A Faster Path-Based Algorithm for Traffic Assignment

R. Jayakrishnan
Wei T. Tsai
Joseph N. Prashker
Subodh Rajadhyaksha

Working Paper
UCTC No. 191

The University of California Transportation Center

University of California
Berkeley, CA 94720

University of California
Transportation Center

108 Naval Architecture Building
Berkeley, California 94720
Tel: 510/643-7378
FAX: 510/643-5456

A Faster Path-Based Algorithm for Traffic Assignment

R. Jayakrishnan
Wei T. Tsai
Joseph N. Prashker
Subodh Rajadhyaksha

Institute of Transportation Studies
University of California at Irvine
Irvine, CA 92717-3600

Working Paper
January 1994

presented at the 73rd annual meeting of the Transportation Research Board, Washington, D.C.

UCTC No. 191

The University of California Transportation Center
University of California at Berkeley

A FASTER PATH-BASED ALGORITHM FOR TRAFFIC ASSIGNMENT

R. Jayakrishnan[1], Wei K. Tsai[2], Joseph N. Prashker[3], Subodh Rajadhyaksha[4]
Institute of Transportation Studies, University of California at Irvine
Irvine, California 92717, USA

ABSTRACT

This paper takes a fresh look at the arguments against path-enumeration algorithms for the traffic assignment problem and provides the results of a gradient projection method. The motivation behind the research is the orders of magnitude improvement in the availability of computer storage over the last decade. Faster assignment algorithms are necessary for real-time traffic assignment under several of the proposed Advanced Traffic Management System (ATMS) strategies, and path-based solutions are preferred. Our results show that gradient projection converges in 1/10 iterations than the conventional Frank-Wolfe algorithm. The computation time improvement is of the same order for small networks, but reduces as the network size increases. We discuss the computer implementation issues carefully, and provide schemes to achieve a 10-fold speed-up for larger networks also. We have used the algorithm for networks of up to 2000 nodes on a typical computer work station, and we discuss certain data structures to save storage and solve the assignment problem for even a 5000 node network.

INTRODUCTION

As is well known, traffic assignment is the process of finding the flow pattern in a given network with a given travel demand between the origin-destination pairs. Equilibrium assignment finds flow patterns under user equilibrium, when no driver can unilaterally change routes to achieve better travel times. Optimal assignment determines the flow patterns such that the total travel time cost in the network is minimum, usually under external control. Assignment has long been an essential step in the transportation planning process. See Sheffi(1) for detailed discussions on traffic assignment.

---

1 Assistant Professor, Civil and Environmental Engineering, University of California, Irvine.

2 Associate Professor, Electrical and Computer Engineering, University of California, Irvine.

3 Professor, Technion Israel Institute of Technology, Haifa, Israel.

4 Graduate Student, Civil and Environmental Engineering, University of California, Irvine.

Real-time traffic assignment could be a part of potential ATIS (Advanced Traveller Information Systems) or ATMS (Advanced Traffic Management System) strategies (see Kaysi and Ben-Akiva (2), for a discussion of such strategies). The applications of network assignment in such real-time contexts could be in the on-line estimation of origin-destination demand matrices, or in an on-line dynamic assignment framework. The conventional approach, used in planning applications, is to solve the assignment problem with the Frank-Wolfe algorithm (denoted as F-W in the rest of the paper), also known as the convex-combinations algorithm (1). However, real-time applications typically require path-based solutions (see Mahmassani and Peeta (3)), which are not available with the link-flow based F-W algorithm. Faster convergence is also a very desirable feature for a real-time algorithm.

In this paper, we report our investigation of the Goldstein-Levitin-Poljak gradient projection algorithm, as formulated by Bertsekas(4). This algorithm falls under the set of algorithms called 'path-enumeration' algorithms, which have traditionally been discarded by transportation researchers as too memory-intensive and slow for large networks. In light of the orders of magnitude improvement in the availability of computer memory in recent years, we feel such algorithms deserve a fresh look. In this paper we describe our implementation of the algorithm and the extremely encouraging results. We discuss the assignment formulation for the sake of completeness in the next section, and follow it by a literature review and further qualitative discussions of the algorithms. We then proceed to discuss the computer implementation issues. The paper concludes with the results on comparative performance of the algorithms and pointers for future research.

THE STATIC USER-EQUILIBRIUM ASSIGNMENT PROBLEM

As is well known, the static assignment user equilibrium problem is stated as

$$\min Z = \sum_a \int_0^{x_a} t_a(\omega)\, d\omega \tag{1}$$

subject to the demand and non-negativity constraints given by

$$\sum_k f_k^{rs} = q_{rs} \qquad \forall \quad r,\ s,\ k \in K_{rs} \tag{2}$$

$$f_k \geq 0 \tag{3}$$

where,

x_a	= flow on link a (sum of the flows on the paths sharing link a),
$t_a(\omega)$	= cost(travel time) on link a for a flow of ω,
f_k^{rs}	= flow on path k connecting origin r and destination s,
q_{rs}	= the total traffic demand between r and s,
K_{rs}	= the set of paths with positive flow between r and s.

The above problem or variations of the same has appeared in some of the recently proposed dynamic assignment algorithms with time-varying demands, such as the bi-level algorithm of Janson(5) and the instantaneous dynamic assignment algorithm of Ran et al.(6). Note also that a system optimal assignment problem reduces to a user equilibrium problem with transformed (marginal) cost functions (1), and hence algorithms developed for user equilibrium assignment are applicable to the system optimal assignment as well.

REVIEW OF RELEVANT LITERATURE

There has been extensive work done on network optimization approaches to address the traffic equilibrium assignment problem. A detailed discussion of the conventional approaches to it is presented by Sheffi(1). A detailed study by Lupi(7) showed that the Frank-Wolfe algorithm is superior to most other algorithms. Nagurney(8) compared the Frank-Wolfe algorithm with the Dafermos-Sparrow algorithm (9) and found the latter to be in general more efficient. There has been some research to improve the efficiency of the F-W algorithm. Arezki and Van Vliet(10) presented an analytic implementation of the PARTAN technique as applied to the Frank-Wolfe algorithm and presented results indicating improvements over the original algorithm. Leblanc et al.(11) and Florian et al.(12) showed how the PARTAN method could be applied to the traffic network equilibrium assignment problem and showed improved convergence in real networks. Weintraub et al.(13) investigated a method of improving the convergence of the Frank-Wolfe algorithm by making modifications on the step size. One of the most recent improvements was by Larsson and Patriksson(14) who employed simplicial decomposition approaches to the original F-W algorithm.

Algorithms for assignment based on Benders decomposition have also been developed by Florian(15) and Barton et al.(16). Projection-based algorithms that have been developed in the past include those by Pang and Chan(17) and Dafermos(18). There has however not been much drawn from advances made in the parallel field of optimal flow assignment in computer communication networks. The gradient projection algorithm popularized by Bertsekas(19) is one such algorithm that we investigate in this study. In computer communication, the networks are usually smaller in size compared to the large urban networks where traffic assignments are carried out for planning purposes, and this may have been why transportation researchers have not paid enough attention to the research in that field.

SELECTION OF ALGORITHMS: HISTORICAL PERSPECTIVE

The choice of an appropriate algorithm for the traffic equilibrium assignment problem is guided by several criteria for selection depending on the specific needs of the application, the overriding criteria often being the memory requirements of the algorithm and its speed of convergence. These concerns become increasingly critical as the network size increases. We provide the following discussion to reveal the motivations behind this research.

The conventional choice for the traffic assignment problem so far has been the Frank-Wolfe algorithm. This choice has been guided largely by the memory requirements criterion. Since the F-W algorithm at any one iteration deals with only a single path between each origin-destination pair, its storage requirements are well within the capabilities of most ordinary computers. However, it has the drawback that typically the convergence becomes very slow as it approaches the optimal solution. It shows a tendency to flip-flop as it gets close to the optimum. The reason is that the algorithm is driven more by the constraint corners and less by the actual descent direction of the objective function surface, once it is close to the solution. This was not considered a serious problem in earlier applications of the traffic assignment, as the problem was being addressed from a transportation planning viewpoint. Under this scenario, assignment is used for forecasting purposes where the origin-destination demand data itself are derived using the extrapolation of current values or statistical models. This inherent inexactness in the process renders the exact estimation of link volumes unimportant and so practitioners are often content to stop the algorithm after a few iterations when it reaches within 5-10% of the solution. The memory requirement criterion was of importance too. When the Frank-Wolfe algorithm was introduced to the transportation field in the late 70s, computers were incapable of handling the larger memory requirements of path-enumeration algorithms. Recent advances in computing equipment have placed vastly increased computing power in smaller and smaller machines. Computer work stations with 16M storage are only about as expensive as a PC with 128K storage in the mid 80s, and have more storage than the largest mainframes of the late 70s. Given these possibilities it is important that we rethink our choice of traffic assignment algorithms and take advantage of the technological edge provided by current and future improvements in computer hardware.

Another aspect of F-W algorithm is that it does not automatically find the intersection turning movements. This has traditionally been found by microcoding the intersections with specific turning links, which usually increases the number of nodes and links in the networks considerably. Path flow solutions automatically provide such turning counts without any microcoding of the network, thus keeping the network sizes small. However, if separate flow-cost functions are to be used for turning movements, microcoding may be necessary with path-based algorithms also.

The recent advent of IVHS (Intelligent Vehicle-Highway Systems), brings up needs for real-time traffic assignments with different requirements than in the case of planning applications. Such assignments may be part of dynamic assignment frameworks or real-time O-D demand estimation frameworks. Faster convergence becomes important, and path-based solutions may be necessary. Moreover, there is increasing emphasis placed on estimating the fuel consumption and modelling the air quality over specific routes. Solutions based on path flows provide speed profiles over the network paths which are conceivably useful (though, still very approximate) for such applications. Link flow solutions from the F-W algorithm are much poorer in this regard.

THE GRADIENT PROJECTION ALGORITHM (GP)

The gradient projection algorithm is extensively used in computer communication networks, where path flow solutions are essential for optimal flow routing. However, the networks in these applications are typically much smaller than urban traffic networks and the path-enumeration issues have not been serious concerns. Moreover, the network structures are also somewhat different in these two applications. We adapt the basic Goldstein-Levitin-Poljak gradient projection algorithm formulated by Bertsekas(4) to the traffic assignment problem, and concentrate on the practical convergence and computer implementation issues.

In contrast to the Frank-Wolfe algorithm which finds auxiliary solutions that are at corner points of the linear constraint space, the gradient projection algorithm makes successive moves in the direction of the minimum of a Newton approximation of a transformed objective function. The objective function includes the demand constraints also, and thus the feasible space for gradient projection is defined only by the non-negativity constraints, as opposed to both non-negativity and demand constraints in the case of F-W. Should the move to the minimum in the negative gradient direction result in an infeasible solution point, a projection is made to the constraint boundaries. As a result of the redefinition of the problem, infeasibility occurs only when a variable violates the non-negativity constraint and thus the projection is easily accomplished by making that variable zero. We describe this in detail below.

The formulation of the algorithm focusses on the traffic demand constraints

$$\sum_{k \in K_{rs}} f_k = q_{rs}$$

where K_{rs} is the set of paths (with positive flow) between origin r and destination s.

If we express the shortest path flows $f_{\bar{k}_{rs}}$ in terms of other path flows

$$f_{\bar{k}_{rs}} = q_{rs} - \sum_{\substack{k \in K_{rs} \\ k \neq \bar{k}_{rs}}} f_k \tag{4}$$

the standard optimization problem (equations 1 through 3) can be restated as,

$$\min \quad \tilde{Z}(\tilde{f}) \tag{5}$$

$$\text{subject to} \qquad f_k \geq 0 \qquad \forall\, f_k \in \tilde{f} \tag{6}$$

where $\tilde{Z}$ is the new objective function and $\tilde{f}$ is the set of non-shortest path flows between all the O-D pairs.

For each O-D pair, while at any feasible (non-optimal) solution, a better solution can be found by moving in the negative gradient direction. This gradient is calculated with respect to the flows on the non-shortest paths (which are the only independent variables now), and a move-size is found using the second derivatives with respect to these path-flow variables. Once the flows on these non-shortest paths are updated, the flow on the shortest path is appropriately updated so that the demand constraint is satisfied.

The gradient of the objective function written in terms of the non-shortest path variables can be found using,

$$\frac{\partial \tilde{Z}}{\partial f_k} = \frac{\partial Z}{\partial f_k} - \frac{\partial Z}{\partial f_{\bar{k}_{rs}}}, \qquad \text{where } k \in K_{rs},\ k \neq \bar{k}_{rs} \tag{7}$$

which results from the definition of $\tilde{Z}$. Thus each component of the gradient vector is the differences between the first derivative lengths of a path and the corresponding shortest path (14). In the case of equilibrium assignment, the objective function is in terms of integrals and the first derivative lengths are simply the path costs at that flow solution.

A small increase in the flow on a path k results in an equal decrease in the flow on the corresponding shortest path. This results in no change in the flow on the common part of the two paths. Thus the second derivative is simply the sum of the second derivative lengths of the links on either path k or path $\bar{k}_{rs}$, but not both[5]. Once the second derivatives of $\tilde{Z}$ with respect to each path flow are calculated, we assume a diagonal Hessian matrix and the inverse of each second derivative gives an approximate quasi-Newton step size for updating each path flow.

For the remainder of the paper, when we refer to the 'first derivative lengths', we mean the first derivatives of the objective function, which is composed of link costs at specific path flows (i.e., $t_a(x_a)$). Similarly, 'second derivative lengths' refers to the second derivative of the objective function, and is composed of first derivatives of link costs. (i.e., $\frac{\partial t_a}{\partial x}$ at $x = x_a$).

5 A small increase in the flow on path k causes an equal decrease in the flow on the shortest path $\bar{k}_{rs}$. The flows on the common links on these paths do not change. The increase in flow on the other links on k causes positive second derivatives. The decrease in flow on the other links on $\bar{k}_{rs}$ also causes positive second derivatives, as it increases the negative first derivatives.

Based on the above discussion, the gradient projection algorithm can be formalized as follows:

Step 0: Initialization. Set $t_a = t_a(0)$, $\forall a$ and perform all-or-nothing assignments. This yields path flows f_1^{rs}, $\forall$ r,s and link flows x_a^1, $\forall a$. Set iteration counter n=1. Initialize the path-set K_{rs} with the shortest path for each O-D pair rs.

Step 1: Update. Set $t_a^n = t_a(x_a^n)$, $\forall a$. Update the first derivative lengths d_k^n (i.e., path-costs at current flow) of all the paths k in K_{rs}, $\forall$ r,s.

Step 2: Direction finding. Find the shortest paths $\bar{k}_{rs}^n$ from each origin r to each destination s, based on $\{t_a^n\}$. If different from all the paths in the existing path-set K_{rs} (no need for path comparison here; just compare d_k^n), add it to K_{rs} and record $d_{\bar{k}_{rs}^n}$. If not, tag the shortest among the paths in K_{rs} as $\bar{k}_{rs}^n$.

Step 3: Move. Set the new path flows.

$$f_k^{n+1} = \max\{0,\ f_k^n - \frac{\alpha^n}{s_k^n}(d_k^n - d_{\bar{k}_{rs}^n})\} \qquad \forall \quad r,s,\ k \in K_{rs},\ k \neq \bar{k}_{rs}^n$$

where,

$$s_k^n = \sum_a \frac{\partial t_a^n}{\partial x_a^n} \qquad \forall \quad k \in K_{rs},$$

a denotes links that are on either k or $\bar{k}_{rs}$, but not on both, and

α^n is a scalar step-size modifier (say, α^n = 1).

Also,

$$f_{\bar{k}_{rs}^n}^{n+1} = q_{rs} - \sum_k f_k^{n+1} \qquad \forall \quad k \in K_{rs},\ k \neq \bar{k}_{rs}^n$$

Assign the flows on the trees and find the link flows x_a^{n+1}.

Step 4: Convergence test. If the convergence criterion is met, stop.

Else, set $n = n+1$ and go to step 1.

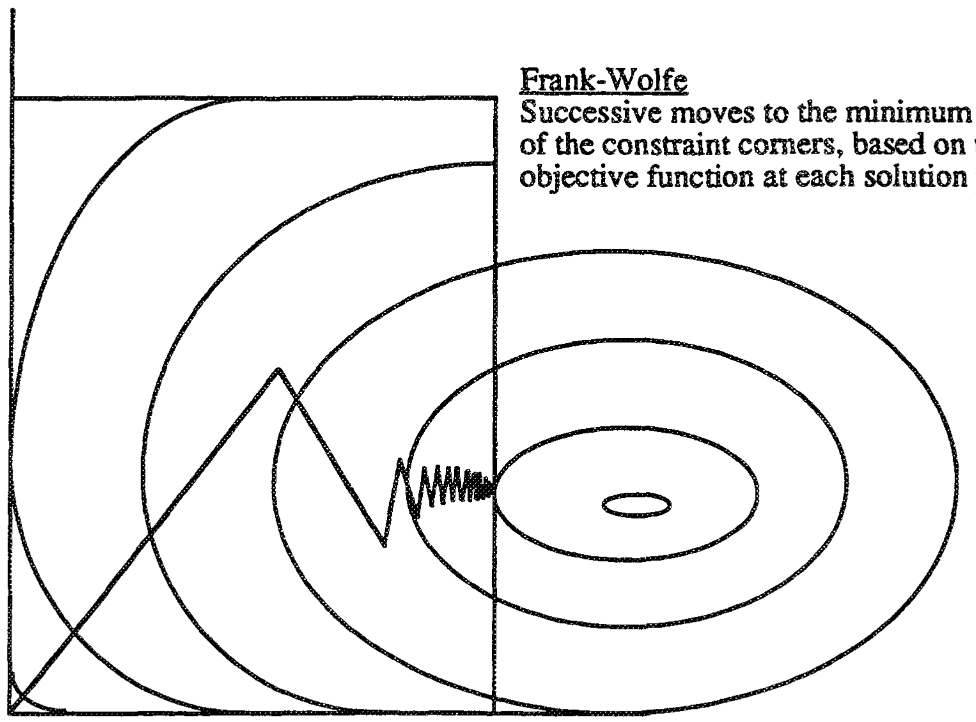
It is better to keep α^n a constant (i.e., $\alpha^n = \alpha, \forall n$). It can be shown that given any starting set of path flows, there exists an $\bar{\alpha}$ such that if $\alpha \in (0, \bar{\alpha}]$, the sequence generated by this algorithm converges to the optimum (Bertsekas(1)), provided the link cost functions are convex. Our experience shows that $\alpha = 1$ achieves very good convergence rate, and all the results shown in this paper use this value of α. The solutions reached are unique in terms of the link flows, but the path-flow solution, while it is optimal is not necessarily unique (See Sheffi(1) for a discussion on why the path-flow solutions need not be unique).

A qualitative graphical comparison of GP and F-W algorithms is shown in figure 1. In this case, F-W moves in directions which are almost orthogonal to the descent direction, once it is close to the optimum, because the moves are towards constraint corners to avoid infeasibility. GP still moves in the descent direction when it is closer to the optimum. Note that this is just an example, and is provided only to illustrate the qualitative reason behind the faster convergence of GP. The actual nature of the objective functions and the constraints in the network assignment context are quite different.

GP distributes flows from existing paths to the shortest path during every iteration, different fractions of flow being taken out of the alternative paths between an O-D pair. A careful look at the F-W algorithm will show that it also implicitly redistributes flows from alternative paths. However, the fraction taken out from the paths are all same, and the path flow solutions are never kept track of.

COMPUTATIONAL STORAGE CONSIDERATIONS

The GP algorithm is a path-enumeration algorithm and the paths need to be carefully stored to prevent memory problems. This is an issue that has rarely been addressed in the computer communication applications, but we address this here due to the larger size of the traffic networks that we need to deal with. In GP, one shortest path tree is built during each iteration, from each


Frank-Wolfe
Successive moves to the minimum in the direction of the constraint corners, based on the linearized objective function at each solution point.

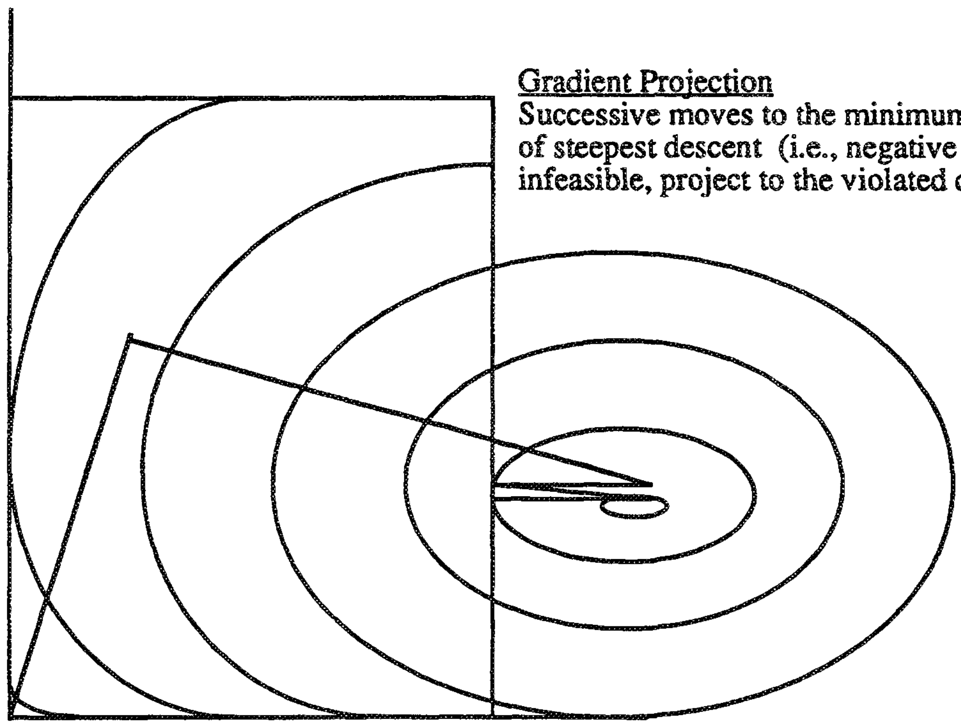

Gradient Projection
Successive moves to the minimum in the direction of steepest descent (i.e., negative gradient). If infeasible, project to the violated constraint.

Figure 1. Illustrative Comparison of Gradient Projection and Frank-Wolfe Algorithms

origin. The paths between an O-D pair are all generated during different iterations, and each path is part of one of the shortest path trees built from the corresponding origin node. It is important not to store the paths as node-lists, but rather as predecessor trees (note that each iteration produces shortest paths, which are invariably trees). This avoids double storage of common portions of different paths found from each origin in each iteration. As we store one predecessor node number for each node, each tree in a network of N nodes requires N storage locations. This results in N_o*N storage locations in each iteration, where N_o is the number of origins. Thus the main memory requirement of the algorithm is of the order of N_o*N*N_i, where N_i is the number of iterations. We do not see the kind of combinatorial explosion that is expected to occur with path-enumeration algorithms. The fact that the paths in each iteration are part of trees is thus a very handy feature of the GP algorithm.

However, for the F-W algorithm, the memory requirements are fixed by network size and are not affected by the number of iterations till convergence. Typically, its requirements are of the order of N_o*N, as it stores only one tree (the shortest paths for the all-or-nothing assignment) every iteration. Thus, GP does have a storage disadvantage compared to F-W, but it is able to provide a 'richer' solution for precisely the same reason, as its gives us path-flows as opposed to link-flows in F-W.

The memory requirement is hardly a significant concern, based on our experience. With a simple predecessor array data structure the authors were able to run networks up to 1200 nodes with 22,300 O-D pairs on a SUN work station. If only about 10 iterations of GP are attempted (which itself generally finds better solutions than 100 iterations of F-W, as our results shown in this paper indicate), we can run networks of more than 2000 nodes. We briefly describe a new data structure that allows us to run networks of up to 5000 nodes and 560,000 O-D pairs till 32 iterations. The purpose for attempting to run such networks is to show that the storage problem that kept researchers away from applying path enumeration algorithms to larger networks is really not a problem anymore, at least in the case of GP.

An efficient data structure for larger network problems: The shortest path trees built in successive iterations often have several identical branches. When these are stored as separate trees, it results in a great amount of duplication of storage, as we find several nodes to have the same predecessor arcs in successive trees. Rather than storing the predecessor arcs for all the nodes in every iteration, we store a shorter list of nodes for which the predecessor arcs change in an iteration (change being defined from the predecessor in the 'anchor' iteration, say, the first). The information regarding the iteration numbers where predecessor of each node changes, is stored in terms of the bits of a number. A '1' in bit-location i of this number for a node means that its predecessor changed in iteration i, and a '0' would indicate that no change occurred in iteration i. In a computer with 32 bit numbers, this lets us store the information with just N numbers. To find the predecessor for this node during, say, iteration j, we need to find the bit-location of the last '1'. This can be efficiently done at the hardware or software level, and yields us the appropriate iteration number, i. We go to the short list of changed predecessors corresponding to iteration i to find the predecessor for the node of concern. This approach will not achieve good computation time results unless it is carefully implemented, and we leave out the complicated implementation details in this paper. With this data structure we were able to reduce the storage requirement for the trees from the N_o*N*N_i above, to about N_o*N*C, where C is about 5-10 for up to even 100 iterations (i.e., N_i=100). This is because the predecessors of each node changes only less than 10 times during 100 iterations of GP, based on our assignment runs on realistic traffic networks.

COMPUTATION TIME CONSIDERATIONS

A careful implementation is absolutely essential for the GP algorithm to perform well. As we found that the algorithm converges generally about 10 times faster than F-W in terms of the number of iterations, our intention was to ensure that the GP algorithm achieves similar speed in computation time also. While F-W requires no other operations of computational intensity comparable to the shortest path determination during each iteration, GP has other procedures during its iterations which can be more time intensive than the shortest paths determination for larger networks. Our studies (as discussed in the next section) show that GP converges 10 times faster than F-W for a 100 node network, but only 60 percent faster for a 900 node network, though the number of iterations needed is still less than 1/10th. As we find that almost all of the time in F-W is spent on the shortest path routine for larger networks, this means that there are routines in GP whose computational intensity increases faster than that of the shortest path routine as the network size increases. We identify three such key operations, and these have to be carefully implemented: (1) assignment of the flow on each path to the links along its length to find the total link flows, (2) finding the first derivative lengths for all the paths between each O-D pair, and (3) finding the second derivative lengths for each path. We have been successful in developing efficient schemes for the first two, but have not been able to tackle the third problem without modifying the algorithm itself. The results that we provide in this paper are based on a program that includes only the techniques for the first derivative lengths. However, we discuss all three aspects here to show the potential of the algorithm to show even better results than we have provided, if our suggestions are implemented. Our early results with all three of the following procedures are indeed very encouraging.

Implementation of flow assignment procedure: This refers to assigning the path flow to all the links on each path, after the path flows are updated during each iteration. There are N_o*N_o O-D pairs in a network (where N_o is typically 10 to 20 percent of N), and the expected number of active paths between an O-D pair, N_p, is typically about 5 to 10 at convergence. Each path in a traffic network has roughly $O(N^{1/2})$ links on them. Thus, this operation could be of $O(N_o*N_o*N^{1/2}*N_p)$ effort in each iteration, if each path is considered independently (i.e., the link-level operations are repeated for paths that share common portions). In contrast to this, an efficient implementation (say, using a heap) of a routine to find the shortest paths from all origins to all nodes results in $O(N_o*N\log N)$ or better computational intensity. Clearly the flow assignment step becomes much more time consuming for larger networks. We have developed an efficient tree-traversal procedure to assign the flows, instead of doing it path by path. The procedure starts from a leaf-node of the tree and goes up assigning the flow on the links till a node is reached where there is another branch with no flow assigned. Once the flows are added on all the branches at a node, we find the total flow to assign on the predecessor link of the node, and move up. This procedure goes only once over each arc in the tree and hence it is an O(N) operation for each tree (a maximum of 4 additions per arc in a typical traffic network). This results in only $O(N_o*N)$ operations for all origins, all destinations, and all paths in each iteration. We leave out further details of this procedure for brevity. Suffices to say that this assigns flows of multiple paths sharing common portions without repeated calculations at the links. This is a significant improvement as the computations now do not depend on the number of paths or the number of nodes on the paths. The computational intensity drops to below that of the shortest path determination with this method.

Finding the first derivative path lengths: Here the link costs are added up on different paths. Similar to above, we need to avoid path-based computations of $O(N_o*N_o*N^{1/2}*N_p)$ order in this case also. Here, we again perform a tree-traversal procedure. We go up from any node in the tree till the root (origin) node, and add the link costs on the links to find the first derivative length till that node. Then we go up from that node once more (second pass), subtracting one link-cost at a time to find the costs till each node along the way[6]. Then we move to any node not yet considered and repeat the procedure, this time starting the second pass after reaching the origin node *or* a node with an already-computed path length. As each node is reached strictly twice, this results in only $O(N_o*N)$ operations in every iteration, which is much faster than the shortest path determination in larger networks.

The second derivative length calculations: This requires the addition of second derivative lengths of links not common between each path and the corresponding shortest path. If this is done path-by-path, adding the second derivatives on each link with the shortest path, this also results in $O(N_o*N_o*N^{1/2}*N_p)$ computations. We have so far not been able to find an $O(N_o*N)$ technique for this, without changing the GP algorithm itself to some extent. The difficulty arises because the path under consideration is on another tree, different from the tree that the current shortest path is part of. It is possible that we can improve the situation only by changing the algorithm substantially. We suggest using a line-search rather than the second derivatives to find the step size in the negative gradient direction. An auxiliary path-flow solution can be easily found in the negative gradient direction, and then an unconstrained line-search can be used to determine the step size to reach the minimum in this direction. This line search can be performed fast in the link-flow domain (using the link flows at the current and auxiliary path flow solutions), and based on the optimal step size, a path-flow update is performed. The flow update would be based on path flows. Our early experience with this technique has been encouraging.

ASSIGNMENT RESULTS

The assignment studies compare the performance of GP algorithm with the F-W algorithm. In order to make conclusions on comparative performance of the algorithms, it is necessary that they be tested under sufficiently diverse networks. We studied the algorithms on grid networks of different sizes generated using a random network generator program that we developed, as well as on the network of major arterials and freeways in Anaheim, California.

6 The two passes are needed only because we cannot move down the tree, when the tree is stored with predecessor representation. A threaded-tree storage will let us do a one-pass traversal, but there may be additional overheads involved. Another option is to keep the tree-traversal order right after each tree is built, but this requires as many storage locations as the tree itself, and doubles the storage requirements.

The test networks are grids only in terms of the connectivity of the links, the link lengths being determined randomly. There are two links each way between the nodes. The link lengths are randomly picked from a uniform distribution between 500 and 5000 feet. The free flow speed on the links are randomly picked from a uniform distribution between 22 mph and 40 mph. The capacity of the links are based on the number of lanes (1, 2 or 3), each lane having a capacity of 1800 vehicles/hour. Certain nodes from the network are randomly picked as origin/ destination centroids. This is done based on a set of rules that attempts to create a network representative of real-world traffic networks. First, approximately 12.5% (1/8th) of the total nodes in the network are picked to be centroid nodes, which is about the fraction of zone-centroid nodes in typical assignment applications. There are at least three links between any two centroid nodes to ensure that they are not too close to each other. Once the centroids are set up, the origin-destination flow matrix is generated. Each centroid generates demand at a pre-specified rate (9600 veh/hr was used in our studies) and the generated traffic is distributed to other nodes based on the inverse squared distances to develop the O-D matrix.

The Anaheim network has 416 nodes (of which 38 are origin/destination centroids), 914 arcs, and 1406 O-D pairs. A static O-D demand matrix was estimated using the COMEST program, based on some link counts in the network. The demand data refers to the evening peak period in the network, which moderately high level levels of congestion. No microcoding of the intersections was attempted for this network.

The assignments were carried out using a BPR link cost function, $t = t_0(1 + 0.15\ (x/c)^4)$, where t is the link travel time cost, t_0 is the free-flow cost, x is the flow and c is the link capacity. Both GP and F-W programs included identical shortest path routines, which is based on a binary heap data structure. The line-search routine for the F-W algorithm uses an efficient Bolzano search (see Sheffi(1)). The programs were implemented in FORTRAN-77 on a Sun SPARC-II work station with 64M storage. The flows and costs were floating point variables.

Table 1 shows the results from assignments on networks of varying sizes. For all the network sizes. This table shows the number of iterations required by F-W to find the objective function value that GP finds in 2, 4, 6, 8 and 10 iterations, as well as the corresponding computation time. We see that F-W requires 30 to 160 iterations to reach the solutions found by

Square Grid Networks	Gradient Projection			Frank-Wolfe		
	Iterations	Obj. function	Time (sec)	Iterations	Obj. function	Time (sec)
36 nodes	2	5385.2	0.08	4	5376.7	0.08
120 arcs	4	4890.2	0.14	27	4889.2	0.51
12 OD pairs	6	4794.5	0.17	54	4793.9	1.02
	8	4747.9	0.22	123	4747.8	2.32
	10	4728.8	0.28	390	4728.8	7.35
100 nodes	2	13076.9	0.38	5	13035.8	0.44
360 arcs	4	12562.6	0.76	30	12561.9	2.64
132 OD pairs	6	12526.0	1.14	168	12526.0	14.81
	8	12522.1	1.52	618	12522.1	68.58
	10	12521.4	1.90	1282	12521.4	198.32
225 nodes	2	33872.1	1.87	7	33757.4	2.73
840 arcs	4	32979.2	4.13	36	32977.7	14.03
756 ODs	6	32912.4	6.26	176	32912.4	68.58
	8	32904.8	8.44	509	32904.8	198.32
	10	32902.3	10.52	1611	32902.3	627.20
400 nodes	2	70849.9	5.78	6	70835.1	6.90
1520 arcs	4	68797.2	13.86	21	68774.8	24.14
2450 ODs	6	68408.2	22.67	65	68407.5	74.71
	8	68343.9	31.33	185	68343.9	212.63
	10	68326.8	38.97	537	68326.8	617.19
625 nodes	2	124874	14.51	7	124225	19.03
2400 arcs	4	120151	38.56	23	120144	62.52
6006 ODs	6	119322	64.04	48	119315	130.49
	8	119100	88.77	88	119100	239.22
	10	119028	112.39	156	119028	424.08
900 nodes	2	206379	30.90	7	206097	39.56
3480 arcs	4	198483	86.53	26	198436	146.93
12432 ODs	6	197974	148.76	31	197929	175.19
	8	196668	207.98	80	196665	452.10
	10	196508	271.74	123	196508	695.10

Table 1. Comparative performance of the Gradient Projection and Frank-Wolfe algorithms

GP in just 6 iterations. For all the networks, we found that GP converges between 10 and 15 iterations to solutions that F-W takes between 300 and 2000 iterations to reach. Clearly, there is at least an order of magnitude improvement in terms of the number of iterations.

We see that the computation times are also improved similar to the reduction in iterations, for smaller networks. This is expected, as the main computational step is the shortest path determination for both GP and F-W in small networks. However, the computation times with GP is about 40 percent of that with F-W for 10 iterations of GP in a 1000 node network. This shows that procedures other than the shortest path determination use up significant time in GP for larger networks, as explained in the previous section. It should be stressed that these assignments were carried out with an implementation of GP that has not yet include most of the procedures that we explained before. Thus, even though a 60 percent improvement is significant, the computation times for GP can be reduced even further than those shown in table 1, especially for larger networks.

Figures 2 to 5 show the results of the assignments on the network of Anaheim, for various demand levels. The demands generated by the O-D matrix estimated from actual link counts is denoted as demand level = 1.0. For other demand levels, the cells in the trip table were all multiplied by appropriate fractions. These assignments were carried out to examine the effect of the demand level on the relative performance of GP and F-W. Again we see that for all the cases, F-W takes more than 10 to 15 iterations to reach the solutions found by GP in 1 to 3 iterations. There does not appear to be a significant change in the performance of GP in comparison with F-W, as the demand level increases. Both algorithms require more iterations to converge for higher demand levels, but GP still shows 5 to 10 times faster convergence.

We have not compared the PARTAN version of F-W algorithm to GP. However, published results on PARTAN (10, 11) indicate that this typically finds solutions about twice as fast as ordinary F-W, in number of iterations. Based on the improvements that we find with GP, we decided that the effort was not warranted at this time. Moreover, PARTAN is still not commonly used for assignments by practitioners. We do intend, however, to carry out these comparisons in the future.

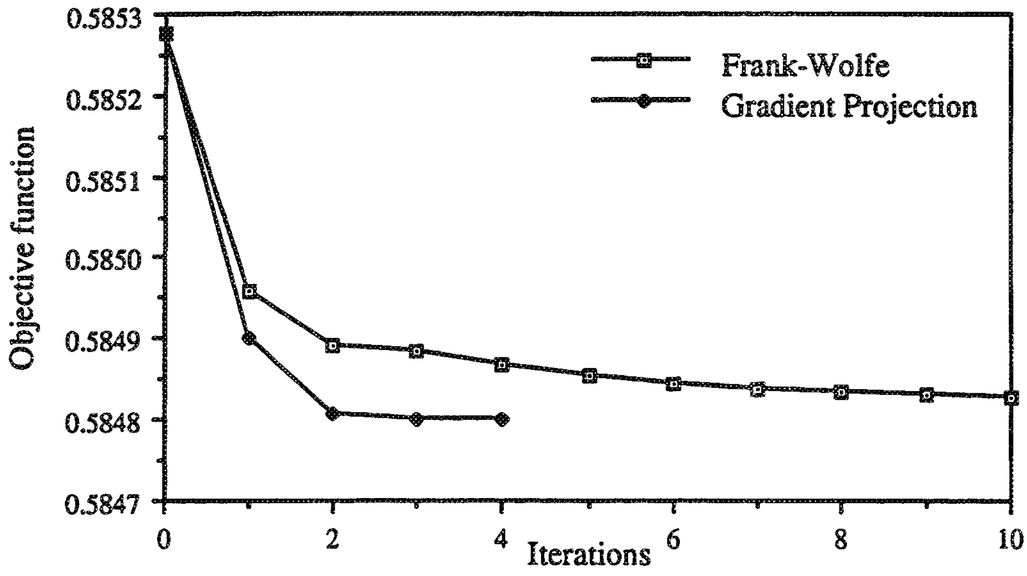


Figure 2. Comparison of F-W and GP on Anaheim Networks (Volume level = 0.5)

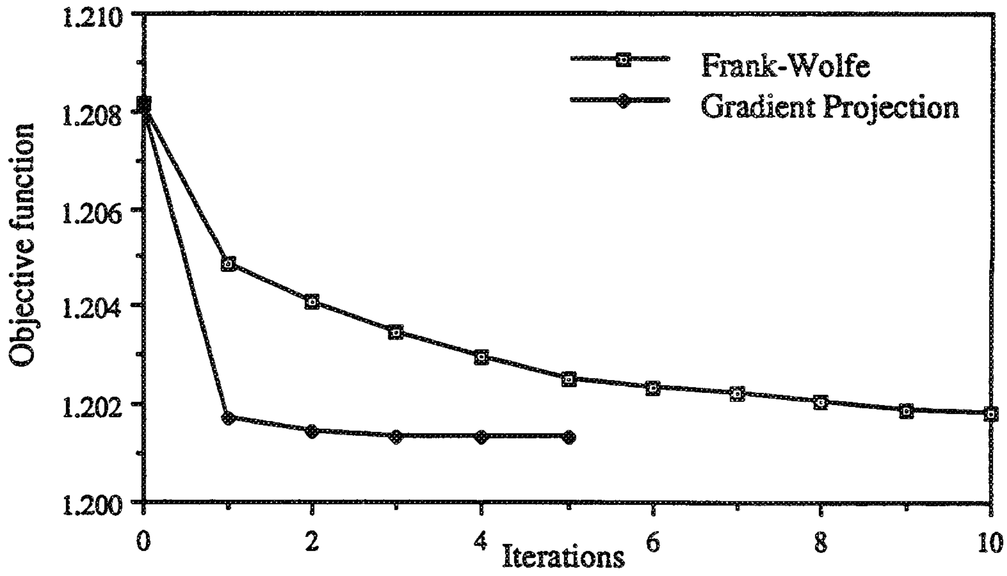


Figure 3. Comparison of FW and GP on Anaheim Network (Volume level = 1.0)

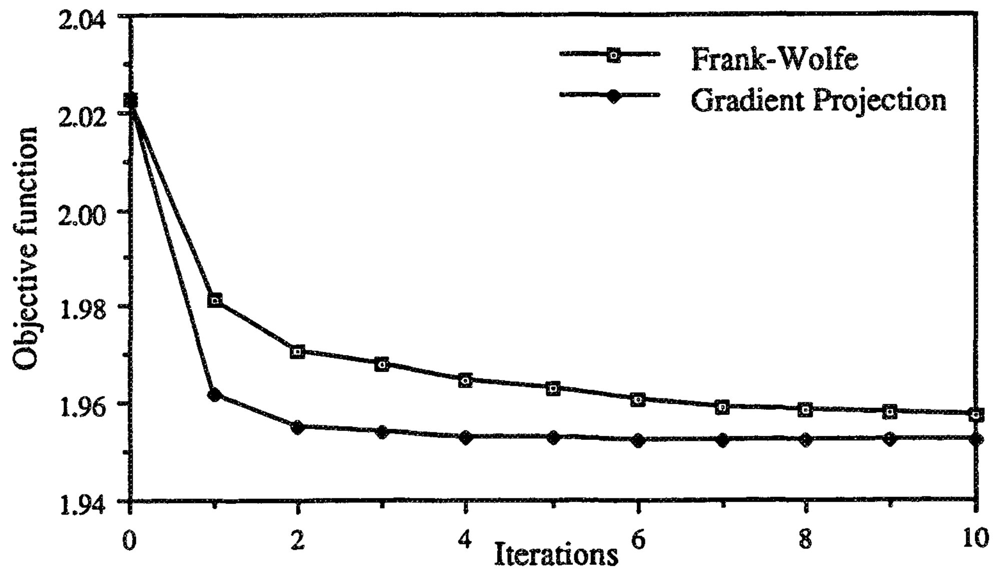


Figure 4. Comparison of F-W and GP on the Anaheim Network (Volume level = 1.5)

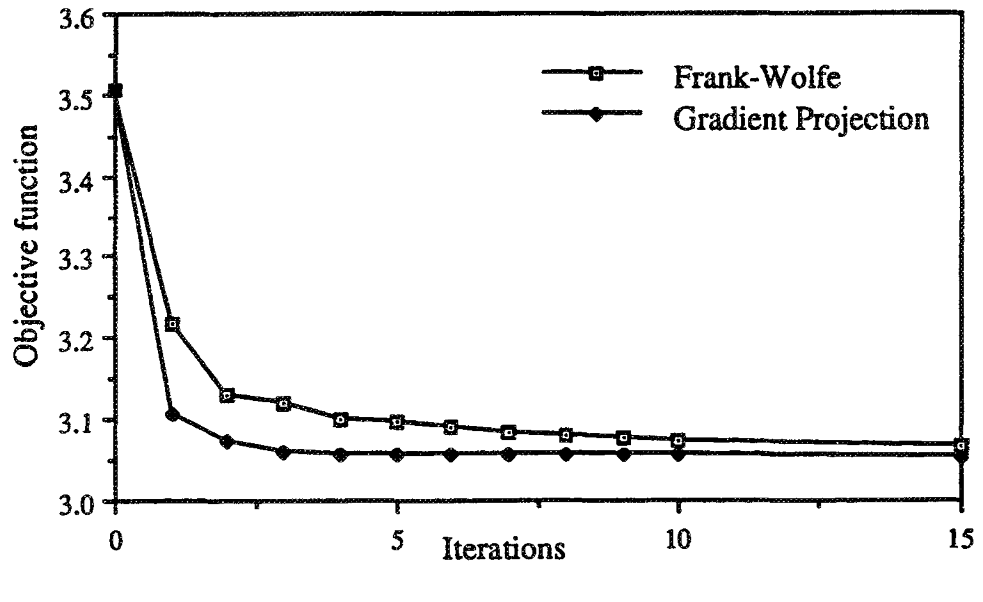


Figure 5. Comparison of F-W and GP on the Anaheim Network (Volume level = 2.0)

CONCLUSIONS

In this paper, we have provided a detailed discussion, and supportive results to show that path-enumeration algorithms such as gradient projection deserve a fresh look for applications in traffic assignment. There were two main motivations behind the research: (1) the tremendous improvement in recent years of the availability of computer memory, and (2) the need for fast assignment algorithms for certain possible IVHS strategies for optimal routing and guidance based on dynamic assignment frameworks, real-time trip table estimation, etc. We show that path-based algorithms can be applied to networks of thousands of nodes. We provide data structures to handle path-based storage problems, and we suggest techniques to achieve fast completion of path-based procedures in the algorithm. These techniques are applicable to other path-based algorithms also. Our implementation of gradient projection converges in an order of magnitude fewer iterations than the conventional Frank-Wolfe algorithm, and can be made to show similar computation time speed-up, if implemented carefully.

There are advantages with the path-based solutions generated by the GP algorithm. Such solutions can be directly used in path-based routing frameworks. Another advantage is the direct determination of node turning counts without microcoding the intersections and increasing the network size. In addition, we can find the link-to-link flow variation on each path. This may provide some opportunities to find approximate estimates of fuel consumption, environmental impacts etc., for selected paths or O-D pairs.

Several aspects of the algorithm require further study. One important aspect is the convergence rates under different link cost functions. Though we have carried out some research in this area and have found the results to be reasonably robust, our research has by no means been exhaustive. Application to other related problems such as dynamic assignment and variable demand assignment would provide more insights on the algorithm's performance. Research is also underway at the University of California, Irvine, on developing gradient projection with hierarchical decomposition schemes, for traffic network assignment.

ACKNOWLEDGEMENT

The authors wish to thank Professor Wilfred Recker, Director of the Institute of Transportation Studies at the University of California, Irvine, for his constant encouragement. The research was supported by the California Department of Transportation under the Anaheim ATMS testbed research program, and we acknowledge the generous support.

REFERENCES

(1) Y. Sheffi, *Urban Transportation Networks*. Englewood Cliffs, NJ: Prentice-Hall, 1985.

(2) I. Kaysi and M. Ben-Akiva, *An Integrated Approach to Vehicle Routing and Congestion Prediction for Real-time Driver Guidance*. Paper presented at the 72nd Annual Meeting of the Transportation Research Board, Washington, D.C., January, 1993.

(3) H. S. Mahmassani and S. Peeta, *Network Performance under System Optimal and User Equilibrium Dynamic Assignment*. Paper presented at the 72nd Annual Meeting of the Transportation Research Board, Washington, D.C., January 1993.

(4) D. P. Bertsekas, *On the Goldstein-Levin-Poljak Gradient Projection Method*. IEEE Trans. Automat. Control, Vol. AC-21, pp. 174-184, Apr. 1976.

(5) B. Janson, *Dynamic Traffic Assignment With Schedule Delay*. Paper # 920142 presented at the Transportation Research Board annual meeting, Washington,D.C., January 1992.

(6) B. Ran, D.E. Boyce, and L.J. Leblanc, *A New Class of Instantaneous Dynamic Assignment Models*. Operations Research, Vol. 41, No.1, 1993.

(7) M. Lupi, *Convergence of the Frank-Wolfe Algorithm in Transportation Networks*. Civil Engineering Systems, Vol.19, pp. 7-15, 1985.

(8) A. B. Nagurney, *Comparative Tests of Multimodal Traffic Equilibrium Methods*. Transportation Research B, Vol 18, No.6, pp. 469-485, 1984.

(9) S. C. Dafermos and F. T. Sparrow, *The Traffic Assignment Problem for a General Network*. Journal of Research of the National Bureau of Standards - B. Vol 73B, No. 2, pp. 191-117, 1969.

(10) Y. Arezki and D. Van Vliet, *A Full Analytical Implementation of the PARTAN/Frank-Wolfe Algorithm for Equilibrium Assignment*. Transportation Science, Vol 24, pp. 58-62, 1990.

(11) L. J. LeBlanc, R. V. Helgason and D. E. Boyce, *Improved Efficiency of the Frank-Wolfe Algorithm for Convex Network Problems*. Transportation Science, Vol 19, pp. 445-462, 1985.

(12) M. Florian, J. Guélat and H. Spiess, *An Efficient Implementation of the PARTAN Variant of the Linear Approximation Method for the Network Equilibrium Problem*. Networks, Vol 17, pp. 319-339, 1987.

(13) A. Weintraub, C. Ortiz and J. Gonzales, *Accelerating Convergence of the Frank-Wolfe Algorithm*. Transportation Research B, Vol 19, pp. 113-122, 1985.

(14) T. Larsson and M. Patriksson, *Simplicial Decomposition with Disaggregated Representation for the Traffic Assignment Problem*. Transportation Science, Vol 26, pp. 4-17, 1992.

(15) M. Florian, *A Traffic Equilibrium Model of Travel by Car and Public Transit Modes*. Transportation Science, Vol 11, pp. 166-179, 1977.

(16) R. B. Barton, D. W. Hearn, and S. Lawphongpanich, *The Equivalence of Transfer and Generalized Benders Decomposition Methods for Traffic Assignment*. Transportation Research-B, Vol 23B, pp. 61-73, 1989.

(17) J. S. Pang and D. Chan, *Iterative Methods for Variational and Complementarity Problems*. Math. Programming, Vol. 24, pp. 284-313, 1982.

(18) S. C. Dafermos, *A Variable Inner-product Projection Algorithm for Variational Inequalities with Application to the Traffic Equilibrium Problem*. Working paper, Lefschetz Center for Dynamical Systems, Brown University, Providence, 1983.

(19) D.P. Bertsekas and R.G. Gallager, *Data Networks*. Englewood Cliffs, NJ: Prentice-Hall, 1987.