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A FAST DISCRETE FOURIER TRANSFORM WITH UNEQUALLY-SPACED FREQUENCIES

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Realization of the discrete Fourier transform (DFT) as the fast Fourier transform (FFT) has a widespread application in practice.
There are, however, such problems in technology where the application of the FFT method does not justify satisfactory results. Such problems comprise e.g. the analysis of short sections of fast decaying vibrations or the analysis of instantaneous values of nonstationary signal parameters. Solving the problems arising in this type of signals analysis has contributed to development of the algorithm of the DFT fast computing for transient vibrations. The developed method enables one to obtain high resolution for low frequencies by increasing the density of sampling of the analyzed signal. The algorithm in terms of the computer program (Lenort (1989)) has been tested both on the model data and on the real signal records.

1. Introduction

When evaluating the aircraft structure in flight response to the impulse excitation, some problems arose in the modal analysis of transient signals. Forced vibrations of e.g. a wing were decaying within one second, half a second and even a shorter period of time (see Fig.1). This response should constitute a basis for defining natural frequencies of vibration components, damping coefficients and vibration amplitudes, respectively.

Analysis of the signals having frequencies from a few Hz up to tens of Hz is, if carried out by means of the FFT, very complicated.

The resolution of FFT is well known as

$$\Delta f = \frac{1}{T}[Hz]$$

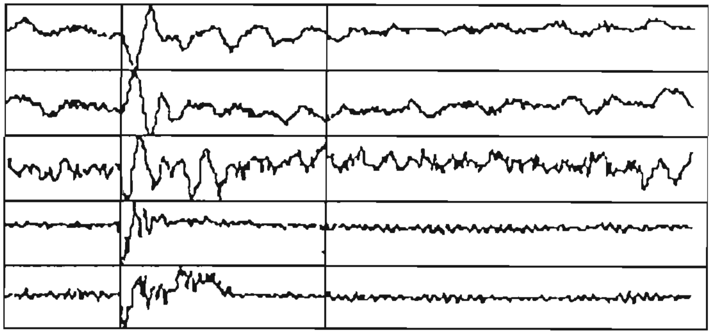

Fig. 1. The impulse response of an aircraft with underwing stores, measured at the speed $V = 870$ kph. Time of the response decaying marked by vertical lines does not exceed 0.3 sec. Measurement done by means of strain gauges

where T is the length of evaluated signal. For $T = 1$ second the resolution is $\Delta f = 1$ Hz, such a definition of the natural frequency is therefore of low precision. The so-called 'making up' of short signals by supplementing them by zeros, used sometimes, results in the decrease in sampling signal density, i.e. the loss in information and deformation of the analysis results.

Neglecting the speed and many other advantages of the FFT, modal frequencies may be defined more precisely by considering sufficiently small values of Δf and computing the DFT as follows

$$Y(k\Delta f) = \frac{2}{N} \sum_{n=0}^{N-1} y(n\Delta t) \exp\Big(-\mathrm{j}2\pi \frac{\Delta f}{f_s} kn\Big) \tag{1.1}$$

where

- $y(n\Delta t)$ – sampled signal under consideration
- f_s – frequency of sampling signals $y(t)$, $f_s = 1/\Delta t$
- Δf – optionally selected transform resolution, for example $\Delta f = 0.01$ Hz; limitations in selecting Δf and k arise from the Nyquist criterion and other rules of a correct signals processing (cf Mańczak (1971)).

By means of this transformation more precise natural frequencies of the object can be defined, despite the possibility of the so-called 'side areas' appearance, which are connected results of with the analysis of finite sections of vibrations.

As an example Fig.2 presents comparison of the FFT resolution and realization of the DFT in accordance with Eq (1.1) for the model signal having the form of a sine curve with the frequency $f = 10.5$ Hz, for $T = 1$ sec, $f_s = 1024$ samples per second and $\Delta f = 0.1$ Hz.

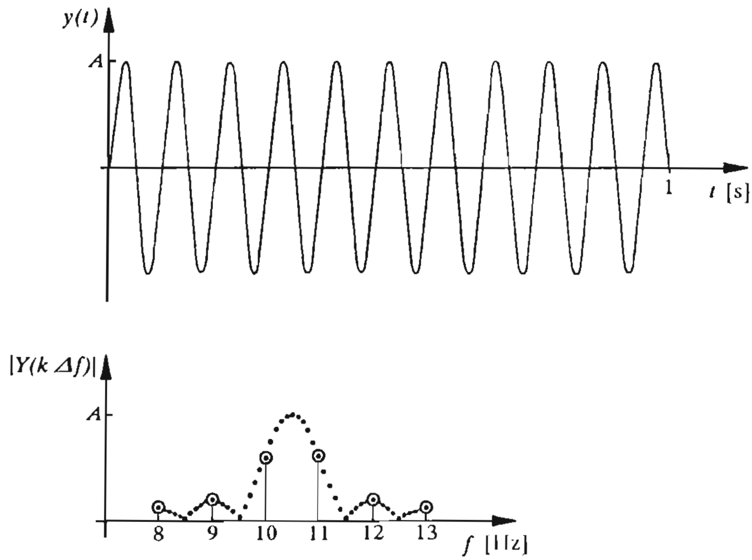

Fig. 2. Some results of the frequency and amplitude analysis of the mode signal $y(t)$ having frequency $f = 10.5$ Hz carried out in terms of the Fourier transformation for $T = 1$ sec and $f_s = 1024$ samples per second; $\odot$ – computed by the FFT method, $\bullet$ – computed according to Eq (1.1)

This example shows that the FFT represents the true frequency of a signal and its amplitude, only if the number of signal cycles on the evaluated section is complete (cf Brigham (1974)).

On the other hand, realization of the DFT in accordance with Eq (1.1) for $\Delta f = 0.1$ Hz may in this case provide the precise determination of the frequency and the amplitude of that signal. The time of computing is, however, several hundred times longer and the analysis of experimental data collected during one flight, carried out in this way, is impossible due to time-consuming computations of thousands of transforms.

The aforementioned disadvantages admitted the necessity for formulation (for such a type of signals and a group of tasks) of a new kind of algorithm for the DFT computation, revealing high resolution at low frequencies and the computation speed comparable to that of FFT.

2. Algorithm for fast computation of the DFT revealing high resolution at low frequencies

The FFT is a well-known, very fast procedure for computation of a transform applying the formula

$$Y(k\Delta f) = \frac{2}{N}\sum_{n=0}^{N-1} y(n\Delta t)\exp\Big(-\mathrm{j}\frac{2\pi}{N}kn\Big) \qquad k = 0,1,2,...,N-1 \quad (2.1)$$

Taking the above into consideration, the algorithm for the fast computation of DFT is based on the formula

$$Y(f_k) = \frac{2}{N}\sum_{n=0}^{N-1} y(n\Delta t)\exp\Big(-\mathrm{j}\frac{2\pi}{k}n\Big) \qquad k = k_{st},...,k_{end} \qquad (2.2)$$

Frequencies f_k of the computed transform lines can be written as

$$f_k = \frac{f_s}{k}$$

Lines on the frequency axis are spaced unequally. On the other hand the period for successive lines is constant and equals

$$T_p = k\Delta t = \frac{k}{f_s}$$

The resolution of this realization of DFT is (according to Lenort (1989))

$$\Delta f = \frac{f^2}{f_s}$$

The increase in frequency of the time signal sampling results in the resolution improvement. On the contrary, the very good resolution for low frequencies deteriorates with the square of a high value of f.

Fig.3. show the transform of the signal from Fig.2, realized in accordance with Eq (2.2) for $f_s = 1024$ samples per second. The real frequency 10.5 Hz of a signal $y(t)$ has been determined by means of the DFT employing Eq (2.2), with accuracy of 0.05 Hz. It should be noticed that also the signal amplitude has been determined more precisely than when applying the FFT. In this case the use of FFT produces the amplitude decrease by approximately 38 per cent and the transform line decrease to 1 Hz.

A very high speed of the FFT computing has been reached by lowering the number of multiplications (cf Brigham (1974)).

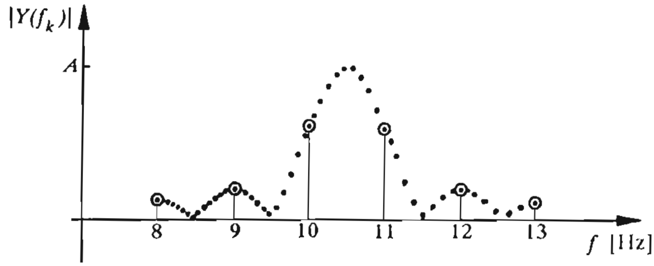

Fig. 3. Determination of the model signal $y(t)$ (Fig.2) frequency and amplitude by means of the DFT in accordance with Eq (2.2); ⊙ – computed by the FFT method, • – computed in accordance with Eq (2.2)

Similarly one can reach a very high speed of computations when realizing the DFT in accordance with Eq (2.2). For even values of k the decrease in number of multiplications is $4fT$-times (cf Lenort (1989)), while for odd values is is $2fT$-times, respectively.

There exists the possibility to compute the single lines only, what considerably decreases the time necessary e.g. for the damping factor computation (cf Lenort (1984)). Due to the algorithm procedure when employing the FFT one should always compute the set of lines.

The possibility of reducing the number of multiplications when realizing the DFT in accordance with Eq (2.2) is shown below.

If it is assumed in Eq (2.2) that

$$n = n_0 + kn_1 \qquad n_0 = 0,1,2,...,k-1 \qquad n_1 = 0,1,2,...,[N/k]$$

where $[N/k]$ is a function "entier", $[N/k] = \mathrm{E}(N/k)$

$$W_k = \exp\Big(-\mathrm{j}\frac{2\pi}{k}\Big)$$

Eq (2.2) can be rewritten as

$$\begin{aligned} Y(f_k) &= \frac{2}{N}\sum_{n_0=0}^{k-1}\sum_{n_1=0}^{[N/k]} y(n_0+kn_1)W_k^{(n_0+kn_1)} = \\ &= \frac{2}{N}\sum_{n_0=0}^{k-1}\sum_{n_1=0}^{[N/k]} y(n_0+kn_1)W_k^{n_0}W_k^{kn_1} = \\ &= \frac{2}{N}\sum_{n_0=0}^{k-1}\sum_{n_1=0}^{[N/k]} y(n_0+kn_1)W_k^{n_0} = \frac{2}{N}\sum_{n_0=0}^{k-1} W_k^{n_0}\sum_{n_1=0}^{[N/k]} y(n_0+kn_1) \end{aligned} \tag{2.3}$$

because

$$W_k^{kn_1} = \left(W_k^k\right)^{n_1} = (1)^{n_1} = 1$$

As it can be seen from the formula (2.2) the number of multiplications required to compute the k-line has been reduced n_1 times, i.e. $[N/k]$ times. For $N = 2000$ and $k = 20$ the the number of multiplications has been reduced 100 times and proportionally has been limited the time of the transform computing.

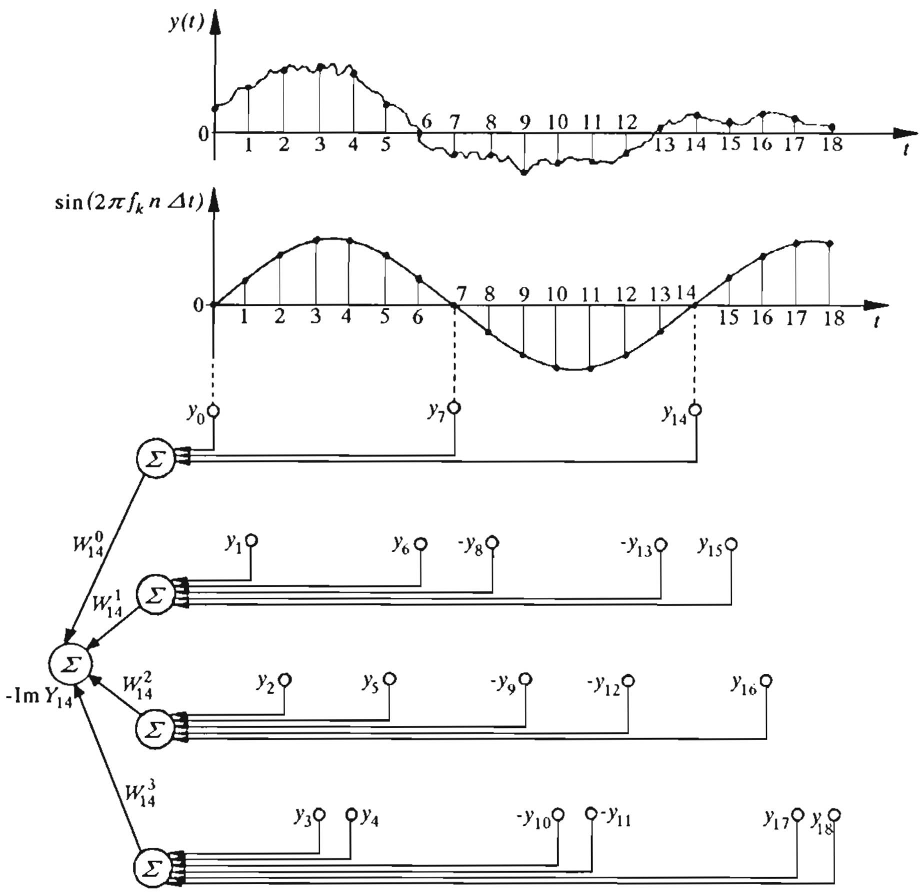

Fig. 4. Graphical presentation of the principles applicable to the realization of DFT computed in accordance with Eq (2.2) for $N = 19$ and $k = 14$. Determination of the imaginary part of a transform.

In order to simplify the presentation of the possibilities of further decrease in the number of multiplications for k as an even number one may employ the graphic method. For better understanding let us draw only the scheme of method of the imaginary part of k-line computing (Fig.4).

Instead of multiplying the successive values of the $y(n\Delta t)$ function by the values of sine function at these points (what would require N multiplications) we add up at first the values of $y(n\Delta t)$ function at the points where the values of sine function are identical and only then we multiply this sum by the appropriate value of sine function.

As an example we may write, in accordance with Fig.4

$$y_1 \sin\alpha_1 + y_6 \sin\alpha_6 + y_8 \sin\alpha_8 + y_{13} \sin\alpha_{13} + y_{15} \sin\alpha_{15} =$$
$$= (y_1 + y_6 - y_8 - y_{13} + y_{15}) \sin\alpha_1$$

In this way, for even values of k the $4fT$-times decrease in the number of multiplications is obtained as well as the appropriate decrease in the time of the transform computation.

Some features of the described DFT are presented below on the background of the FFT features.

Fast Fourier transform (FFT)

Works for $N = 2^l$.

Resolution $\Delta f = 1/T = f_s/N$.

Equal spacing of lines along the frequency axis $f = k\Delta f$, k is the number of cycles on the T-signal section.

For $k > N/2$ the Nyquist criterion is not met, for $k = 0$ computed the average value of signal is computed.

$2N/l$ times decrease in the number of multiplications.

Computed real and imaginary parts constitute the coefficients of Fourier series, with integration replaced by summing-up.

The worked out algorithm for computing the complete set of lines.

In order to increase the resolution, the so called 'making-up' of a signal with zeros is used.

The method for very wide range of applications.

Discrete Fourier transform (DFT)

For every N.

Resolution $\Delta f = f^2/f_s$.

Equal spacing of lines along the axis of vibration period $T_p = k\Delta t$. Unequal spacing of lines along the frequency axis $f = f_s/k$, k is the number of samples in a cycle.

For $k \geq 2$ the Nyquist criterion is met, for $k = 1$ the average value of a signal is computed.

For even values of k, $4fT$-times decrease in the number of multiplications appears, for odd values of k, $2fT$-times decrease appears.

Results of the discrete transformation are close to the continuous Fourier transformation for signals with a finite length.

Fast computing of a single transform line is possible.

In order to increase resolution single zero values may be placed between signal samples to obtain the resolution $\Delta f = f^2/(2f_s)$ or e.g. triple zero values to obtain the resolution $\Delta f = f^2/(4f_s)$.

The practical method for the modal analysis, useful in determining natural frequencies, damping coefficients, amplitudes and phases of short signals with low frequency.

3. Concluding remarks

The presented algorithm for fast computation of the discrete Fourier transformation has been tested on the model and real data (cf Lenort (1989)).

This algorithm is available in the form of computer codes in Fortran language. One of them is presented by Lenort (1989).

The described procedure of computing the DFT is the suitable and fast tool for the analysis of the signals like e.g. vibrations appearing during flight tests of the flutter tendency of aircraft structures. The possibility of a single transform line fast computation is very useful when determining the damping coefficient of a noise-disturbed signal. High resolution of this transform at low frequencies enables one to define properly the modal frequencies even from the short-lasting, fast-vanishing signals.

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Szybka dyskretna transformata Fouriera o nierównomiernym rozłożeniu częstości

Sreszczenie

Bardzo szerokie zastosowanie w praktyce znalazła realizacja dyskretnego przekształcenia Fouriera (DFT) w postaci szybkiej transformaty Fouriera (FFT). Powszechne zastosowanie metody FFT do obliczania DFT najlepiej świadczy o jej zaletach.

Istnieją jednak w technice zagadnienia, do których zastosowanie metody FFT nie daje zadawalających wyników. Do takich zagadnień można zaliczyć analizę krótkich odcinków, szybko zanikających drgań lub analizę chwilowych własności sygnału niestacjonarnego.

Problemy analizy tego typu sygnałów przyczyniły się do opracowania algorytmu szybkiego obliczania DFT dla krótkich realizacji drgań. Opracowana metoda pozwala na uzyskanie dużej rozdzielności dla małych częstości poprzez zwiększenie gęstości próbkowania sygnału analizowanego. Algorytm w postaci programu na EMC (Lenort (1989)) został przetestowany na danych modelowych i na rzeczywistych przebiegach sygnałów.

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