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A fast method for fully nonlinear water wave computations

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Abstract

A fast computational method for fully nonlinear non-overturning water waves is derived in two and three dimensions. A corresponding computational scheme is developed in the two-dimensional case. The essential part of the method is a fast converging iterative solution procedure of the Laplace equation. One part of the solution is obtained by Fast Fourier Transform, while another part is highly nonlinear and consists of integrals with kernels that decay quickly in space. The number of operations required is asymptotically $\mathcal{O}(N \log N)$, where N is the number of nodes at the free surface. The scheme is so rapidly convergent that one iteration is sufficient in practical computations, while maintaining high accuracy. This constitutes an explicit approximation of the scheme. Any accuracy of the computations is achieved by a continued iteration of the equations. Illustrative examples complement the derivations.

1 Introduction

Fully nonlinear models for water waves are employed to make advanced studies of various complex wave phenomena. These include the important topic of nonlinear inviscid potential flow with a free surface, where recent reviews may be found in e.g. Tsai & Yue (1996) and Dias & Kharif (1999). A common drawback of the existing fully nonlinear methods, however, is that the computational schemes are slow. This means that long time simulations of wave-fields with appreciable size are unrealistic. While the integration of the prognostic equations can be made fast, the bottleneck is the solution of the Laplace equation which is required at each time step. Thus, a fully nonlinear model for water waves can only be fast provided that the Laplace equation solver is fast. Fully nonlinear and fast wave models that in a realistic way can be used to analyse highly nonlinear wave phenomena, such as freak waves, steep transient waves or steep irregular wave fields, are lacking (ISSC-report 2000). This is the motivation of the present study where the primary focus is to derive a fast and robust Laplace equation solver, and thereby a computationally fast model for fully nonlinear water waves.

Computationally fast fully or highly nonlinear methods exist, but they have various drawbacks that limit their usefulness. The method by Fornberg (1980), applying a conformal mapping, is efficient but has the disadvantage that the spatial resolution becomes poor at the crest of steep waves, where high resolution is required. Conversely, computational nodes become dense at the troughs. Further, the method can only be used for one-dimensional wave propagation. The high-order spectral methods outlined by Dommermuth & Yue (1987) and West *et al.* (1987) are based on Taylor series expansion of the velocity potential at the free surface about the mean water line (or about another reference level). They are computationally efficient when the series converge. For steep waves, however, the methods involve high-order derivatives of the velocity field. The models then become numerically unstable. The gain by the added terms is lost by the necessity of stronger smoothing than for the low order version of the method. In practice, few terms are used (Dommermuth & Yue 1987, Yasuda & Mori 1994). For highly nonlinear waves the methods do not converge. We note that slowly modulated waves may successfully be modelled by means of the nonlinear Schrödinger equation (NLS), where a narrowbanded spectrum is assumed. The modifications by Dysthe (1979) and Trulsen & Dysthe (1996) increase the bandwidth of the NLS somewhat, but the resulting equations are generally not sufficient to provide a sound model of a realistic sea.

We here derive a novel rapid method for fully nonlinear non-overturning water waves. The method is outlined for two and three dimensions. A fast Laplace equation solver is obtained by means of integral equations. One part of the solution is obtained by Fast Fourier Transform of the potential at the free surface and products betwen this potential, or its horizontal derivatives, and the wave elevation. The other part is highly nonlinear and consists of integrals that may be evaluated in a rapid way since the integrands are quickly decaying in space. Computations for one-dimensional wave propagation show that integration over a distance of one characteristic wavelength is sufficient for these integrals, even for highly nonlinear waves. The resulting implicit equation for the unknown function forms a basis for an iterative scheme with rapid convergence. The number of operations required is asymptotically $\mathcal{O}(N \log N)$, where N is the number of computational nodes. One iteration of the scheme is found to be sufficient for practical purposes. We shall term this approximation to be the explicit approximation of the method. However, any accuracy is achieved by a continued iteration of the equations.

For simplicity, all derivations are made for water of infinite depth. The method may, however, easily be extended to a fluid layer bounded by a bottom, where the latter may be varying. Generalizations of the method are further elaborated in the final part of the paper and in the appendix.

2 Two-dimensional motion

We first consider the two-dimensional problem of a fluid which is homogeneous, incompressible and inviscid. The wave induced motion is irrotational and the depth is infinite. Let x, y, t be the horizontal, upward vertical and time variables, and let $\eta(x,t)$ be the surface elevation relative to the mean level $y=0$. These assumptions imply the existence of a velocity potential ϕ and a stream function ψ. These quantities are linked, for $-\infty \le y \le \eta$, by the Cauchy–Riemann relations, $\phi_x = \psi_y$, $\phi_y = -\psi_x$. The functions ψ and ϕ_y decay to zero for $y \to -\infty$. The surface impermeability gives $\phi_y = \eta_t + \phi_x \eta_x$ at $y = \eta$. The pressure is either zero or prescribed at the surface, and the Bernoulli equation gives that $g\eta + \phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \widetilde{p} = 0$ at $y = \eta$, where g is the acceleration due to gravity and $\widetilde{p}$ the (given) pressure at the free surface (normalized by the density of the fluid). For non-overturning waves, this set of equations can be reformulated with quantities at the surface only

$$\eta_t + \widetilde{\psi}_x = 0, \qquad \widetilde{\phi}_t + g\eta + \frac{1}{2}\frac{\widetilde{\phi}_x^2 - \widetilde{\psi}_x^2 + 2\,\eta_x \widetilde{\phi}_x \widetilde{\psi}_x}{1+\eta_x^2} + \widetilde{p} = 0, \tag{1}$$

where the 'tildes' denote the functions at $y = \eta$. Other formulations are possible, involving the normal derivative of ϕ, for example. These equations are the evolution equations of η and $\widetilde{\phi}$, and can be integrated once $\widetilde{\psi}$ is known. The harmonic functions ϕ and ψ may be obtained in several ways. In two dimensions the powerful theory of complex functions may be used. Using the Cauchy integral formula, split into real and imaginary parts, the following equations are deduced (Baker, Meiron & Orzag 1982), in our notation

$$\widetilde{\phi} = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{D\,(\widetilde{\phi}' - \eta_x' \widetilde{\psi}') - \widetilde{\psi}' - \eta_x' \widetilde{\phi}'}{1+D^2} \frac{\mathrm{d}x'}{x'-x}, \tag{2}$$

$$\widetilde{\psi} = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\widetilde{\phi}' - \eta_x' \widetilde{\psi}' + D\,(\widetilde{\psi}' + \eta_x' \widetilde{\phi}')}{1+D^2} \frac{\mathrm{d}x'}{x'-x}, \tag{3}$$

where $\widetilde{\phi} = \widetilde{\phi}(x,t)$, $\widetilde{\phi}' = \widetilde{\phi}(x',t)$, etc. In (2)–(3) the function $D = (\eta' - \eta)/(x' - x)$ is introduced, where D decays according to $|x' - x|^{-1}$ for $|x' - x| \to \infty$ and $D \to \eta_x$ for $x' \to x$. The equation (3), or equations that are similar, are commonly used to determine $\widetilde{\psi}$, given $\widetilde{\phi}$ and η. $\widetilde{\psi}$ is then determined implicitly, and the equation is typically solved iteratively with $\mathcal{O}(N^2)$ operations. This is the intensive part of the computations. An alternative, however, is to determine $\widetilde{\psi}$ from equation (2). This leads to a significantly faster iterative scheme, as we shall see, than working with equation (3).

When the surface is horizontal, the integral equations are convolution products and can therefore be computed very quickly *via* Fast Fourier Transform, for example. For a non-horizontal surface it is then tempting to reformulate these integrals obtaining the form of convolutions. Splitting (2) into singular and regular integrals we obtain after one integration by parts

$$\begin{aligned}\tilde{\phi} = &-\frac{1}{\pi}\, -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\tilde{\psi}'}{x'-x}\,\mathrm{d}x' + \frac{1}{\pi}\, -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\eta'\,\tilde{\phi}'_x}{x'-x}\,\mathrm{d}x' - \frac{\eta}{\pi}\, -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\tilde{\phi}'_x}{x'-x}\,\mathrm{d}x' \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty} [\arctan(D) - D]\,\tilde{\phi}'_x\,\mathrm{d}x' + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{D\,(D-\eta'_x)\,\tilde{\psi}'}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}. \end{aligned} \tag{4}$$

Applying the Hilbert transform (i.e. $\mathcal{H}\{f\} \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x'$, $\mathcal{H}^{-1} = -\mathcal{H}$), equation (4) becomes

$$\begin{aligned}\tilde{\psi} = &\,\mathcal{H}\left\{\tilde{\phi}\right\} + \eta\,\tilde{\phi}_x + \mathcal{H}\left\{\eta\,\mathcal{H}\left\{\tilde{\phi}_x\right\}\right\} \\ &- \mathcal{H}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} [\arctan(D) - D]\,\tilde{\phi}'_x\,\mathrm{d}x' + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{D\,(D-\eta'_x)\,\tilde{\psi}'}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}\right\}. \end{aligned} \tag{5}$$

This is another equation for $\tilde{\psi}$. In (5), the singular integrals are convolutions and can thus be computed quickly, with computational burden $\mathcal{O}(N \log N)$. The remaining regular integrals have kernels that decrease rapidly, as $|x'-x|^{-3}$ and $|x'-x|^{-2}$, respectively. Therefore, integrations over $(-\infty, +\infty)$ can be approximated by integrations over a limited interval $(x-\lambda, x+\lambda)$. The parameter λ is choosen in accordance with the precision needed and depends on the wave characteristics and not on the length of the computational domain (see below). Moreover, the contribution on the right hand side of (5) involving $\tilde{\psi}$, is cubic in nonlinearity, while in equation (3) the corresponding term is quadratic. For nonbreaking waves, where cubic terms are smaller than quadratic ones, iterations with (5) converge faster than iterations with (3).

Note that analog transformations of (3), i.e. in a way that the kernels of the regular integrals decrease at least as $|x'-x|^{-2}$, give

$$\begin{aligned}\tilde{\psi} = &\,\mathcal{H}\left\{\tilde{\phi}\right\} + \mathcal{H}\left\{\eta\,\tilde{\psi}_x\right\} - \eta\,\mathcal{H}\left\{\tilde{\psi}_x\right\} \\ &+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{D\,(D-\eta'_x)\,\tilde{\phi}'}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x} + \frac{1}{\pi}\int_{-\infty}^{\infty} [\arctan(D) - D]\,\tilde{\psi}'_x\,\mathrm{d}x'. \end{aligned} \tag{6}$$

This equation is of the form $\tilde{\psi} = F(\tilde{\psi}_x)$ and is numerically unstable. This is in contrast to equation (5) which is of the form $\tilde{\psi} = F(\tilde{\psi})$ and is stable. However, equation (6) can be applied to obtain $\tilde{\phi}$ when η and $\tilde{\psi}$ are given.

3 Successive approximations

An iterative scheme is initialized by the explicit quadratic approximation

$$\tilde{\psi}_1 = \mathcal{H}\left\{\tilde{\phi}\right\} + \eta\,\tilde{\phi}_x + \mathcal{H}\left\{\eta\,\mathcal{H}\left\{\tilde{\phi}_x\right\}\right\}. \tag{7}$$

Applying one analytical iteration, neglecting integrals being of quartic nonlinearity, we get another approximation

$$\tilde{\psi}_{2,\lambda} = \tilde{\psi}_1 - \mathcal{H}\left\{\frac{1}{\pi}\int_{x-\lambda}^{x+\lambda} \frac{D\,(D-\eta'_x)\,\tilde{\psi}'_1}{1+D^2}\,\frac{\mathrm{d}x'}{x'-x}\right\}. \tag{8}$$

The latter is explicit and does not involve transcendental functions. It is very accurate (see below) and quickly computable. We term (8) by the explicit approximation of (5). To estimate the regular integrals in (5) we consider a steady periodic linear wave approximation given by $\eta \simeq a\cos kx$, $\widetilde{\phi} \simeq a\,\omega/k \sin kx$, $\widetilde{\psi} \simeq a\,\omega/k \cos kx$, $\omega^2 \simeq gk$. First the width of a truncated integration is investigated by considering

$$I_\lambda = \mathcal{H}\left\{ \frac{1}{\pi} \int_{x-\lambda}^{x+\lambda} \frac{D\,(\,D - \eta'_x\,)\,\widetilde{\psi}'}{x' - x}\,\mathrm{d}x' \right\}. \tag{9}$$

An integration limited to one wavelength ($\lambda = \pi/k$) gives

$$\max\left(\left| \frac{I_{\frac{\pi}{k}} - I_\infty}{I_\infty} \right| \right) = \frac{8}{\pi^2} - \frac{\pi + 3\,\mathrm{Si}(\pi) - 8\,\mathrm{Si}(2\pi) + 3\,\mathrm{Si}(3\pi)}{\pi} \simeq 0.05, \tag{10}$$

which indicates that an integration over one wave length gives a sufficiently accurate approximation of the integral. In (10) Si denotes the sine integral. Even when the computational domain includes several characteristic wavelengths, the truncated integration provides an accurate and fast result.

The magnitude of the final integral in (5) is investigated. With the linear approximation this becomes

$$J = \frac{1}{\pi} \int_{-\infty}^{\infty} [\,\arctan(D) - D\,]\,\widetilde{\phi}'_x\,\mathrm{d}x' \simeq -\tfrac{1}{24}\,a^4\,k\,\omega \sin 2kx. \tag{11}$$

The relative contribution of this integral is thus $\max(|\mathcal{H}\{J\}/\widetilde{\psi}|) \simeq a^3k^3/24$. For a limiting wave ($ak \simeq 0.44$, Fenton 1990), this means 0.4%. For unsteady simulations, however, the relative contribution from this integral is much smaller, since unsteady waves become unstable somewhat below the limiting slope of Stokes waves. It follows that this integral can be neglected for most of applications.

If more precision is needed, the full equation (5) has to be solved. Starting iterations with $\widetilde{\psi}_1$ and a given λ (increased at each iteration), we have obtained a rapidly convergent and fast Laplace solver. For long tank simulations the number of operations needed in evaluating the regular integrals in (3) or (8) has been reduced from $\mathcal{O}(N^2)$ to $\mathcal{O}(M \times N)$ operations, where $M \ll N$. This means that the number of operations is $\mathcal{O}(N \log N, NM) \sim \mathcal{O}(N \log N)$ for $N \to \infty$.

4 Numerical comparisons

We first compare our formulae with an exact solution of Stokes waves. Fenton (1988) gave a program for the latter, and we have used this program to compute waves with $ak \simeq 0.41$ which is 93% of the limiting wave, $2a$ being the total wave height. This is the steepest wave we were able to obtain with Fenton's program. The cubic approximation $\widetilde{\psi}_{2,\lambda}$, given by (8), has been evaluated with an integration over one wavelength ($\lambda = \pi/k$). It is found that $\widetilde{\psi}_{2,\lambda}$ is a very accurate approximation, while $\widetilde{\psi}_1$ is not sufficiently precise (Fig. 1). These results are more pronounced when the horizontal derivative of the stream function is considered.

We also apply the approximation $\widetilde{\psi}_{2,\lambda}$ in unsteady wave simulations. An accurate numerical model is developed where the x-axis is discretized with a constant step length Δx. A staggered grid is used. The temporal evolution equations are integrated by a fourth-order Runge–Kutta scheme, and spatial derivatives are evaluated *via* FFT. The scheme is developed in a similar way as the one by Baker *et al.* (1982). The waves are generated with a pneumatic wavemaker in one end of the tank, and damped by a spongious absorber at the other end. More details of the wave generation and absorption can be found in Clamond & Grue (2000). In the test the generation of a wave train from rest is investigated comparing simulations using (3) and (8). The latter is evaluated with $\lambda = \pi/k_0$ where $k_0 = \omega^2/g$, ω the angular frequency of the wave maker. The results show that the two methods are almost identical, where the differences measured by the root mean square is less than 1% (Fig. 2). The novel method gave the results about fifteen times faster than by using equation (3), for a computational domain of about thirty characteristic wavelengths. If a longer tank is used the gain increases. We note that the wave broke a few time-steps later than the profile shown in Fig. 2. This illustrates the high nonlinearity of the unsteady steep wave. (Similar wave breaking is also observed in a physical wave tank.)

5 Generalization to three dimensions

Generalization to three dimensions is then considered. In this case the Laplace equation has to be solved in a different way than by using complex theory, since the latter is limited to two-dimensional flows. Green's theorem is applied for this purpose. The y-coordinate is kept as above, while $\mathbf{x} = (x, z)$ are the two horizontal Cartesian coordinates.

The prognostic equations in three dimensions are similar to those in the two-dimensional case as given in equation (1) (see Tsai & Yue 1996). These equations update the velocity potential $\widetilde{\phi}$ at the free surface and the elevation η when the (outgoing) normal velocity ϕ_n is known. The latter is obtained from the solution of the Laplace equation when $\widetilde{\phi}$ and η are given on the free surface. As in the previous paragraphs we assume that the depth of the water is infinite. From Green's theorem we have

$$\iint_S \frac{1}{r}\frac{\partial \phi'}{\partial n'}\,\mathrm{d}S = 2\pi\widetilde{\phi} + \iint_S \widetilde{\phi}'\,\frac{\partial}{\partial n'}\frac{1}{r}\,\mathrm{d}S, \tag{12}$$

where $\widetilde{\phi} = \widetilde{\phi}(\mathbf{x})$, $\widetilde{\phi}' = \widetilde{\phi}(\mathbf{x}')$, $r = [\mathbf{R}^2 + (y' - y)^2]^{\frac{1}{2}}$, $\mathbf{R} = (\mathbf{x}' - \mathbf{x})$ and S denotes the free surface. The element of the latter is given by $\mathrm{d}S = \sqrt{1 + (\nabla'\eta')^2}\,\mathrm{d}x'\mathrm{d}z'$ where $\nabla = (\partial_x, \partial_z)$ denotes the horizontal gradient, giving

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V'}{R\sqrt{1+D^2}}\,\mathrm{d}x'\mathrm{d}z' = 2\pi\widetilde{\phi} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+D^2)^{-\frac{3}{2}} \left(\frac{\mathbf{R}\cdot\nabla'\eta'}{R^3} - \frac{\eta' - \eta}{R^3}\right)\widetilde{\phi}'\,\mathrm{d}x'\mathrm{d}z', \tag{13}$$

where $R^2 = \mathbf{R}\cdot\mathbf{R}$, $V = \phi_n\sqrt{1 + (\nabla\eta)^2}$ and $D = (\eta' - \eta)/R$ are introduced. We have that $D \sim R^{-1}$ for $R \to \infty$ and $D \to \eta_R$ for $R \to 0$. We now exploit that

$$\frac{\mathbf{R}\cdot\nabla'\eta'}{R^3} - \frac{\eta' - \eta}{R^3} = -\nabla'\cdot\left[(\eta' - \eta)\nabla'\frac{1}{R}\right]. \tag{14}$$

By application of Gauss theorem we may partially rewrite the last term in (13). The modified and reorganized version of the equation reads

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V'}{R}\,\mathrm{d}x'\mathrm{d}z' = {} & 2\pi\widetilde{\phi} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta' - \eta)\nabla'\widetilde{\phi}'\cdot\nabla'\frac{1}{R}\,\mathrm{d}x'\mathrm{d}z' \\ & - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\phi}'\left[(1+D^2)^{-\frac{3}{2}} - 1\right]\nabla'\cdot\left[(\eta' - \eta)\nabla'\frac{1}{R}\right]\mathrm{d}x'\mathrm{d}z' \\ & - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V'}{R}\left[(1+D^2)^{-\frac{1}{2}} - 1\right]\mathrm{d}x'\mathrm{d}z'. \end{aligned} \tag{15}$$

A decomposition $V = V_1 + V_2 + V_3 + V_4$ is then introduced, where V_1, V_2, V_3, V_4 satisfy, respectively,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V_1'}{R}\,\mathrm{d}x'\mathrm{d}z' = 2\pi\widetilde{\phi}, \tag{16}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V_2'}{R}\,\mathrm{d}x'\mathrm{d}z' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\eta' - \eta)\nabla'\widetilde{\phi}'\cdot\nabla'\frac{1}{R}\,\mathrm{d}x'\mathrm{d}z', \tag{17}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V_3'}{R}\,\mathrm{d}x'\mathrm{d}z' = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\phi}'\left[(1+D^2)^{-\frac{3}{2}} - 1\right]\nabla'\cdot\left[(\eta' - \eta)\nabla'\frac{1}{R}\right]\mathrm{d}x'\mathrm{d}z', \tag{18}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V_4'}{R}\,\mathrm{d}x'\mathrm{d}z' = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{V'}{R}\left[(1+D^2)^{-\frac{1}{2}} - 1\right]\mathrm{d}x'\mathrm{d}z'. \tag{19}$$

A Fourier transform is then applied to the equations. For the left hand sides of (16)–(19) we get

$$\mathcal{F}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{V_j'}{R}\,\mathrm{d}x'\mathrm{d}z'\right\}=\frac{2\pi}{\nu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}V_j'\,e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}\,\mathrm{d}x'\mathrm{d}z'=\frac{2\pi\mathcal{F}\{V_j\}}{\nu},\quad j=1,...,4,\qquad(20)$$

where $\mathcal{F}$ denotes Fourier transform, $\nu=\sqrt{\mathbf{k}\cdot\mathbf{k}}$, and we have exploited that $\mathcal{F}\{1/R\}=(2\pi/\nu)e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}$. The transformed equation (16) becomes $\mathcal{F}\{V_1\}=\nu\mathcal{F}\{\widetilde{\phi}\}$ giving

$$V_1=\mathcal{F}^{-1}\left\{\nu\,\mathcal{F}\left\{\widetilde{\phi}\right\}\right\}.\qquad(21)$$

The Fourier transform of (17) leads to $\mathcal{F}\{V_2\}=-\nu\mathcal{F}\{\eta V_1\}-\mathrm{i}\mathbf{k}\cdot\mathcal{F}\{\eta\nabla\widetilde{\phi}\}$, giving

$$V_2=-\mathcal{F}^{-1}\left\{\,\nu\,\mathcal{F}\left\{\,\eta V_1\,\right\}\right\}-\nabla\cdot\left(\eta\,\nabla\widetilde{\phi}\right).\qquad(22)$$

Further, from (18)–(19) we obtain

$$V_3=-\frac{1}{2\pi}\mathcal{F}^{-1}\left\{\nu\,\mathcal{F}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\phi}'\left[(1+D^2)^{-\frac{3}{2}}-1\right]\nabla'\cdot\left[(\eta'-\eta)\nabla'\frac{1}{R}\right]\mathrm{d}x'\mathrm{d}z'\right\}\right\},\qquad(23)$$

$$V_4=-\frac{1}{2\pi}\mathcal{F}^{-1}\left\{\nu\,\mathcal{F}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{V}{R}\left[(1+D^2)^{-\frac{1}{2}}-1\right]\mathrm{d}x'\mathrm{d}z'\right\}\right\}.\qquad(24)$$

We note that the (negative) x-derivative of (7) corresponds to the one-dimensional version of V_1+V_2 given in (21)–(22). The kernels of the inner integrals of (23) and (24) decay like R^{-4} and R^{-3}, respectively. These integrals may, similarly as in the two-dimensional formulation, be evaluated over a very limited region of the xz-plane, still keeping high accuracy. While V_1, V_2 and V_3 are determined by known functions at the free surface, V_4 is determined implicitly. The latter may be determined iteratively like in the two-dimensional case, where in the first iteration V is replaced by $V_1+V_2+V_3$ on the right of (24). The iteration procedure may then be continued until desired accuracy is achieved. In practical computations, however, one iteration may be sufficient, like we found in the two-dimensional numerical examples described above. The contribution from V_3 becomes very small in most cases and may be left out in practical computations like in the two-dimensional case.

6 Discussion

A novel fast procedure for computations of fully nonlinear non-overturning ocean surface waves is developed. The method is derived in both two and three dimensions. A corresponding computational scheme is developed in the two-dimensional case. The essential part of the method is the rapid iterative scheme whereby solution of the Laplace equation is obtained. The scheme converges fast. The number of operations required is asymptotically $\mathcal{O}(N\log N)$. In fact, the scheme is so fast that one iteration is sufficient in practical computations, while still keeping high accuracy. This constitutes an explicit version of the scheme. However, any accuracy is achieved by a continued iteration of the equations.

Preliminary computations with the scheme are promising. Results with the explicit version of the scheme cannot be distinguished from reference computations of Stokes waves with almost highest wave slope. Further, simulations with the explicit scheme, of a nonlinear wave train generated by a periodic wave maker, are almost indistinguishable from computations using the full equations. These simulations were continued until the waves broke. (The simulations were similar to experiments in a physical wave tank where the same wave breaking occurred.)

The fast novel method may be applied to fully nonlinear simulations of irregular waves including bi-directional and short-crested waves. Previous weakly nonlinear computations, e.g. horse shoe patterns (Dias & Kharif 1999), may be extended to include full nonlinearity of the waves with the present method. Further, the method may be employed to study the formation of freak waves.

This is a topic that is poorly understood and receives considerable interest from both scientific and engineering communities.

For simplicity all derivations are given for infinite water depth. The method is, however, easily extended to a fluid with constant depth, as outlined in the appendix for the two-dimensional case. Similar extension is straight forward also in three dimensions. An extension of the method may also include a variable bottom with finite slope. We finally note that boundary integral methods are employed to investigate stratified flows with several homogeneous layers (Grue *et al.* 1997). The present fast Laplace eqaution solver represents also in this case a tool for accelerated computations of non-breaking interfacial waves.

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A Appendix

The two-dimensional method outlined in section 2 is generalized here to a fluid with horizontal bottom at $y = -h$. The Cauchy integral yields in this case

$$\begin{aligned}\widetilde{\phi} = & \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{-(x'-x)(\widetilde{\psi}' + \eta'_x \widetilde{\phi}') + (\eta' - \eta)(\widetilde{\phi}' - \eta'_x \widetilde{\psi}')}{(x'-x)^2 + (\eta'-\eta)^2} \, \mathrm{d}x' \\ & - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x'-x)(\widetilde{\psi}' + \eta'_x \widetilde{\phi}') - (2h + \eta' + \eta)(\widetilde{\phi}' - \eta'_x \widetilde{\psi}')}{(x'-x)^2 + (2h + \eta' + \eta)^2} \, \mathrm{d}x'. \end{aligned} \tag{25}$$

Introducing $d = (\eta' + \eta)/(x' - x)$, $H = 2h/(x' - x)$ and the operators

$$\begin{aligned} \mathcal{H}_{\pm}\{f\} &= \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(x')}{x'-x} \, \mathrm{d}x' \pm \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x'-x)\, f(x')}{(x'-x)^2 + 4\, h^2} \, \mathrm{d}x' \\ \mathcal{I}\{f\} &= f(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2\, h\, f(x')}{(x'-x)^2 + 4\, h^2} \, \mathrm{d}x', \end{aligned}$$

equation (25) may be rewritten along the lines of the infinite depth case, giving

$$\begin{aligned} \mathcal{H}_+ \left\{ \widetilde{\psi} \right\} = & - \mathcal{I} \left\{ \widetilde{\phi} \right\} + \mathcal{H}_+ \left\{ \eta\, \widetilde{\phi}_x \right\} - \eta\, \mathcal{H}_- \left\{ \widetilde{\phi}_x \right\} \\ & + \frac{1}{\pi} \int_{-\infty}^{\infty} [\arctan(D) - D]\, \widetilde{\phi}'_x \, \mathrm{d}x' + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{D\,(D - \eta'_x)\, \widetilde{\psi}'}{1 + D^2} \frac{\mathrm{d}x'}{x'-x} \\ & + \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \arctan\left[\frac{d}{1 + d\, H + H^2} \right] - \frac{d}{1 + H^2} \right\} \widetilde{\phi}'_x \, \mathrm{d}x' \\ & - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\,H + d\,)\, \eta'_x\, \widetilde{\psi}'}{1 + (\,H + d\,)^2} \frac{\mathrm{d}x'}{x'-x} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\,(\,2\,H + d\,)\, \widetilde{\psi}'}{(\,1 + H^2\,)\,[\,1 + (\,H + d\,)^2\,]} \frac{\mathrm{d}x'}{x'-x} \end{aligned} \tag{26}$$

(where $D = (\eta' - \eta)/(x' - x)$ as in section 2). The operators can be computed and inverted easily in the Fourier space where

$$\widehat{\mathcal{H}}_{\pm} = \mathrm{i}\,\mathrm{sgn}(k) \left[1 \pm \widehat{\mathcal{I}}/4h^2 \right], \qquad \widehat{\mathcal{I}} = 1 - \exp(-2\,|k|\,h).$$

An analogous procedure can be derived with the Green function method in three dimensions.

References

Baker, G. R., Meiron, D. I. & Orszag, S. A. 1982. Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.* **123**, 477.

Clamond, D. & Grue, J. 2000. Dynamics of the transient leading part of a wave train. *Proc. 15th Int. Workshop on Water Waves and Floating Bodies*, Caesarea, Israel, Eds. T. Miloh & G. Zilman, Faculty of Engng., Tel Aviv Univ., 28–31.

Dias, F. & Kharif, C. 1999. Nonlinear gravity and capillary–gravity waves. *Annu. Rev. Fluid Mech.* **31**, 301–346.

Dommermuth, D. & Yue, D. K. P. 1987. A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid Mech.* **184**, 267–288.

Dysthe, K. B. 1979. Note on a modification to the nonlinear Schrödinger equation for application to deep water. *Proc. R. Soc. Lond.* A **369**, 105–114.

Fenton, J. D. 1988. The numerical solution of steady water wave problems. *Computers & Geosciences* **14**, **3**, 357–368.

Fenton, J. D. 1990. Nonlinear wave theories. *The Sea*. **9** : Ocean Eng. Sci., **A**, 3–25. Wiley.

Fornberg, B. 1980. A numerical method for conformal mapping. *SIAM J. Sci. Stat. Comp.* **1**, 386–400.

Grue, J., Friis, A., Palm, E. & Rusås, P. O. 1997. A method for computing unsteady fully nonlinear interfacial waves. *J. Fluid Mech.* **380**, 257–252.

ISSC 2000. Report of the Environment Committee. *14th Int. Ship and Offshore Structures Congress*, **1**, Nagasaki, Japan.

Trulsen, K. & Dysthe, K. B. 1996. Freak Waves – A Three-Dimensional Wave Simulation. *Proc. 21st Symp. on Naval Hydrodynamics* Trondheim, 550-558.

Tsai, W. T. & Yue, D. K. P. 1996. Computation of nonlinear free-surface flows. *Annu. Rev. Fluid Mech.* **28**, 249–278.

West, B. J., Brueckner, K. A., Janda, R. S., Milder, D. M. & Milton, R. L. 1987. A new numerical method for surface hydrodynamics. *J. Geophys. Res.* **92**, 11, 11803–11824.

Yasuda, T. & Mori, N. 1994. High Order Nonlinear Effects on Deep-Water Random Wave trains. *Proc. Int. Symp.: Waves - Physical and Numerical Modelling* University of British Columbia, Vancouver, Canada, 823–832.

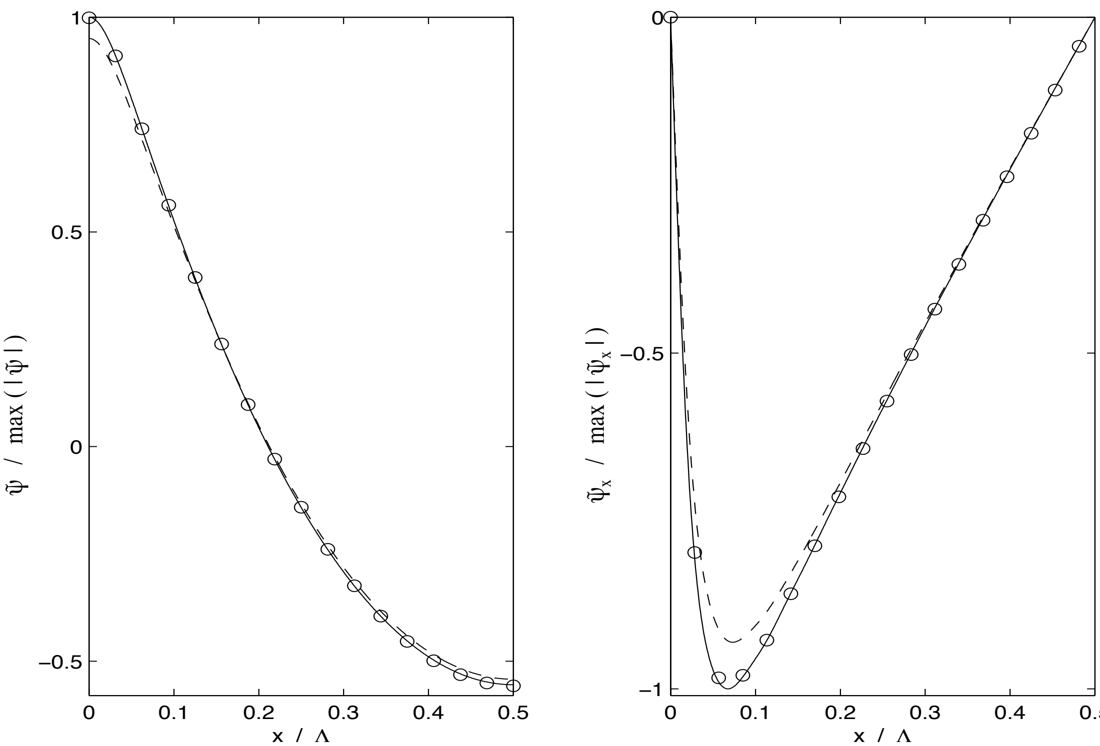

Figure 1: *Comparison of approximations for* $ak = 0.41$.

— exact, $--$ $\widetilde{\psi}_1$ (eq. 7), o $\widetilde{\psi}_{2,\lambda}$ (eq. 8 with $\lambda = \Lambda/2 = \pi/k$).

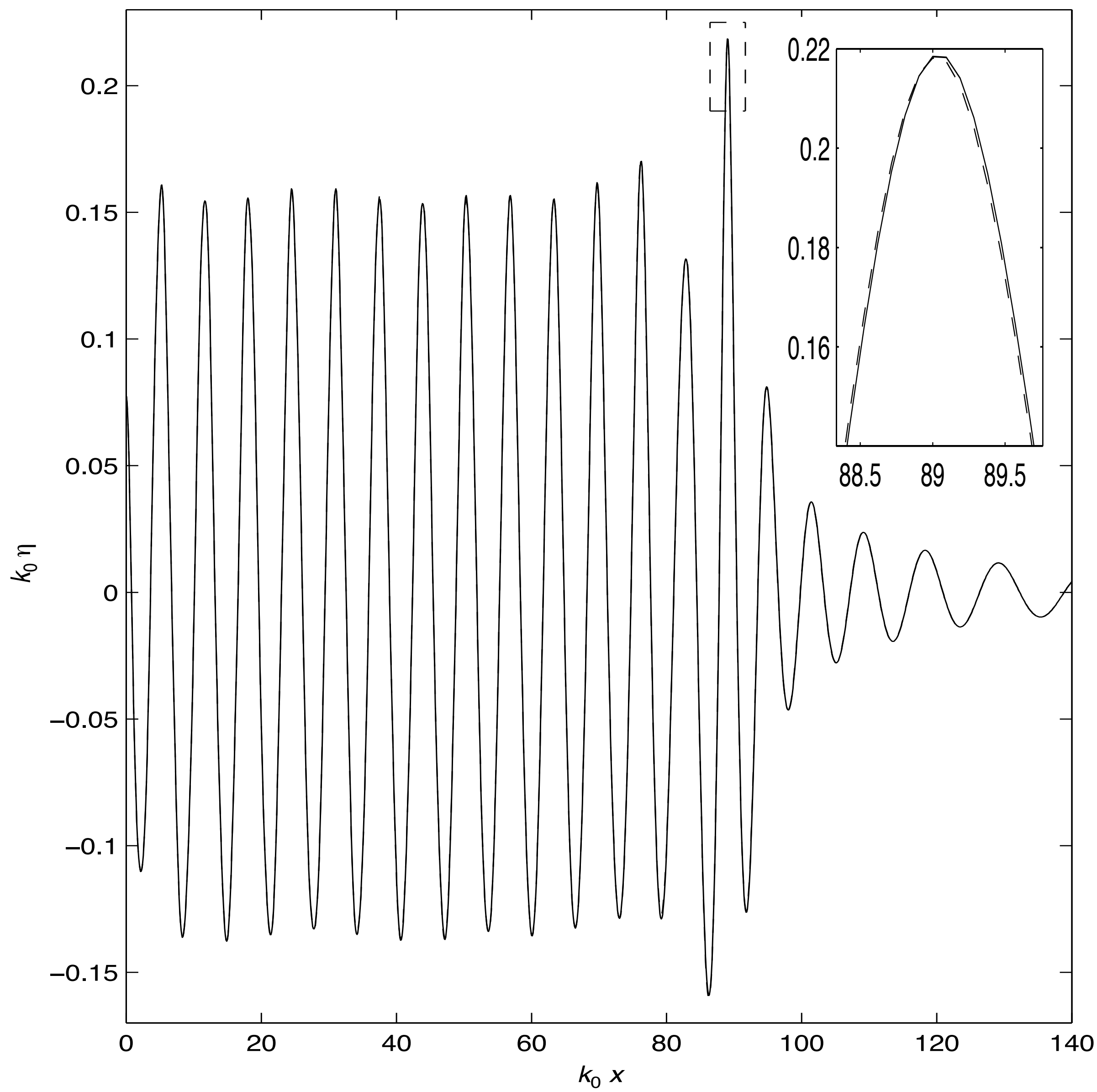

Figure 2: *The transient leading part of a wave train.*

— exact, - - $\widetilde{\psi}_{2,\lambda}$ (eq. 8 with $\lambda = \pi/k_0$, $k_0\Delta x = 0.095$).