is constant on a line of force. Since q is a function of A, we obtain

$$j=c\,\mathrm{e}^{-\alpha z}\,dq/dA \tag{9}$$

by dividing (6) or (8) by $\partial A/\partial x$ or $\partial A/\partial z$ respectively, so that (9) is true unless H vanishes. Equation (9) shows that $\mathrm{e}^{\alpha z}j$ is a function of A.

4. *A particular family of solutions.*—The family of solutions described here is defined by the lines of force. The condition is imposed that

for any magnetic field having these lines of force, a distribution of the gas exists which will be in equilibrium with this field. (A)

The lines of force may be defined by the vector potential A of any field having these lines of force; the lines of force are the lines of constant A by (2). Hence, if B is the vector potential of another field having the same lines of force, B is a function of A, and, if the suffixes A and B refer to quantities derived from the vector potentials A and B respectively, we obtain from (5) and (4)

$$j_B=j_A\frac{dB}{dA}-\frac{c}{4\pi}H_A{}^2\frac{d^2B}{dA^2}. \tag{10}$$

Now the condition (A) requires that (9) should be true for j_B and since B is a function of A, $\mathrm{e}^{\alpha z}j_B$ must then be a function of A. Then (10) shows that $\mathrm{e}^{\alpha z}H_A{}^2$ must be a function of A. This is in fact the condition that, if the magnetic field vanishes outside a region bounded by a line of force, the magnetic force on the bounding surface can be balanced by a discontinuous change in gas pressure proportional to $\mathrm{e}^{-\alpha z}$. It now follows that there will be a function C of A which satisfies the equation

$$H_C{}^2=\mathrm{e}^{-\alpha z}. \tag{11}$$

It is convenient to use C to determine the lines of force. From (11) and (4) we may put

$$\left.\begin{aligned}\mathrm{e}^{-\alpha z/2}\sin\psi&=-\frac{\partial C}{\partial z}=H_{Cx},\\ \mathrm{e}^{-\alpha z/2}\cos\psi&=\frac{\partial C}{\partial x}=H_{Cz},\end{aligned}\right\} \tag{12}$$

and then
$$\frac{\partial}{\partial x}(\mathrm{e}^{-\alpha z/2}\sin\psi)=-\frac{\partial^2 C}{\partial x\partial z}=-\frac{\partial}{\partial z}(\mathrm{e}^{-\alpha z/2}\cos\psi)$$

giving
$$\left(\cos\psi\frac{\partial}{\partial x}-\sin\psi\frac{\partial}{\partial z}\right)\psi=\tfrac{1}{2}\alpha\cos\psi. \tag{13}$$

Now for a line of force $dx/dz=\tan\psi$ and for a curve orthogonal to the lines of force $dx/dz=-\cot\psi$. The left-hand side of (13) is the curvature $d\psi/ds$ of such an orthogonal, so that the orthogonals can be obtained by integrating

$$\frac{d\psi}{dx}=\tfrac{1}{2}\alpha. \tag{14}$$

Equation (14) shows that the orthogonals and hence also the lines of force are periodic in x with period $4\pi/\alpha$. We may therefore use the conformal transformation

$$\xi+\mathrm{i}\eta=\mathrm{e}^{-\alpha(z+\mathrm{i}x)/2};$$

ξ and η are periodic in x. Then

$$\frac{\partial}{\partial z}-\mathrm{i}\frac{\partial}{\partial x}=-\tfrac{1}{2}\alpha(\xi+\mathrm{i}\eta)\left(\frac{\partial}{\partial\xi}-\mathrm{i}\frac{\partial}{\partial\eta}\right)$$

and from (5) $$j_C = -\frac{c\alpha^2}{16\pi} e^{-\alpha z}\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{\partial^2 C}{\partial \eta^2}\right) \tag{5'}$$

and (11) becomes $$(\partial C/\partial \xi)^2 + (\partial C/\partial \eta)^2 = 4\alpha^{-2}. \tag{11'}$$

Instead of (12) we put $$\left.\begin{aligned} \sin\phi &= -\tfrac{1}{2}\alpha\,\partial C/\partial\eta, \\ \cos\phi &= \tfrac{1}{2}\alpha\,\partial C/\partial\xi, \end{aligned}\right\} \tag{15}$$

and obtain $$(\cos\phi\,\partial/\partial\xi - \sin\phi\,\partial/\partial\eta)\phi = 0. \tag{16}$$

Equation (16) shows that the orthogonals are straight lines in the (ξ, η) plane.

Also from (15) $$\tfrac{1}{2}\alpha\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{\partial^2 C}{\partial \eta^2}\right) = -\left(\sin\phi\frac{\partial}{\partial\xi} + \cos\phi\frac{\partial}{\partial\eta}\right)\phi. \tag{17}$$

Now (9), (5′) and (17) yield
$$\left(\sin\phi\frac{\partial}{\partial\xi} + \cos\phi\frac{\partial}{\partial\eta}\right)\phi = \frac{8\pi}{\alpha}\frac{dq}{dC}. \tag{18}$$

The left-hand side of (18) is the curvature of a line of force in the (ξ, η) plane and, since the right-hand side is a function of C, this must be constant for a line of force. The lines of force are therefore circles in the (ξ, η) plane and, since their orthogonals are straight lines, they are concentric circles. It is easily verified that if A is any function of $F = (\xi - \xi_0)^2 + (\eta - \eta_0)^2$, where ξ_0 and η_0 are constants, (9) is satisfied, since
$$\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial \eta}\right)^2 = 4F$$
and
$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} = 4.$$

Changes in ξ_0 and η_0 correspond to movement of the origin of x and z, and it is convenient to put $\xi_0 = 1$ and $\eta_0 = 0$, giving
$$F = e^{-\alpha z} - 2e^{-\alpha z/2}\cos\tfrac{1}{2}\alpha x + 1. \tag{19}$$

The lines of force determined by F = constant are shown in Fig. 2. They are closed loops for $F < 1$ and infinite wavy lines for $F > 1$.

Since A is now a function of F we have from (4)
$$H_A{}^2 = \alpha^2 e^{-\alpha z} F (dA/dF)^2 \tag{20}$$

and from (5) $$j_A = -\frac{c\alpha^2}{4\pi} e^{-\alpha z}\left(\frac{dA}{dF} + F\frac{d^2 A}{dF^2}\right). \tag{21}$$

Then from (9) $$\frac{\alpha^2}{4\pi}\left(\frac{dA}{dF} + F\frac{d^2A}{dF^2}\right) = -\frac{dq/dF}{dA/dF}$$

or, multiplying by dA/dF, using (20) and replacing q by $pe^{\alpha z}$,
$$\frac{d}{dF}\{e^{\alpha z}(p + H^2/8\pi)\} = -e^{\alpha z}H^2/8\pi F. \tag{22}$$

Given H, equation (22) determines p apart from an arbitrary constant which must be chosen large enough for p to be positive everywhere. Since from (19) F is positive, the right-hand side of (22) is negative and $e^{\alpha z}(p + H^2/8\pi)$ increases as F decreases.

5. *Discussion.*—It has already been pointed out that solutions in which the magnetic field vanishes outside a certain region are of interest; thus we may consider a solution in which the magnetic field is confined to the region between

two of the infinite horizontal cylinders generated by projecting the closed loops in the y-direction. The situation may be specified by the quantity of matter inside the inner cylinder, and in the region of the magnetic field by the quantity of matter inside any cylinder, as a function of the value of A on that cylinder. It is probable that the equilibrium configuration for a system specified in this way is unique and, since the family of solutions discussed above contains a solution for any such specification, this would then be the unique solution.

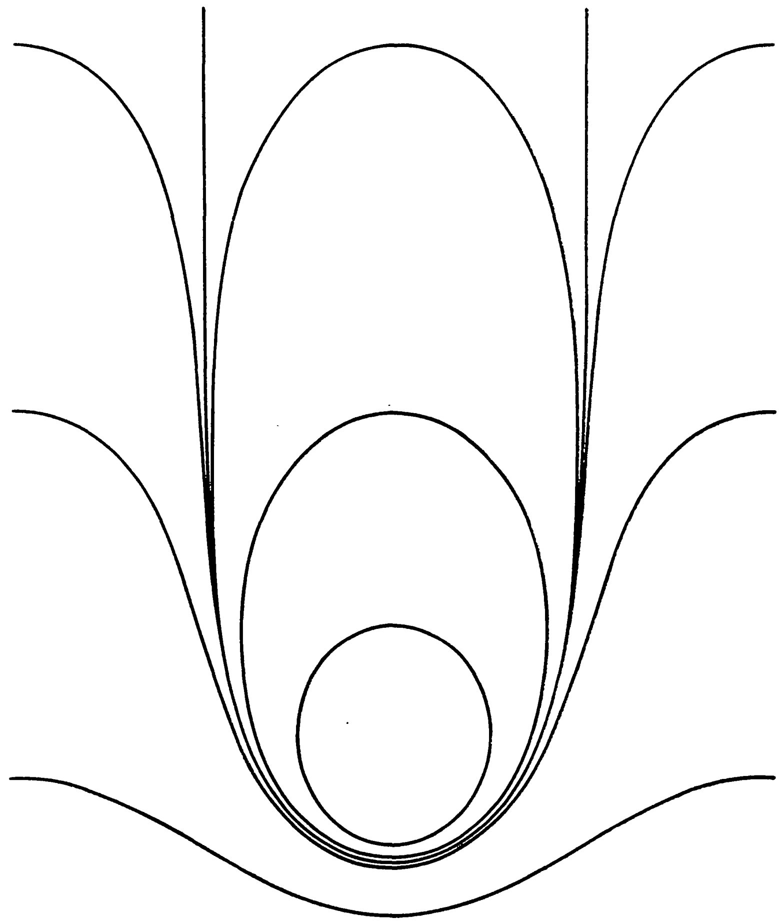


FIG. 2.

Equation (22) shows that the total pressure at a given height increases with decreasing F, so that the total pressure is largest for the inner cylinders. This is due to the first term in (2), which represents a tension in the lines of force and so increases the pressure on the concave side of the lines of force. On a horizontal line passing through the above model the gas density just inside the outer cylinder is reduced by the magnetic field, but in the inner cylinder the density is increased above the normal value. Mechanical equilibrium requires that the total mass contained by the outer cylinder should be the same as it would be in the absence of a magnetic field; otherwise it would float upwards or sink. Consequently the gas pressure inside the inner cylinder may be substantially larger than the normal pressure at the same height. This effect is similar to the constriction of a discharge, which has been discussed by Alfvén (1).

If this model is to be relevant to solar phenomena, the dimensions of the latter must be comparable with the scale height. At the photosphere g is

$2 \cdot 8 \times 10^4$ cm sec^{-2} and the temperature varies from 5 000 deg. K in the reversing layer to about 10^6 deg. K in the corona. The radius of the photosphere r_0 is 7×10^{10} cm and the scale height h varies from $r_0/5\,000$ in the reversing layer to $r_0/25$ in the corona. The horizontal width of the closed loops in Fig. 2 cannot exceed $2\pi h$, and closed loops which are much smaller than this are approximately circular.

If the magnetic field causes a notable variation in the gas pressure, (22) shows that the magnetic pressure must be comparable with the gas pressure, that is $H^2 \sim 8\pi p$. Sunspot fields are known to be of this order at the level of the photosphere; in the upper chromosphere where $p \sim 1$ dyn/cm^2, a field of about 5 gauss is needed.

Prominences are almost certainly regions of high pressure; this increases the radiation loss and explains why prominences are cooler than the surrounding corona. The high pressure may be caused by a magnetic field in the way illustrated by our model. Most quiescent prominences appear on the disk as dark filaments; M. and Mme. d'Azambuja (**4**) have given as typical dimensions: width $\sim r_0/100$, length $\sim r_0/3$ and height $\sim r_0/20$. Since their length is so much greater than their width a rough model for these might be obtained by taking a suitable length of the cylinders in our model and neglecting the lack of equilibrium at the ends. The filament would then be the inner cylinder, where the pressure is high, and the y-axis would have the direction of the filament. On the other hand the temperature of prominences is about 2×10^4 deg. K and since the surrounding corona is at about 10^6 deg. K the isothermal model must be seriously wrong in some respects. Other types of prominence, besides being relatively short-lived, are wispier in appearance and are usually composed of many streamers.

There seems to be better justification for applying our model to the coronal arches observed above prominences, because these are likely to be more nearly isothermal. They are seen in white light at eclipses and are common features of the corona. They are difficult to photograph, but can be seen in drawings (**5**); there are usually several concentric arches alternately dark and bright, with a bright centre. The brightness in the corona is proportional to the line-of-sight integral of the electron density, so that the dark arches have a low density and probably a low gas pressure; this can be explained by the existence of magnetic fields in the dark arches.

A system of arches resembling those observed can be obtained from the upper parts of the closed loops in our model, with a suitable choice of A. The arches are bounded by cylinders generated by projecting the lines of force in the y-direction, and hence the y-direction must be nearly the direction of the line of sight. The model explains why the arches are associated with a prominence and predicts that this is a filament directed in the line of sight, that is east–west on the Sun. No data are available to check this, but the orientation of filaments is closely correlated with their heliographic latitude and filaments lying nearly east–west occur at latitudes greater than 50°; although arches can be found at all latitudes the most pronounced occur near latitude 60°. Assuming a temperature of 10^6 deg. K, the model predicts that the width of the arches should not exceed $r_0/4$; arches are observed with widths up to twice this value. Photometric measurements by von Klüber (**6**) show that the electron density in the dark arches must be an order of magnitude less than the density in the normal surroundings, and that the extension of the dark arches along the line of sight must be a considerable fraction of the solar radius.

In conclusion, the model described in this paper is useful as an illustration of the effect of the magnetic force. Although it is too simple to apply directly to prominences, the points of similarity encourage the belief that the same physical effects are important in prominences.

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University of Sydney :
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