A FATOU-JULIA DECOMPOSITION OF TRANSVERSALLY HOLOMORPHIC FOLIATIONS

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Abstract. A Fatou-Julia decomposition of transversally holomorphic foliations of complex codimension one was given by Ghys, Gomez-Mont and Saludes. In this paper, we propose another decomposition in terms of normal families. Two decompositions have common properties as well as certain differences. It will be shown that the Fatou sets in our sense always contain the Fatou sets in the sense of Ghys, Gomez-Mont and Saludes and the inclusion is strict in some examples. This property is important when discussing a version of Duminy's theorem in relation to secondary characteristic classes. The structure of Fatou sets is studied in detail, and some properties of Julia sets are discussed. Some similarities and differences between the Julia sets of foliations and those of mapping iterations will be shown. An application to the study of the transversal Kobayashi metrics is also given.

Résumé. Une décomposition de Fatou-Julia de feuilletages transversalement holomorphes de codimension complexe un est donée par Ghys, Gomez-Mont et Saludes. Dans cet article, nous proposons une autre décomposition en utilisant des familles normales. Deux décompositions ont des propriétés communes également différences certaines. Il est montré que l'ensembles de Fatou à notre sense contiennent toujours ceux au sense de Ghys, Gomez-Mont et Saludes, et aussi que l'inclusion peut être stricte dans quelques exemples. Cette propriété est importante en discutant une version du théorème de Duminy relié aux classes caractéristiques secondaires. Quelques similitudes et différences entre les ensembles de Julia de feuilletages et ceux d'itérations d'applications sont présentées. Une application aux études de la métriques transversale de Kobayashi est aussi donée.

1. Introduction

The Fatou-Julia decomposition is one of the most basic and important notions in complex dynamical systems. It has been expected that there also exists the Fatou-Julia decomposition of transversally holomorphic foliations.

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Such a decomposition of complex codimension-one foliations was firstly introduced by Ghys, Gomez-Mont and Saludes in [11]. We call the decomposition the *GGS-decomposition* for short. The GGS-decomposition is given according to the existence of certain sections to the complex normal bundles of foliations, and it enjoys several significant properties. For example, foliations restricted to the GGS-Fatou sets are transversally C^ω-Hermitian, namely, they admit transversal Hermitian metrics transversally of class C^ω and invariant under holonomies. This implies that foliations have simple dynamics on the GGS-Fatou sets.

On the other hand, the dynamical properties of the GGS-Julia set is complicated in general, and the Julia sets are expected to play a role of minimal sets for real codimension-one foliations. Indeed, a weak version of Duminy's theorem for real codimension-one foliation [9] is known, namely, the non-triviality of certain characteristic classes implies the non-vacancy of the GGS-Julia sets [3]. However, there are transversally C^ω-Hermitian foliations of which the GGS-Julia set is the whole manifold. The characteristic classes of these foliations are trivial. From the viewpoint as above, it is preferable if this kind of Julia sets can be avoided. One way to exclude such foliations is using characteristic classes. On the other hand, it will be also possible by replacing the Julia sets with smaller ones.

In this paper, we will propose another Fatou-Julia decomposition defined in a certain analogy to that of complex dynamical systems (Section 2). The foliation restricted to the Fatou set is transversally Hermitian of class C^ω. In this sense, our decomposition has the same property as the GGS-decomposition. Moreover, there is a description similar to that of the GGS-Fatou sets. The structure of transversally Hermitian foliations is well-studied by Molino, Haefliger, Salem et. al. [21], [14], [13], [28]. The classification of the Fatou components will be done by showing that foliations restricted on the Fatou set are locally given by actions of Lie groups and then repeating well-developed arguments as above. On the other hand, two decompositions are different in some examples. In fact, it will be shown that the Fatou sets in our sense always contain the GGS-Fatou sets. These properties of the Fatou sets are studied in Sections 3 and 4.

Some properties of the Julia sets are also studied (Section 5). It will be shown that some basic notions concerning the Julia sets of mapping iterations work well also in our context. In particular, a version of the Patterson-Sullivan measure is introduced by using invariant metrics.

In relationship with characteristic classes, a weak version of Duminy's theorem for complex codimension-one foliation will be shown valid also for our decomposition (Section 6).

The GGS-decomposition is also related with deformations of foliations. Indeed, the definition of the GGS-decomposition is directly related with deformations and the GGS-Julia set is largely decomposed into two parts according to the existence of invariant Beltrami coefficients. On the other hand, it is not quite clear how the decomposition in this paper is related with deformations. Certain GGS-Julia sets which admit invariant Beltrami coefficients are contained in the Fatou set in our sense so that the relationship to deformations of foliations is not necessarily the same.

To say about invariant metrics, our construction is not canonical. Many canonical invariant metrics and distances are known in complex geometry, and some of them can be translated in the foliation theory. Among them, the transversal Kobayashi distance is previously studied by Duchamp and Kalka [8]. We will discuss the transversal Kobayashi metric and show an analogous result (Section 7).

Some examples in [11] together with some other ones are examined in the last section (Section 8). Constructions are done in terms of compactly generated pseudogroups throughout the paper, however, examples are mostly given by using foliations.

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2. Definitions

For generalities of pseudogroups we refer readers to [12], [13] and [15]. Throughout this paper, compactly generated pseudogroups of local biholomorphic diffeomorphisms of $\mathbb{C}$ are studied. Examples in mind are the holonomy pseudogroups of transversally holomorphic foliations of complex codimension one. Compactly generated pseudogroups are defined as follows [13].

Definition 2.1. A pseudogroup (Γ, T) is compactly generated if there is a relatively compact open set U in T which meets every orbit of Γ, and a finite collection of elements $\{\gamma_1, \dots, \gamma_r\}$ of Γ of which the sources and the targets are contained in U such that

1) $\{\gamma_1, \dots, \gamma_r\}$ generates $\Gamma|_U$,
2) each γ_i is the restriction of an element of Γ defined on a neighborhood of the closure of the source of γ_i.

$(\Gamma|_U, U)$ is called a *reduction* of (Γ, T). A reduction of (Γ, T) will always be denoted by (Γ', T').

Remark 2.2. It is easy to see that we can choose a reduction in a way such that $T' = \coprod_{i \in I} T'_i$, where I is a finite set and each T'_i is an open disc in $\mathbb{C}$. We may furthermore assume that the closures $\overline{T'_i}$ of T'_i are mutually disjoint by parallel translations. Indeed we will choose reductions always in this way.

Example 2.3.
- Let G be a finitely generated group which acts on a closed manifold M. Then (G, M) is naturally a compactly generated pseudogroup. Such a pseudogroup is called the pseudogroup generated by G, and is realizable as the holonomy pseudogroup of a foliation of a closed manifold by taking suspensions.
- The holonomy pseudogroup of a transversally holomorphic foliation of a closed manifold is compactly generated.

We adopt the following notation.

Notation 2.4. Let (Γ, T) be a pseudogroup.

1) For $\gamma \in \Gamma$, the source (the domain of definition) of γ is denoted by $\operatorname{dom}\gamma$.
2) Let $x \in T$. Then, $\Gamma_x = \{\text{the germ of } \gamma \in \Gamma \text{ at } x \mid \operatorname{dom}\gamma \ni x\}$. By abuse of notation, elements of Γ_x are considered as elements defined on a neighborhood of x. For $\gamma \in \Gamma_x$ and $x \in T$, $\gamma(x)$ is also denoted by γx.
3) The Γ-orbit of a subset X of T is by definition $\Gamma(X) = \bigcup_{x \in X} \Gamma_x x$.
4) Regarding T as a subset of $\mathbb{C}$, we define the derivative of an element γ of Γ in the natural way and denote it by γ'. The absolute value of γ' is denoted by $|\gamma'|$, and $|\gamma'(x)|$ is denoted also by $|\gamma'|_x$.
5) The Euclidean disc of radius r and centered at x is denoted by $D_x(r)$. In general, if K is a compact set then $D_K(r)$ denotes the r-neighborhood of K with respect to the Euclidean metric.

The following notion can be found in [14].

Definition 2.5. A subset X of T is called Γ-*connected* if X satisfies the following condition: let $X = \coprod_{\lambda \in \Lambda} X_\lambda$ be the decomposition of X into its connected components, then for any $\lambda, \lambda' \in \Lambda$, there exists a sequence $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_r = \lambda'$ such that $\Gamma(X_{\lambda_i}) \cap X_{\lambda_i+1} \neq \varnothing$ holds for $i = 0, \dots, r-1$.

Remark 2.6. T is Γ-connected if and only if $\Gamma \backslash T$ is connected with the quotient topology. If $X \subset T$, then $\Gamma \backslash X \subset \Gamma \backslash T$ is connected if X is Γ-connected. The converse also holds if X is Γ-invariant, and not always true. Indeed, let $T = T_1 \sqcup T_2$, where $T_1 = T_2 = \mathbb{R}$, and equip T with the natural topology. Let Γ be the pseudogroup generated by $\gamma\colon T_1 \to T_2$ given by $\gamma(x) = x$, $X_1 = (-\infty, 0] \subset T_1$, $X_2 = (0, \infty) \subset T_2$ and $X = X_1 \cup X_2$. Then X is not Γ-connected but $\Gamma \backslash X = \Gamma \backslash T = \mathbb{R}$.

If (Γ, T) is the holonomy pseudogroup of a foliation, then Γ-connected components of Γ-invariant sets correspond to connected components of saturated sets.

The Fatou set is defined as a subset of T as follows.

Definition 2.7. Let (Γ, T) be a compactly generated pseudogroup and let (Γ', T') be a reduction.

1) A connected open subset U of T' is called a *Fatou neighborhood* if the following conditions are satisfied:
   (a) The germ of any element of Γ'_x, $x \in U$, extends to an element of Γ defined on the whole U.
   (b) Let
   $$\Gamma_U = \left\{ \gamma \in \Gamma \,\middle|\, \begin{array}{l} \operatorname{dom}\gamma = U, \text{ and } \gamma \text{ is the extension of the} \\ \text{germ of an element of } \Gamma' \text{ as above} \end{array} \right\}.$$
   Then, Γ_U is a normal family.
2) The union of Fatou neighborhoods is called the *Fatou set* of (Γ', T') and denoted by $F(\Gamma')$. The complement of the Fatou set is called the *Julia set* of (Γ', T') and denoted by $J(\Gamma')$.
3) The *Fatou set* of (Γ, T) is the Γ-orbit of $F(\Gamma')$, namely, $F(\Gamma) = \Gamma(F(\Gamma'))$. The *Julia set* of (Γ, T) is the complement of $F(\Gamma)$ and denoted by $J(\Gamma)$.
4) Γ-connected components of $F(\Gamma)$ and $J(\Gamma)$ are called the *Fatou components* and *Julia components*, respectively.

If $x \in F(\Gamma)$, then any Fatou neighborhood $U \subset F(\Gamma')$ which contains x is called a Fatou neighborhood of x, where (Γ', T') is a reduction of (Γ, T) such that $x \in T'$.

Remark 2.8. 1) $F(\Gamma)$ is open and Γ-invariant. $J(\Gamma)$ is closed and Γ-invariant.
2) The condition (b) in 1) is always satisfied by virtue of Montel's theorem because we choose T' as a disjoint union of finite number of discs in $\mathbb{C}$

(see Remark 2.2). On the other hand, it is necessary to fix a domain of definition in order to speak of normal families. This leads to the condition (a) in 1) of Definition 2.7.

3) $J(\Gamma) = \Gamma(J(\Gamma'))$.

We recall the notion of equivalence [14].

Definition 2.9. Let (Γ, T) and (Δ, S) be pseudogroups. A *holomorphic étale morphism* $\Phi\colon \Gamma \to \Delta$ is a collection Φ of biholomorphic diffeomorphisms of open sets of T to open sets of S such that

i) if $\varphi \in \Phi$, $\gamma \in \Gamma$ and $\delta \in \Delta$, then $\delta \circ \varphi \circ \gamma \in \Phi$,
ii) the sources of the elements of Φ form a covering of T,
iii) if $\varphi, \varphi' \in \Phi$, then $\varphi' \circ \varphi^{-1} \in \Delta$.
iv) Φ is maximal in the following sense.
  1) If $\varphi \in \Phi$ and U is an open subset of $\operatorname{dom}\varphi$, then $\varphi|_U \in \Phi$.
  2) Suppose that φ is a biholomorphic diffeomorphism from an open set of T to an open set of S. If there is an open covering $\{U_\alpha\}$ of $\operatorname{dom}\varphi$ such that $\varphi|_{U_\alpha} \in \Phi$, then $\varphi \in \Phi$.

If $\Phi^{-1} = \{\varphi^{-1}\}_{\varphi\in\Phi}$ is also a holomorphic étale morphism, then Φ is called an *equivalence*.

Remark 2.10. 1) Any reduction (Γ', T') is equivalent to (Γ, T).
2) If (Γ, T) and (Δ, S) are compactly generated, then Φ is finitely generated in the following sense. Let (Γ', T') be a reduction of (Γ, T) and Φ' the restriction of Φ to T'. Then there is a finite collection $\{\varphi_i\} \subset \Phi'$ such that $\{\operatorname{dom}\varphi_i\}$ is an open covering of T' and any $\varphi \in \Phi$ is locally of the form $\delta \circ \varphi_i \circ \gamma$ for some $\gamma \in \Gamma$ and $\delta \in \Delta$. If $\varphi \in \Phi'$, then γ can be chosen from Γ'. We call $\{\varphi_i\}$ a finite set of *generators* of Φ.

If Φ is an étale morphism, then we set $\Phi^{-1}(X) = \bigcup_{\phi\in\Phi} \phi^{-1}(X)$ for $X \subset S$.

Lemma 2.11. *The Fatou set is well-defined on the equivalence classes of pseudogroups, namely, the decomposition $T = F(\Gamma) \sqcup J(\Gamma)$ is independent of the choice of the reduction (Γ', T').*

Proof. Let $\{(\Gamma_n, T_n)\}$ be a sequence of pseudogroups such that $\overline{T_n} \subset T_{n+1}$, $\Gamma_n = \Gamma|_{T_n}$, $T = \cup T_n$ and every (Γ_n, T_n) is a reduction of (Γ, T). Note that T_{n+1} is naturally a subset of $\mathbb{C}$ so that it is equipped with the standard Hermitian metric. It is clear from the definition that $F(\Gamma_{n+1}) \cap T_n \subset F(\Gamma_n)$. To show the converse, let $\widetilde{\Phi}$ be an equivalence from T to T_n and let Φ be the equivalence from T_{n+1} to T_n obtained by restricting $\widetilde{\Phi}$ to T_{n+1}. Φ is generated by a finite collection $\{\varphi_i\}$ as above and there is a $\delta > 0$ such that $D_x(\delta)$ is contained in at least one of dom φ_i, where $x \in T_{n+1}$. Moreover, there is a $\delta' > 0$ independent of i and x such that the image of φ_i as an element of $\widetilde{\Phi}$ contains $D_{\varphi_i(x)}(\delta') \subset T_{n+1}$. Let $U \subset F(\Gamma_n)$ be a Fatou neighborhood, and Γ_U be the subset of Γ which consists of extension of elements of $(\Gamma_n)_x$, $x \in U$. Then we may assume by shrinking U that $\gamma(U)$ is always contained in a disc of radius $\delta'/2$ for any $\gamma \in \Gamma_U$. If $x \in U$ and $\gamma \in (\Gamma_{n+1})_x$, then $\varphi_i\gamma \in (\Gamma_n)_x$ for some i. Hence $\zeta = \varphi_i\gamma$ is defined on U and $\zeta(U) \subset D_{\varphi_i\gamma(x)}(\delta')$. Therefore, $\varphi_i^{-1}\zeta$ is defined on U and is an extension of γ as an element of Γ. Let Γ'_U be the subset of Γ which consists of extension of elements of $(\Gamma_{n+1})_x$ as above and let $\{\gamma_k\} \subset \Gamma'_U$. Then for each γ_k there is a $\varphi_{i(k)}$ such that $(\zeta_k)_x \in (\Gamma_n)_x$, where $x \in U$ and $\zeta_k = \varphi_{i(k)}\gamma_k$. The family $\{\zeta_k\}$ is a subfamily of Γ_U so that we can find a convergent subsequence, which we denote again by $\{\zeta_k\}$. Since $\Phi = \{\varphi_i\}$ is a finite collection, we can find a subsequence of $\{\zeta'_l\}$ of $\{\zeta_k\}$ and $\varphi_i \in \Phi$ such that $\varphi_i^{-1}\zeta'_l$ is always defined. The family $\{\varphi_i^{-1}\zeta'_l\}$ is a convergent subsequence of $\{\gamma_k\}$. Consequently U is a Fatou neighborhood for Γ_{n+1} so that $F(\Gamma_n) \subset F(\Gamma_{n+1}) \cap T_n$. It follows that $F(\Gamma) = \cup F(\Gamma_n) = \Gamma(F(\Gamma_n))$. If (Γ', T') is a reduction, then $T' \subset T_n$ for some n so that $\Gamma(F(\Gamma')) = \Gamma(F(\Gamma_n))$. □

Lemma 2.12. *The Fatou-Julia decomposition has a naturality in the following sense.*

1) *Let* $\Phi\colon (\widehat{\Gamma}, \widehat{T}) \to (\Gamma, T)$ *be a holomorphic étale morphism. Then* $F(\widehat{\Gamma}) \supset \Phi^{-1}(F(\Gamma))$.
2) *If* $(\widehat{\Gamma}, \widehat{T})$ *is a Galois covering of* (Γ, T) *with finite Galois group* [14], *then* $F(\widehat{\Gamma}) = p^{-1}(F(\Gamma))$, *where* $p\colon \widehat{T} \to T$ *is the projection.*
3) *If* (Γ, T) *and* (Δ, S) *are compactly generated pseudogroups and if* Φ *is an equivalence from* (Γ, T) *to* (Δ, S), *then* $\Phi(F(\Gamma)) = F(\Delta)$.

Proof. First we show 1). Let $(\widehat{\Gamma}', \widehat{T}')$ be a reduction and $\{\varphi_i\}$ a finite set of generators of Φ. We may assume that there is a $\delta_1 > 0$ such that at least one φ_j is defined on $D_{\widehat{x}}(2\delta_1)$ for any $\widehat{x} \in \widehat{T}'$. Then there is an ϵ independent of j and $\widehat{x}$ such that $\varphi_j(D_{\widehat{x}}(\delta_1)) \supset D_{\varphi_j(\widehat{x})}(2\epsilon)$. Let $\widehat{x} \in \widehat{T}'$ and assume that $x = \varphi_i(\widehat{x}) \in F(\Gamma)$. Let (Γ', T') be a reduction of (Γ, T) such that $x \in T'$. Then we may assume that there is a Fatou neighborhood U of x in T' such that $\gamma(U) \subset D_{\gamma x}(\epsilon)$ for any $\gamma \in \Gamma_U$. We may also assume that φ_i^{-1} is defined on U by shrinking U if necessary, and set $\widehat{U} = \varphi_i^{-1}(U)$. Let $\widehat{\gamma}' \in \widehat{\Gamma}_{\widehat{y}}$, where $\widehat{y} \in \widehat{U}$, and let φ_j be such that φ_j is defined on $D_{\widehat{\gamma}\widehat{y}}(2\delta_1)$. Since $x \in F(\Gamma)$, $\varphi_j \circ \widehat{\gamma} \circ \varphi_i^{-1}$ is well-defined on U as an element γ of Γ. Note that $\varphi_j^{-1} \circ \gamma \circ \varphi_i(\widehat{U}) \subset D_{\widehat{\gamma}\widehat{y}}(\delta_1)$ because $\gamma \circ \varphi_i(\widehat{U}) = \gamma(U) \subset D_{\gamma x}(\epsilon) \subset D_{\varphi_j(\widehat{\gamma}\widehat{y})}(2\epsilon)$. Fix now a finite set $\{\widehat{\gamma}_1, \ldots, \widehat{\gamma}_r\}$ of generators of $\widehat{\Gamma}'$ and denote by $\widehat{\Gamma}'(k)$ the subset of $\widehat{\Gamma}'$ which consists of elements obtained by composing at most k generators, then the germ of any element of $\widehat{\Gamma}'$ is the germ of an element of $\widehat{\Gamma}'(k)$ for some k. We may assume by decreasing δ_1 and shrinking $\widehat{U}$ that if $\widehat{y} \in \widehat{T}'$ then all the generators are defined on $D_{\widehat{y}}(\delta_1) \subset \widehat{T}$ as an element of $\widehat{\Gamma}$. Suppose inductively that if $\widehat{\gamma} \in \widehat{\Gamma}'_{\widehat{y}}$ is the germ of an element of $\widehat{\Gamma}'(k)$, then $\widehat{\gamma}$ is defined on $\widehat{U}$ as an element of $\widehat{\Gamma}$ and $\widehat{\gamma}(U) \subset D_{\widehat{\gamma}\widehat{y}}(\delta_1)$. This holds certainly for $k = 1$. If $\widehat{\gamma} \in \widehat{\Gamma}'_{\widehat{y}}$ is the germ of an element of $\widehat{\Gamma}'(k+1)$, then $\widehat{\gamma} = \widehat{\gamma}_i \circ \widehat{\zeta}$ for some i in the germinal sense, where $\widehat{\zeta} \in \widehat{\Gamma}'(k)$. By the hypothesis, $\widehat{\zeta}$ is well-defined on $\widehat{U}$ as an element of $\widehat{\Gamma}$ and $\widehat{\zeta}(\widehat{U}) \subset D_{\widehat{\gamma}\widehat{y}}(\delta_1)$. Then by the choice of δ_1, $\widehat{\gamma}_i \circ \widehat{\zeta}$ is well-defined on $\widehat{U}$. Moreover, from what we have shown first, $\widehat{\gamma}_i \circ \widehat{\zeta}(\widehat{U}) \subset D_{\widehat{\gamma}\widehat{y}}(\delta_1)$. Thus $\widehat{U}$ is a Fatou neighborhood of x. This completes the proof of 1). 2) can be shown by slightly modifying the proof of Lemma 2.11 so that omitted. 3) follows from 1) at once. □

Lemmas 2.11 and 2.12 justify the following definition. Let $\mathcal{F}$ be a complex codimension-one transversally holomorphic foliation of a closed manifold M and let (Γ, T) be the holonomy pseudogroup of $\mathcal{F}$. We may assume that T is embedded in M.

Definition 2.13. The *Fatou set* of $\mathcal{F}$ is the saturation of $F(\Gamma) \subset T \subset M$, and denoted by $F(\mathcal{F})$. The *Julia set* is the complement of $F(\mathcal{F})$ and denoted by $J(\mathcal{F})$. The connceted components of the Fatou set and the Julia set are called the *Fatou components* and the *Julia components*, respectively.

It is clear that $J(\mathcal{F})$ is the saturation of $J(\Gamma)$.

The following is an immediate consequence of Lemma 2.12.

Corollary 2.14. *Let M and N be closed manifolds and let $\mathcal{F}$ be a complex codimension-one transversally holomorphic foliation of M. Let $f\colon N \to M$ be a smooth mapping transversal to $\mathcal{F}$ and let $\mathcal{G} = f^*\mathcal{F}$ be the induced foliation of N. Then $F(\mathcal{G}) \supset f^{-1}(F(\mathcal{F}))$. If f is a (regular) finite covering, then $F(\mathcal{G}) = f^{-1}(F(\mathcal{F}))$.*

It is easy to see that $F_{\mathrm{GGS}}(\mathcal{G}) \supset f^{-1}(F_{\mathrm{GGS}}(\mathcal{F}))$ but the equality for coverings does not hold in general (Example 4.3).

The existence of reductions is essential for the definition of the Fatou-Julia decomposition as follows.

Example 2.15. Let $D(r)$ be the disc in $\mathbb{C}$ of radius r and let $\mathcal{F}$ be the foliation of $M = (-1,1) \times D(1)$ with leaves $(-1,1) \times \{z\}$. If M itself is regarded as a foliation atlas, then the Fatou set should be the whole M. On the other hand, let $i \in \mathbb{Z}$ and define a foliation atlas as follows. For $i > 0$, let $\{V^{(i)}_j\}_{j=1,2,\dots}$ be an open covering of $D(1)$ by discs of radius 2^{-i}. Let $W^{(i)}_j =$

$(-1+1/2^{-i+1}, -1+1/2^{-i-1}) \times V_j^{(i)}$ and $T_j^{(i)} = \{-1+1/2^{-i}\} \times V_j^{(i)}$. Giving an order to $\{W_j^{(i)}\}$, let $\{W_j^{(i)}\} = \{W_1', W_2', \ldots\}$ and $\{T_j^{(i)}\} = \{T_1', T_2', \ldots\}$. Set then $U_0 = (-1/2, 1/2) \times D(1)$, $T_0 = \{0\} \times D(1)$, and $U_i = W_{|i|}'$, $T_i = T_{|i|}'$ for $i \neq 0$. Simply applying the definition without taking reduction, the Fatou set should be empty. Note that this construction can be done in a foliation chart.

In what follows, we usually fix a reduction (Γ', T') and work on it.

We will show some fundamental properties of the Fatou-Julia decomposition.

Lemma 2.16. *Suppose that (Γ, T) is C^0-Hermitian, namely, there is a continuous Hermitian metric on T which is invariant under Γ, then $T = F(\Gamma)$.*

Proof. The proof is an application of arguments found in [10]. If h is the invariant metric and if g is the Euclidean metric on $T \subset \mathbb{C}$, then there is a constant $C \geq 1$ such that $C^{-1}g \leq h \leq Cg$ on $\overline{T'}$ (see Definition 3.6 for the notation). Let $\{\gamma_1, \ldots, \gamma_r\}$ be a set of generators of Γ'. Then, there is a positive real number $\delta > 0$ such that any germ of γ_i at any point $x \in T'$ extends to an element of Γ defined on $D_x(\delta)$. If we denote by $\Gamma'(k)$ the subset of Γ' which consists of elements which can be realized by composing at most k generators, then the germ of any element of Γ' is the germ of an element of $\Gamma'(k)$ for some k. Let $x \in T'$ and let $U = D_x\left(\frac{\delta}{2C^2}\right)$, and assume that germs of elements of $\Gamma'(k)$ at $u \in U$ extend to elements of Γ defined on U. The assumption certainly holds if $k = 1$. If γ is the germ of an element of $\Gamma'(k+1)$ at $u \in U$, then $\gamma = \gamma_i \circ \zeta$ for some $\zeta \in \Gamma'(k)$. By the induction hypothesis, ζ extends to an element of Γ defined on U. Then, $\gamma(U) \subset D_{\zeta(x)}(\delta)$. On the other hand, γ_i is defined on $D_{\zeta(x)}(\delta)$ by the choice of δ. Therefore, γ extends to an element of Γ defined on U. This implies that U is a Fatou neighborhood of x. □

The above lemma can be slightly strengthen. See Remark 7.10.

Definition 2.17. Let $x \in T'$ and assume that $\gamma(x) = x$ for some $\gamma \in \Gamma_x$. The fixed point x is called

1) *hyperbolic* if $|\gamma'|_x \neq 1$,
2) *parabolic* if $(\gamma_x')^k = 1$ for some $k \in \mathbb{Z}$ but $\gamma^{\circ m} \neq \mathrm{id}$ for any $m \in \mathbb{Z}$, where $\gamma^{\circ m}$ denotes the m-th iteration of γ (in a germinal sense),
3) *irrationally indifferent* if $|\gamma'|_x = 1$ but $(\gamma_x')^k \neq 1$ for any $k \in \mathbb{Z}$.

Remark 2.18. It is easy to see that none of the above cases is exclusive. For instance, let Γ be a subgroup of $\mathrm{PSL}(2;\mathbb{C})$ generated by g_1, g_2 and g_3, where $g_1(z) = 2z$, $g_2(z) = z+1$ and $g_3(z) = e^{2\pi\sqrt{-1}\theta}z$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then Γ acts on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and ∞ is hyperbolic, parabolic and irrationally indifferent.

The Julia set has the following fundamental property as usual.

Lemma 2.19. *Let $x \in T$. If there is an element $\gamma \in \Gamma_x$ which has x as a parabolic or hyperbolic fixed point, then $x \in J(\Gamma)$.*

It is difficult to tell if a given point belongs to the Fatou set or the Julia set in general. However, we have the following lemma which is significant in the sequel.

Lemma 2.20. *Let $x \in F(\Gamma')$ and let $\{\gamma_i\}$ be a family of elements of Γ' defined on a neighborhood V of x. Assume that $\{\gamma_i(x)\}$ converges to a point $y \in \overline{T'} \subset T$.*

1) *If $\{|\gamma_i'|_x\}$ admits a subsequence which is bounded away from 0, then y belongs to $F(\Gamma)$. Moreover, $\{|\gamma_i'|_x\}$ is bounded and bounded away from 0.*
2) *If $\{|\gamma_i'|_x\}$ admits a subsequence which converges to 0, then $\{|\gamma_i'|_x\}$ converges to 0 and y belongs to $J(\Gamma)$.*

Proof. We may assume that $V = D_x(r)$. Then, $\{|\gamma_i'|_x\}$ is bounded from above because Γ_V is a normal family.

First let $\{\zeta_j\}$ be a subsequence of $\{\gamma_i\}$ such that $\{\left|\zeta_j'\right|_x\}$ is bounded away from 0. Since Γ_V is a normal family, we may assume after slightly shrinking V that $\{\zeta_j\}$ uniformly converges to a function γ on V. As $\{\left|\zeta_j'\right|_x\}$ is bounded away from 0, γ is not a constant function so that $\gamma(V)$ is an open set. It follows that $\zeta_i(V)$ contains y for sufficiently large i. Since $V \subset F(\Gamma')$, y belongs to $F(\Gamma)$.

Second, let $\{\zeta_j\}$ be a subsequence of $\{\gamma_i\}$ such that $\{\left|\zeta_j'\right|_x\}$ converges to 0. As (Γ, T) is equivalent to (Γ', T'), we may assume that $y \in T'$. If $y \in F(\Gamma')$, then there is a Fatou neighborhood U of y. We may assume that U is an open ball centered at y. We may also assume that $\zeta_j(x) \in U$ if $j \geq j_0$. Let $x' = \zeta_{j_0}(x)$ and set $\eta_j = \zeta_j \circ \zeta_{j_0}^{-1}$. Then U is a Fatou neighborhood of x' and $\{\left|\eta_j'\right|_{x'}\}$ converges to 0. By slightly shrinking U, we may assume that $\{\eta_j\}$ uniformly converges to a constant function. Then, the image $\eta_j(U)$ is contained in U for sufficiently large j. Hence η_j has a hyperbolic fixed point in U. This is a contradiction because $U \subset F(\Gamma')$. This completes the proof. □

Remark 2.21.

1) The more can be said about $\gamma_i(V)$ in the proof of 1), where $V = D_x(r)$. Namely, if δ is a positive number such that $|\gamma_i'|_x > \delta$, then $\gamma_i(V) \supset D_{\gamma_i(x)}(r\delta/4)$ by the Koebe 1/4-theorem.

2) It is possible that $x \in F(\Gamma)$ admits a family $\{\gamma_i\}$ which contains a subsequence $\{\zeta_j\}$ with $\zeta_i'(x) \to 0$ but $\{\gamma_i'(x)\}$ does not converge to 0 if $\{\gamma_i(x)\}$ does not converge to a single point. See Example 3.11.

3. Construction of an invariant metric of class $C^{\mathrm{Lip}}_{\mathrm{loc}}$

A metric of the form $g dz \otimes d\bar{z}$ is said to be of class $C^{\mathrm{Lip}}_{\mathrm{loc}}$ if g is locally Lipschitz continuous. We first show the following.

Proposition 3.1. *$(\Gamma|_{F(\Gamma)}, F(\Gamma))$ is $C^{\mathrm{Lip}}_{\mathrm{loc}}$-Hermitian, namely, there is a locally Lipschitz continuous metric g^L on $F(\Gamma)$ invariant under $\Gamma|_{F(\Gamma)}$.*

Remark 3.2. It is known that invariant metrics of class C^ω exist on the GGS-Fatou sets. We will later show that there are invariant metric of class C^ω also on the Fatou sets (Theorem 4.21). It will be also shown that the metric in Proposition 3.1 is of class C^ω along orbit closures (Corollary 4.16).

Proposition 3.1 will be shown in steps. Note that it suffices to construct a Γ'-invariant metric on $F(\Gamma')$. Hence by taking a reduction, we may assume that $T = \coprod_{i \in I} T_i$, where I is a finite set and each T_i is an open disc in $\mathbb{C}$. We may furthermore assume that the closures $\overline{T_i}$ of T_i are mutually disjoint. Let (Γ', T') be a reduction. Then we may also assume that each component T_i' of T' is a slightly small open disc such that $\overline{T_i'} \subset T_i$.

Let h_0 be a metric on T' defined as follows. Let TT' be the holomorphic tangent bundle of T'. Let η_ϵ, $0 < \epsilon < 1$, be a smooth non-negative function on $\mathbb{R}$ such that

1) $\eta_\epsilon(t) = 1$ on $(-\infty, 1-\epsilon]$,
2) η_ϵ is strictly decreasing on $[1-\epsilon, 1]$,
3) $\eta_\epsilon(t) = 0$ on $[1, +\infty)$.

Definition 3.3. Let $c_i \in \mathbb{C}$ and $r_i > 0$ be the center and the radius of T_i', respectively. Set $h_i(z_i) = \eta_\epsilon(|z_i - c_i|/r_i)$ and define a Hermitian metric h_0 on TT' by $h_0|_{T_i'} = h_i(z_i)^2 dz_i \otimes d\bar{z}_i$, where $|\cdot|$ denotes the absolute value. The set of functions $\{h_i\}$ is denoted by h and considered as a function on T'.

In what follows, $\gamma(x)$ is also denoted by γx, where $\gamma \in \Gamma$ and $x \in T$.

Definition 3.4. For $x \in T_i'$, set $g_i(x) = \sup_{\gamma \in \Gamma_x'} h(\gamma x)\,|\gamma'|_x$. The set of functions $\{g_i\}$ is denoted by g and considered as a function on T'.

Remark 3.5. The meaning of g is as follows. Let $x \in T_i'$ and set $\|v\|_x^L = g_i(x)\,\|v\|_x$ for $v \in T_xT'$, where $\|v\|_x$ denotes the Euclidean norm of v multiplied by $h(x)$, then

$$\|v\|_x^L = \sup_{\gamma \in \Gamma'_x} \|\gamma_* v\|_{\gamma x}.$$

We recall the notion of equivalence of metrics:

Definition 3.6. Let $h^1 = \{(h_i^1)^2 dz_i \otimes d\bar{z}_i\}$ and $h^2 = \{(h_i^2)^2 dz_i \otimes d\bar{z}_i\}$ be Hermitian metrics on TT'. If there exists a constant $C > 0$ such that $h_i^1 \leq C h_i^2$ for any i, then we write $h^1 \leq C h^2$. If there exists a constant $C \geq 1$ such that $\frac{1}{C} h^1 \leq h^2 \leq C h^1$, then h^1 and h^2 are said to be equivalent.

The following properties are clear.

Lemma 3.7. 1) $g_i(x) \geq h_i(x) > 0$.
2) *If $\gamma \in \Gamma'_x$, then $g(\gamma x) \left|\gamma'\right|_x = g(x)$.*
3) *Let $\widetilde{h}_0 = \{\widetilde{h}_i^2 dz_i \otimes d\bar{z}_i\}$ be a Hermitian metric on TT'. Assume that $\frac{1}{C} h_0 \leq \widetilde{h}_0 \leq C h_0$ and let $\widetilde{g} = \{\widetilde{g}_i\}$ be the set of functions in Definition 3.4 obtained by replacing h_0 with $\widetilde{h}_0$. Then $\frac{1}{C} g_i \leq \widetilde{g}_i \leq C g_i$.*

Lemma 3.8. *g is lower semicontinuous on T'.*

Proof. Let $x \in T'$. First assume that $g(x)$ is finite, and let $\gamma \in \Gamma'_x$ be such that $g(x) - \epsilon < h(\gamma x) \left|\gamma'\right|_x$. If $y \in T'$ is sufficiently close to x, then γy is defined and $h(\gamma y) \left|\gamma'\right|_y > h(\gamma x) \left|\gamma'\right|_x - \epsilon$ by the continuity of the function $z \mapsto h(\gamma z) \gamma'_z$. It follows that $g(x) - 2\epsilon < h(\gamma y) \left|\gamma'\right|_y \leq g(y)$. If $g(x) = +\infty$, then there is an element $\gamma \in \Gamma_x$ such that $M < h(\gamma x) \left|\gamma'\right|_x$ for any real number M. Then $M - \epsilon < h(\gamma y) \left|\gamma'\right|_y$ so that $g(y)$ is also infinite. □

The following lemma is the essential part of Proposition 3.1.

Lemma 3.9. *g is locally Lipschitz continuous on $F(\Gamma')$.*

Proof. Let $x \in F(\Gamma')$, then $g(x)$ is finite by 1) of Lemma 2.20. We may furthermore assume that $M_x = \sup_{\gamma \in \Gamma} \left|\gamma'\right|_x$ is also finite by taking reduction again. Assume that $D_x(2\delta)$ is a Fatou neighborhood of x and that $x = 0$ after a parallel translation. Recall now the Koebe distortion theorem [1]: if $f\colon D_0(1) \to \mathbb{C}$ is a univalent function such that $f(0) = 0$ and $f'(0) = 1$, then $\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}$ and $\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$. Let φ be a univalent function defined on $U = D_x(\delta)$. Applying the Koebe theorem to the function $z \mapsto \frac{1}{\delta \varphi'_x}(\varphi(\delta z) - \varphi(x))$, we have

$$\frac{\left|\varphi'\right|_x |y|}{(1 + \frac{1}{\delta}|y|)^2} \leq |\varphi(y) - \varphi(x)| \leq \frac{\left|\varphi'\right|_x |y|}{(1 - \frac{1}{\delta}|y|)^2}, \text{ and}$$

$$\frac{1 - \frac{1}{\delta}|y|}{(1 + \frac{1}{\delta}|y|)^3} \leq \frac{\left|\varphi'\right|_y}{\left|\varphi'\right|_x} \leq \frac{1 + \frac{1}{\delta}|y|}{(1 - \frac{1}{\delta}|y|)^3},$$

where $|y| = |y-0| = |y-x|$. It follows from the second inequality that if $\gamma \in \Gamma'_y$ and $|y| < \delta/2$, then $|\gamma'|_y \le 12M_x$. We now show the following

Claim. There are $\epsilon_1 > 0$ and δ_2 such that $\gamma \in \Gamma'_y$ induces an element of Γ'_x defined on $D_x(2\delta_2)$ if the conditions $|y| < \delta_2$ and $h(\gamma y)\,|\gamma'|_y > g(y) - \epsilon_1$ are satisfied.

If ϵ_1 is a positive real number less than $\frac{g(x)}{2}$, then there is a positive real number δ_3 such that $g(y) - \epsilon_1 > \frac{g(x)}{2}$ for $|y| < \delta_3$ by the lower semicontinuity of g. Assume that $h(\gamma y)\,|\gamma'|_y > g(y) - \epsilon_1$, then $h(\gamma y) \ge \frac{g(x)}{24M_x} > 0$. It follows that there is a compact subset K' of T' such that $h(\gamma y)\,|\gamma'|_y > g(y) - \epsilon_1$ holds only if $\gamma y \in K'$. Let $\epsilon_2 > 0$ be a real number such that $D_{K'}(\epsilon_2) \subset T'$. If $|y| < \min\left\{\frac{\delta}{2}, \frac{\epsilon_2}{8M_x}\right\}$, then $\frac{|\gamma'|_x |y|}{(1-\frac{1}{\delta}|y|)^2} \le 4M_x\,|y| < \frac{\epsilon_2}{2}$. Set $\delta_2 = \frac{1}{2}\min\{\frac{\delta}{2}, \delta_3, \frac{\epsilon_2}{8M_x}\}$, then $\gamma \in \Gamma'_y$ induces an element of Γ'_x defined on $D_x(2\delta_2)$ if $|y| < \delta_2$ and $h(\gamma y)\,|\gamma'|_y > g(y) - \epsilon_1$. This completes the proof of Claim.

Let $\epsilon_3 > 0$ be any real number less than ϵ_1 and assume that $|y| < \delta_2$. Let $\gamma \in \Gamma'_y$ such that $h(\gamma y)\,|\gamma'|_y > g(y) - \epsilon_3$. The above claim shows that $\gamma \in \Gamma'_z$ if $z \in D_x(2\delta_2)$. It follows that $h(\gamma z)\,|\gamma'|_z \le g(z)$. Hence $g(y) - g(z) < h(\gamma y)\,|\gamma'|_y - h(\gamma z)\,|\gamma'|_z + \epsilon_3$. Moreover, γ is well-defined on $D_z(\delta) \subset D_x(2\delta)$ as an element of Γ so that the Koebe estimate is valid for γ.

Noticing that each h_i is Lipschitz continuous, let L_h be the maximum of the Lipschitz constants. Then $|h(\gamma y) - h(\gamma z)| \le L_h\,|\gamma y - \gamma z| \le 12L_hM_x\,|y-z|$. By taking δ_2 smaller if necessary, we may assume that $4 - 3\frac{|y-z|}{\delta} + \frac{|y-z|^2}{\delta^2} \le 4$ if $y, z \in D_x(\delta_2)$. We may also assume that $\delta_2 < 1$, then it follows from the Koebe distortion theorem that

$$\frac{|\gamma'|_y}{|\gamma'|_z} - 1 \le \frac{1 + \frac{1}{\delta}\,|y-z|}{(1-\frac{1}{\delta}\,|y-z|)^3} - 1 \le 32\,|y-z|\,.$$

Hence $|\gamma'|_y - |\gamma'|_z \le 12M_x \cdot 32\,|y-z|$. Therefore, if $y, z \in D_x(\delta_2)$ then

$$\begin{aligned} g(y) - g(z) - \epsilon_3 &< h(\gamma y)(|\gamma'|_y - |\gamma'|_z) + (h(\gamma y) - h(\gamma z))\,|\gamma'|_z \\ &\le 32 \cdot 12M_x\,|y-z| + 12L_hM_x\,|y-z|\,12M_x \\ &= 48M_x(8 + 3L_hM_x)\,|y-z|\,, \end{aligned}$$

where the fact that $h \le 1$ is used. Since this estimate is independent of the choice of γ, ϵ_3 can be arbitrarily small. Hence $g(y) - g(z) \le 48M_x(8 + 3L_hM_x)\,|y-z|$.

Let now $\gamma \in \Gamma'_z$ be such that $g(z) - \epsilon_3 < h(\gamma z)\,|\gamma'|_z$. Then $\gamma \in \Gamma'_y$ and $h(\gamma y)\,|\gamma'|_y \leq g(y)$. Hence

$$\begin{aligned} g(z) - g(y) - \epsilon_3 &< h(\gamma z)\,|\gamma'|_z - h(\gamma y)\,|\gamma'|_y \\ &= (h(\gamma z) - h(\gamma y))\,|\gamma'|_z + h(\gamma y)(|\gamma'|_z - |\gamma'|_y) \\ &\leq 144 L_h M_x^2\,|y - z| + 12 M_x \left(1 - \frac{|\gamma'|_y}{|\gamma'|_z}\right). \end{aligned}$$

We may assume that $4 + 3\frac{|y-z|}{\delta} + \frac{|y-z|^2}{\delta^2} \leq 8$, then again by the Koebe distortion theorem, $1 - \frac{|\gamma'|_y}{|\gamma'|_z} \leq 32\,|y - z|$. This estimate is also independent of the choice of γ. Hence $g(z) - g(y) \leq 48 M_x (8 + 3 L_h M_x)\,|y - z|$. This completes the proof. □

The proof of Proposition 3.1 is completed by defining g^L by $g^L|_{T'_i} = g_i^2 dz_i \otimes d\bar{z}_i$. Indeed, the non-degeneracy and Γ'-invariance of g^L follow from the properties 1) and 2) in Lemma 3.7. Moreover, 1) implies that $g^L \geq h_0$. The property 3) in Lemma 3.7 implies that if $\widetilde{g}^L$ is constructed by a metric $\widetilde{h}$ such that $\frac{1}{C} h_0 \leq \widetilde{h} \leq C h_0$, then $\frac{1}{C} g^L \leq \widetilde{g}^L \leq C g^L$.

Remark 3.10. $\|\cdot\|^L$ can be either finite or infinite on J. Indeed, it is clear that $\|\cdot\|^L$ is infinite at hyperbolic fixed points. On the other hand, let γ be the automorphism of $\mathbb{C}P^1$ of which the restriction to $\mathbb{C}$ is given by $\gamma(z) = z + 1$. If we regard $(\{\gamma^n\}_{n\in\mathbb{Z}}, \mathbb{C}P^1)$ as a pseudogroup, then $\|\cdot\|^L$ is finite at the parabolic fixed point $\infty \in \mathbb{C}P^1$.

The metric obtained in this way can be of class C^ω but in general not of class C^1. For simplicity, we adopt the following function as η in Definition 3.3. Let

$$\eta_0(t) = \begin{cases} 0, & t \leq 0, \\ e^{-1/t}, & t > 0. \end{cases}$$

Let $\eta_1(t) = \displaystyle\int_{-\infty}^{t} \eta_0(s)\eta_0(1 - s)ds$, $\eta_2(t) = \eta_1(t)/\eta_1(2)$ and $\eta(t) = \eta_2((1 - t)/\epsilon)$.

Example 3.11. Let z be the inhomogeneous coordinates for $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Let λ, μ and ν are non-zero complex numbers such that $|\lambda| = 1$, $|\mu| = 2$ and $1 < |\nu| < 2$. Assume that $\log|\nu| / \log 2 \notin \mathbb{Q}$. Define automorphisms f_α, where α is one of λ, μ and ν, of $\mathbb{C}P^1$ by $f_\alpha(z) = \alpha z$.

First let N_1 be a closed manifold such that there exists a surjective homomorphism φ_1 from $\pi_1(N_1)$ to $\mathbb{Z}^2 = \langle f_\lambda, f_\mu \rangle$, for example let $N_1 = T^2$. Let $(M_1, \mathcal{F}_1)$ be the suspension of $(\mathbb{C}P^1, \varphi_1)$ and let (Γ_1, T) be the pseudogroup defined as follows. Let $T_0 = T_1 = D_0(\sqrt{2})$ and $T = T_0 \sqcup T_1$. Let Γ_1 be the pseudogroup generated by ρ_0, ρ_1, γ_0, γ_1 and γ_{10}, where $\rho\colon T_i \to T_i$ is given by

$\rho_i(z) = \lambda z$ and $\gamma_i\colon T_i \to T_i$ is given by $\gamma_i(z) = z/\mu$ for $i = 0, 1$, and let γ_{10} be the mapping from $\{z \mid 1/\sqrt{2} < |z| < \sqrt{2}\}$ to $\{z \mid 1/\sqrt{2} < |z| < \sqrt{2}\}$ defined by $\gamma_{10}(z) = 1/z$. Then, the holonomy pseudogroup of $\mathcal{F}_1$ is equivalent to (Γ_1, T).

The Julia set is given by $J(\Gamma_1) = J_0 \cup J_\infty$, where $J_0 = \{0\} \subset T_0$ and $J_\infty = \{0\} \subset T_1$. In terms of $\mathcal{F}_1$, $J(\mathcal{F}_1) = L_0 \cup L_\infty$, where L_0 and L_∞ are the leaves which correspond to 0 and ∞, respectively. Let h_0 be the metric on T as in Definition 3.3. Let α be the unique positive real number greater than 1 such that $\eta(\alpha) = 1/\alpha^2$. Then the metric $g^L = \{g_i^2 dz \otimes d\bar{z}\}$ is given by

$$g_0(z) = g_1(z) = \begin{cases} 2^n, & \frac{1}{2^n\sqrt{2}} \leq |z| \leq \frac{1}{2^n\alpha}, \\ \frac{2^n}{|2^n z|^2}\eta\left(\frac{1}{|2^n z|}\right), & \frac{1}{2^n\alpha} \leq |z| \leq \frac{1}{2^n}, \\ 2^n\eta(|2^n z|), & \frac{1}{2^n} \leq |z| \leq \frac{\alpha}{2^n}, \\ \frac{2^n}{|2^n z|^2}, & \frac{\alpha}{2^n} \leq |z| \leq \frac{1}{2^{n-1}\sqrt{2}}. \end{cases}$$

It is locally Lipschitz continuous and piecewise of class C^ω, but not of class C^1.

Second, let N_2 be a closed manifold such that there exists a surjective homomorphism φ_2 from $\pi_1(N_2)$ to $\mathbb{Z}^3 = \langle f_\lambda, f_\mu, f_\nu\rangle$, for example let $N_2 = T^3$. Let $(M_2, \mathcal{F}_2)$ be the suspension of $(\mathbb{C}P^1, \varphi_2)$, and let (Γ_2, T) be the pseudogroup generated by ρ_0, ρ_1, γ_0, γ_1, γ_{10} and ζ_i, $i = 0, 1$, where $\zeta_i(z) = z/\mu$. The holonomy pseudogroup of $(M_2, \mathcal{F}_2)$ is equivalent to (Γ_2, T) and the metric $g^L = \{g_i^2 dz \otimes d\bar{z}\}$ is given by $g_0(z) = g_1(z) = \frac{\beta}{|z|}$, where $\beta = \max\{b \in \mathbb{R} \mid \text{the graphs of } \eta(t) \text{ and } b/t \text{ have an intersection}\}$.

Note that the metric $\frac{\beta^2}{|z|^2} dz \otimes d\bar{z}$ is also invariant under Γ_1. Moreover, if g is a positive function which satisfies $g(2t) = g(t)/2$, then $g(|z|)^2 dz \otimes d\bar{z}$ is invariant under Γ_1. Hence it is quite easy to find an invariant metric of class C^ω.

4. Comparison with the Fatou-Julia decomposition by Ghys, Gomez-Mont and Saludes, Structure of Fatou components

The Fatou-Julia decomposition for foliations is firstly introduced and studied by Ghys, Gomez-Mont and Saludes [11]. The GGS Fatou-Julia decomposition is originally formulated for foliations but it is also defined for compactly generated pseudogroups [15].

Definition 4.1 ([11])**.** Let $C(\Gamma)$ be the set of continuous Γ-invariant $(1,0)$-vector fields X on T such that its distributional derivative is locally in L^2 and that $\bar{\partial}X$ is essentially bounded. The *Fatou set* $F_{\mathrm{GGS}}(\Gamma)$ in the sense of Ghys, Gomez-Mont, Saludes is by definition given by

$$F_{\mathrm{GGS}}(\Gamma) = \{x \in T \,|\, X(x) \neq 0 \text{ for some } X \in C(\Gamma)\}\,.$$

The Fatou set and the Julia set in this sense are called the GGS-Fatou set and the GGS-Julia set, and denoted by F_{GGS} and J_{GGS}, respectively. The most of results in [11] remain valid for compactly generated pseudogroups [15]. We make use of some properties of GGS-Fatou sets without proofs. We refer to [11] and [15] for the detailed accounts.

These Fatou-Julia decompositions are related as follows.

Proposition 4.2. *$F(\Gamma) \supset F_{\mathrm{GGS}}(\Gamma)$.*

Proof. Let $x \in F_{\mathrm{GGS}}(\Gamma')$, then there is a vector field $X \in C(\Gamma')$ with $X(x) \neq 0$. We may assume that $X \in C(\Gamma)$ and that X is uniquely integrable. By integrating X, we can find a 1-parameter family $\varphi\colon T' \times D \to T$ of homeomorphisms which is (Γ', Γ)-equivariant, where D is a small disc in $\mathbb{C}$. Choosing D small, we may assume that $z \mapsto \varphi(\gamma x, z)$, $\gamma \in \Gamma'$, is a homeomorphism of D into T which satisfies $\varphi(x, D) \subset T'$. By repeating an argument by Ghys [10] (cf. Lemma 2.16), we see that D is a Fatou neighborhood of x. □

The inclusions $F(\Gamma) \supset F_{\mathrm{GGS}}(\Gamma)$ and $J(\Gamma) \subset J_{\mathrm{GGS}}(\Gamma)$ can be strict in general. In fact, the naturality as in Lemma 2.12 fails for the GGS-decomposition.

Example 4.3. Consider $T^2 = \mathbb{C}/\mathbb{Z}^2$ and let $\mathcal{F}$ be the foliation of $S^1 \times T^2$ with leaves $\{S^1 \times \{z\}\}_{z \in T^2}$. Then the GGS-Fatou set is the whole manifold. Let $\sigma : T^2 \to T^2$ be an automorphism induced by $z \mapsto -z$. Then $S^1 \times \{z\} \subset S^1 \times_\sigma T^2$, $z = 0, 1/2, \sqrt{-1}/2, (1+\sqrt{-1})/2$ are the GGS-Julia components. On the other hand, $J(\mathcal{F}) = \varnothing$.

The Fatou components also admit a classification analogous to that of GGS-Fatou components. The rest of this section is mostly devoted to it.

A pseudogroup (Γ, T) is said to be *complete* if for any $x, y \in T$ there are neighborhoods V of x and W of y such that every germ $\gamma \in \Gamma_{x'}$, $x' \in V$ with $\gamma x' \in W$ extends to an element of Γ defined on V.

Lemma 4.4 ([30, Proposition 1.3.1])**.** *$(\Gamma|_{F(\Gamma)}, F(\Gamma))$ is complete.*

Proof. Let $x, y \in T$ and let γ_0 and γ_1 be elements of Γ such that the both $z = \gamma_0 x$ and $w = \gamma_1 y$ belong to T'. Let δ be a positive real number such that γ_1^{-1} is defined on $D_w(2\delta)$ and let W be a neighborhood of y such that $W \subset \gamma_1^{-1}(D_w(\delta))$. Let U be a Fatou neighborhood of z such that the diameter of $\gamma(U)$ is less than δ for any $\gamma \in \Gamma_U$. Such an U exists because $\Gamma_{U'}$ is a normal family for any Fatou neighborhood U'. Finally let V be a neighborhood of x

such that $\gamma_0(V) \subset U$. Let $x' \in V$ and let $\zeta \in \Gamma_{x'}$ be such that $\zeta(x') \in W$. Set $\gamma = \gamma_1\zeta\gamma_0^{-1}$, then the germ of γ at $\gamma_0(x')$ extends to U as an element of Γ because $\gamma_0(x') \in U$. If we denote the extension again by γ, then $\gamma(U)$ is contained in $D_w(2\delta)$ so that $\gamma_1^{-1}\gamma\gamma_0$ is an extension of ζ as an element of Γ which is defined on the whole V. □

It is clear that $(\Gamma'|_{F(\Gamma')}, F(\Gamma'))$ is also complete.

Let $x \in F(\Gamma')$ and let D be an open disc centered at x such that the closure $\overline{D}$ is contained in a Fatou neighborhood of x.

Definition 4.5. Let $\mathcal{O}_D$ be the space of holomorphic maps defined on D equipped with the compact open topology. Set $\Gamma'_D = \{\gamma \in \Gamma' \mid \gamma(D) \cap D \neq \varnothing\} \subset \mathcal{O}_D$ and let G_D be the closure of Γ'_D,

Note that G_D consists of biholomorphic diffeomorphisms by Lemma 2.20. The local group G_D and the closure of Γ'-orbits are related as follows.

Lemma 4.6. *If $x \in D$, then $G_D x = \overline{\Gamma'_D x}$.*

Proof. It is clear that $G_D x \subset \overline{\Gamma'_D x}$. Let $y \in \overline{\Gamma'_D x}$ and let $\{\gamma_n\} \subset \Gamma'_D$ be such that $\{\gamma_n x\}$ converges to y. There is a subsequence of $\{\gamma_n\}$ which converges to an element γ of G_D uniformly on D because $\overline{D} \subset V_x$. It is easy to see that $y = \gamma x$. □

We recall some basic notions of local groups [22] (see also [19] for properties of local groups).

Definition 4.7. A topological space G is called a *local group* if a product xy is defined as an element in G for some pairs x, y in G and the following conditions are satisfied:

1) There is a unique element e in G such that ex and xe are defined for each x in G and $ex = xe = x$.
2) If x, y are in G and xy exists then there is a neighborhood U of x and a neighborhood V of y such that if $x' \in U$, $y' \in V$ then $x'y'$ exists. The correspondence $(x, y) \mapsto xy$ is continuous wherever defined.
3) The associative law holds whenever it has meaning.
4) If $xy = e$ then $yx = e$. An element y satisfying this relation is called an inverse of x and is denoted by x^{-1}. The inverse x^{-1} is unique if it exists, and the correspondence $x \mapsto x^{-1}$ is continuous. Moreover, if x^{-1} exists, then y^{-1} exists on a neighborhood of x.

We will apply some theorems of Cartan [6]. When actions of local groups are discussed in [6], a property related to analyticity is assumed in addition to the usual compatibility conditions. This condition is always satisfied if the local group consists of analytic transformations ([6, page 11], where the term 'pseudo-conforme' is used in place of 'holomorphic'). Hence we have the following

Lemma 4.8. *G_D is a local transformation group on D in the sense of Cartan [6].*

Definition 4.9 ([6, p.18])**.** Let X be a topological space and let D be an open subset of X. Let G be a local transformation group of a continuous transformations defined on D. G is *quasi continuous of order at most d* if there exist a neighborhood U of the unit element of G, a compact subset K of $\mathbb{R}^d$ and a bijection $\varphi\colon K \to U$ such that the mapping $\Phi\colon D \times K \to X$ defined by $\Phi(x;k) = \varphi(k)(x)$ is continuous.

Lemma 4.10. *G_D is quasi continuous of order at most 3. Hence G_D is a quasi-continuous group of analytic transformations (un groupe quasi-continue de transformations analytique) in the sense of Cartan.*

Proof. The G_D-action preserves the metric g^L in Section 3 which is locally Lipschitz continuous. Hence elements of G_D are uniquely determined by their 1-jets at x. By the continuity of solutions with respect to the initial values, G_D is indeed quasi continuous of order at most 3. □

The following result of Cartan is essential. We quote it by adapting terminologies.

Theorem 4.11 ([6, Théorèmes 9 et 10])**.** *A local quasi-continuous group which consists of local biholomorphic diffeomorphisms is a local Lie transformation group.*

Remark 4.12. By a ' local Lie transformation group' we mean not only the group is locally a Lie group but the action is also analytic ([6, pages 20–22]).

Corollary 4.13. *G_D is a local Lie transformation group.*

The above arguments can be summarized as follows.

Theorem 4.14. *G_D is a local Lie transformation group of (real) dimension at most 3. The dimension of connected components of G_D is constant.*

Proof. The first claim essentially follows from Lemma 4.10. Indeed, although the assumption is slightly different, the argument of the proof of Théorème 12 of [6] is still valid so that $\dim_{\mathbb{R}}(G_D)_0$ is at most 3. The last claim follows from the fact that G_D is closed (cf. [19]). □

Remark 4.15.
1) If we denote by G_x the stabilizer of x, then G_x is compact since elements of G_x are determined by their 1-jets. In particular, $(G_x)_0 \backslash G_x$ is a finite group, where $(G_x)_0$ is the identity component of G_x.
2) G_D is not necessarily connected. For example, let f and g be automorphisms of $\mathbb{C}P^1$ given by $f([z:w]) = [\alpha z : w]$ and $g([z:w]) = [w:z]$, where $\alpha = e^{2\pi\sqrt{-1}\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let Γ be the group generated by f and g. If we take a suspension of Γ, then $G_D = \mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z})$.

The following is immediate.

Corollary 4.16.
1) *The closures of $\mathit{\Gamma}$-orbits in the Fatou set are C^ω-submanifolds of $F(\mathit{\Gamma})$.*
2) *The metric g^L constructed in Section 3 is of class C^ω along orbit closures.*

Note that G_D depends on the choice of D as in Remark 4.15 but the dimension does not. Moreover, the natural homomorphism of local groups from G_{D_2} to G_{D_1}, where $D_2 \subset D_1$, is injective by the uniqueness of the solution of ordinary differential equations.

The Fatou components are named after [11].

Definition 4.17. A Fatou component F is called
1) *wandering component* if $\dim G_D = 0$,
2) *semi-wandering component* if $\dim G_D = 1$,
3) *dense component* if $\dim G_D \geq 2$,

where $D \subset F$ is any open set as above.

These components admit description analogous to that of GGS-Fatou components. Let E_F be the principal S^1-bundle associated to the frame bundle over F. E_F can be considered as the unit tangent bundle over F if there are invariant Hermitian metrics. Note that $\mathit{\Gamma}|_F$ acts on E_F so that G_D also locally acts on E_F. We denote $\mathit{\Gamma}|_F$ by $\mathit{\Gamma}_F$. Let $(\overline{\mathit{\Gamma}_F}, F)$ be the pseudogroup generated by $\mathit{\Gamma}_F$ and G_D. Let $(\widetilde{\mathit{\Gamma}_F}, \widetilde{F})$ be the universal covering of $(\mathit{\Gamma}_F, F)$ [14] and let $(\widetilde{\overline{\mathit{\Gamma}_F}}, \widetilde{F})$ be the lift of $(\overline{\mathit{\Gamma}_F}, F)$.

Theorem 4.18. *If F is a wandering component, then the orbit space $\Sigma = \mathit{\Gamma}_F \backslash F$ is a V-manifold (an orbifold). If we denote by S the singular set of Σ, then $\pi^{-1}(\Sigma \setminus S)$ is a GGS-Fatou component, where $\pi\colon F \to \Sigma$ is the projection. The number of wandering Fatou components of which $\Sigma \setminus S$ is $\mathbb{C}P^1$ minus one, two or three points is finite.*

Proof. We work on a reduction $(\mathit{\Gamma}'|_{F'}, F')$ but still denote it by $(\mathit{\Gamma}_F, F)$. First note that $\mathit{\Gamma}_F$ is complete by Lemma 4.4. Hence $\mathit{\Gamma}_F \backslash F$ is possibly non-Hausdorff manifold. Assume that $\Gamma_F\backslash F$ is non-Hausdorff, then there are a sequence $\{x_i\}$ in F and a sequence $\{\gamma_i\}$ of elements of Γ such that $\lim\limits_{i\to\infty} x_i = x \in F$, $\lim\limits_{i\to\infty} y_i = y \in F$, where $y_i = \gamma_i x_i$, but there is no element γ of Γ_F such that $\gamma x = y$. Let D be a Fatou neighborhood of x and let D' be a Fatou neighborhood of y as in Theorem 4.14. We may assume that $x_i \in D$ for all i and that $y_j \in D'$ for all j, then γ_1 is defined on D so that $z_i = \gamma_1 x_i$ makes a sense. Moreover, since $\{x_i\}$ converges to x and $y_1 = \gamma_1 x_1 \in D'$, we may assume that $z_i \in D'$. Let $\xi_i = \gamma_i\gamma_1^{-1}$. Then ξ_i is defined on D' and $\xi_i z_i = y_i$. We may assume that the sequence $\{\xi_i\}$ converges to a mapping ξ in $G_{D'}$. As $\dim(G_{D'}) = 0$, we may furthermore assume that $\xi_i = \xi$ for all i and that $\xi \in \Gamma_F$. It follows that $y = \xi\gamma_1(x)$ and it is a contradiction.

Let F be a Fatou component and let $\pi\colon F \to \Sigma$ be the projection. Let S be the singular set of Σ and set $F' = \pi^{-1}(\Sigma \setminus S)$. Then F' is contained in a GGS-Fatou component, say F''. Indeed, there is a smooth vector field on $\Sigma \setminus S$ which does not vanish at a given point $x \in \Sigma \setminus S$ but trivial out of a small neighborhood of x. Such a vector field gives rise to a vector field which belongs to $C(\Gamma)$. If F' is a proper subset of F'', then $F'' \cap \partial F'$ is non-empty. It is impossible because $F_{\mathrm{GGS}}(\Gamma) \subset F(\Gamma)$ and F' is Γ-connected. Hence $F' = F''$. The last claim follows from [11, Theorem 2]. $\square$

Let $G = \{x \mapsto tx + z \,|\, t, z \in \mathbb{C},\ |t| = 1\} \subset \mathrm{Aff}(\mathbb{C})$ and let $G_\lambda = \{x \mapsto \lambda^n z + b \,|\, n \in \mathbb{Z},\ b \in \mathbb{R}\} \subset \mathrm{Aff}(\mathbb{R})$. G contains $S^1 = \{(t, 0) \,|\, |t| = 1\}$ as a closed subgroup.

Theorem 4.19. *If F is a semi-wandering component, then the closure of all but finite number of Γ-orbits are real codimension-one manifold properly embedded in F. The rest of the orbits are proper. Let $P \subset F$ be the union of proper orbits.*

1) *If $P = \varnothing$, then (Γ, F) is equivalent to a pseudogroup generated by a subgroup H' of a group H, where H is either $\mathbb{C}$ or $\mathrm{Aff}(\mathbb{R})$ and H acts on a strip $S_{\alpha,\beta} = \{z \in \mathbb{C} \,|\, \alpha < \mathrm{Im}\, z < \beta\}$, where $-\infty \le \alpha < \beta \le +\infty$. The closure of Γ-orbits in E_T are finite coverings of Γ-orbits in F. Let*

$$F_0 = \{x \in F \,|\, \text{the closure of } \Gamma x \text{ is simply covered}\}, \text{ and}$$
$$F_1 = \{x \in F \,|\, \text{the closure of } \Gamma x \text{ is doubly covered}\}.$$

Then $F = F_0 \cup F_1$, and F_0 is a GGS-semi-wandering component and F_1 is contained in a GGS-ergodic Julia component. We have the following cases.

2a) $(H, S) = (\mathbb{C}, \mathbb{C})$, $\overline{H'} = \mathbb{R} \times \sqrt{-1}\mathbb{Z}$ *and* $\widetilde{\overline{\Gamma_F}}\backslash\widetilde{F} = S^1$.

2b) $(H, S) = (\mathbb{C}, S_{\alpha,\beta})$, $\overline{H'} = \mathbb{R}$ *and* $\widetilde{\overline{\Gamma_F}}\backslash\widetilde{F} = (\alpha, \beta)$.

2c) $(H, S) = (\mathrm{Aff}(\mathbb{R}), \mathcal{H})$, $\overline{H'} = G_\lambda$ *for some* $\lambda > 0$ *and* $\widetilde{\overline{\Gamma_F}} \backslash \widetilde{F} = S^1$, *where* $\mathcal{H}$ *denotes the upper half space.*

2) *If* $P \neq \varnothing$, *then* $(\Gamma, F \setminus P)$ *is as in 2b) and* $(\widetilde{\Gamma_F}, \widetilde{E})$ *is equivalent to a pseudogroup generated by a subgroup* H' *of the group* G *such that* $\overline{H'} = S^1$. *Let* $(\widetilde{\overline{\Gamma_F}}, \widetilde{E_F})$ *be the universal covering of* $(\overline{\Gamma_F}, E_F)$. *Then* $\widetilde{\overline{\Gamma_F}} \backslash \widetilde{E_F}$ *is either* $\{z \in \mathbb{C} \mid |z| < \alpha\}$, *where* $0 < \alpha \leq +\infty$, *or* $\mathbb{C}P^1$. P *consists of at most two* Γ*-orbits.*

Proof. Let $x \in F$ and let D be a small neighborhood of x. Let X be the vector field generated by the $(G_D)_0$-action. As $\sqrt{-1}X$ is also invariant under $(G_D)_0$-action, we can find a holomorphic vector field Z on D such that $2\,\mathrm{Re}\,Z$ is tangent to the G_D-orbits by repeating the argument in [11, Lemma 5.2]. Moreover, if $D \cap D' \neq \varnothing$, then thus constructed vector fields Z and Z' coincide up to multiplication of a real constant. If Z has no singularities for any D, then $P = \varnothing$. Since $(\widetilde{\overline{\Gamma_F}}, \widetilde{F})$ is simply connected, the argument in [11] can be applied and we have the classification as in the statement. Noticing that the G_D-action induces a 1-dimensional foliation, the covering degree of closures of Γx, $x \in F$, by the closures of Γ-orbits in E_F are at most 2. Note that F_1 is closed in F so that F_0 is open. The action of G_D naturally induces a non-trivial invariant vector field on F_0, on the other hand, such a vector field cannot exist on F_1 but an invariant line field is induced.

Assume now that Z has singularities for some $D \subset F$, then $P \neq \varnothing$. If $x \in F$ is not fixed by the $(G_D)_0$-action, then Z is non-singular at x by construction. Hence the singularities of Z are fixed by the $(G_D)_0$-action. If x is a fixed point, then $(G_D)_0 = (G_x)_0 \cong S^1$ and there is a closed orbit C of $2\,\mathrm{Re}\,Z$. If U is the connected component of $F \setminus C$ which contains x, then the G_x-action preserves U so that there are coordinates on U such that the $(G_x)_0$-action is given by $(t, z) \mapsto tz$, where x corresponds to $z = 0$. Noticing that the standard Hermitian metric on U is invariant under G_x, we identify $E_F|_U$ with the unit tangent bundle over U with respect to the standard Hermitian metric. Then, $E_F|_U$ is naturally identified with $S^1 \times U \subset G$, where G is considered as $S^1 \times \mathbb{C}$ by forgetting the group structure. We denote by φ_U this identification. The S^1-action obtained by lifting the $(G_x)_0$-action is given by the multiplication in G. Since the local holomorphic vector fields are unique up to multiplication of real numbers, we have the case 2b) on $F \setminus P$. Let x be a non-fixed point and choose a neighborhood V of x such that the local holomorphic vector field Z is given by $Z = \frac{\partial}{\partial z}$ and x corresponds to $z = 0$. By using the standard Hermitian metric on V, $E_F|_V$ can be identified with the unit tangent bundle of V and also with $S^1 \times V$ by assuming that E_F is trivial on V. Define

$\varphi_V : S^1 \times V \to G$ by $\varphi_V(t,z) = (te^{2\pi\sqrt{-1}\operatorname{Re} z}, e^{2\pi\sqrt{-1}z})$, then we may assume that φ_V is a diffeomorphism. Since $\varphi_V(t, z+\theta) = (te^{2\pi\sqrt{-1}(\operatorname{Re} z+\theta)}, e^{2\pi\sqrt{-1}(z+\theta)}) = (e^{2\pi\sqrt{-1}\theta}, 0)\cdot\varphi_V(t,z)$, the lifted local G_D-action on $E_F|_V$ is also given by the local action of $S^1 \subset G$. It is easy to see that each transition function of these trivializations is given by multiplication of an element of $S^1 \subset G$. Finally, the mapping from G to $\mathbb{C}$ defined by $(t,z) \mapsto t^{-1}z$ induces a mapping from $\widetilde{\Gamma_F}\backslash\widetilde{E_F}$ to $\mathbb{C}$. The imaginary parts of the local holomorphic vector fields generating the G_D-orbits induce the radial vector field $2\operatorname{Re} z\frac{\partial}{\partial z}$ on $\mathbb{C}$, where $0 \in P$. If $\widetilde{\Gamma_F}\backslash\widetilde{E_F} = \mathbb{C}P^1$, then P consists of at most two orbits, otherwise P consists of a single orbit. □

Theorem 4.20. *If F is a dense component, then one of the following holds:*

1) *The Γ-orbits in E_F are also dense and (Γ_F, E_F) is a Lie pseudogroup of dimension 3, namely, (Γ_F, E_F) is modeled on a 3-dimensional Lie group. F is contained in a recurrent GGS-Julia component.*
2) *(Γ_F, F) is a Lie pseudogroup of dimension 2. The closure of Γ-orbits in E_F are finite coverings of F and the covering degree is constant. If the covering is trivial, then F is a wandering GGS-Fatou component. If the covering is two-fold, then F is contained in an ergodic GGS-Julia component. Otherwise, F is contained in a recurrent GGS-Julia component.*

Proof. First assume that $\dim G_D = 3$, then the action of G_D on E_F is locally free because elements of G_D are determined by their 1-jets. Hence G_D is always connected and the germs of G_D at any points in F are isomorphic. If G_F is the simply connected Lie group locally isomorphic to G_D, then there are local submersions from E_F to G_F and (Γ_F, E_F) is a Lie pseudogroup modeled on G_F. Since the G_D-orbits are locally dense in E_F, there are no non-trivial invariant vector fields nor invariant line fields on F. Hence F is contained in a recurrent GGS-Julia component. Assume that $\dim G_D = 2$, then the G_D-orbits in E_F are transversal to the fibers and G_D-orbits in F are locally dense. It follows that for any $x \in F$, there is a neighborhood U of x such that if $g \in G_D$ satisfies $g(x) \in U$ then g is determined by $g(x)$. Consequently, G_D-action on F is locally free and the germ of G_D at any point $x \in F$ is always isomorphic. Hence there is a Lie group G_F such that (Γ_F, F) is a Lie pseudogroup modeled on G_F. The group $(G_x)_0\backslash G_x$ is also isomorphic for all x. Moreover, the Γ-action preserves the orientation of F so that any Γ-orbits in E_F is some k-fold covering to F. If $k = 1$, then it is clear that there is a non-trivial Γ-invariant vector field on F. If $k = 2$, then the normal directions to G_D-orbits in E_F projects down to a Γ-invariant line filed on F. Otherwise there are no non-trivial invariant vector fields nor invariant line fields. □

The following is now clear.

Theorem 4.21. *There is a Γ-invariant complete metric of class C^ω on each Fatou component. The metric can be constructed in the natural conformal class determined by the transversal holomorphic structure.*

The above results are expressed in terms of pseudogroups of isometries as follows. See [14] and [28] for definitions.

Corollary 4.22. *Let $\underline{\mathfrak{g}}$ be the sheaf of Lie algebras over F with stalk $\mathfrak{g}_x$ being the Lie algebra of G_D. The pseudogroup generated by Γ_F and G_D is the closure $(\overline{\Gamma_F}, F)$ of (Γ_F, F) and it is a Lie pseudogroup with Killing vector fields $\underline{\mathfrak{g}}$.*

The following is a direct consequence of Lemma 2.16.

Corollary 4.23. *If (Γ, T) is C^0-Hermitian, then (Γ, T) is C^ω-Hermitian.*

In the simplest case where $T = F(\Gamma)$, the Γ-orbits are described as follows. See also [21, Section 5].

Theorem 4.24. *Let (Γ, T) be a compactly generated pseudogroup. Assume that $\Gamma\backslash T$ is connected and $T = F(\Gamma)$, then (Γ, T) is C^ω-Hermitian. Let $E = E_T$ be the orthonormal frame bundle of T and let $\mathcal{F}_E$ be the foliation formed by orbits of Γ on E. Then, we have the following possibilities:*

1) *The leaves of $\mathcal{F}_E$ are dense. The whole T forms a single recurrent GGS-Julia component. In particular, all Γ-orbits on T are dense and there are neither invariant Beltrami coefficients nor non-trivial invariant continuous sections of TT.*
2) *The closures of the leaves of $\mathcal{F}_E$ form a real codimension-one foliation $\overline{\mathcal{F}_E}$ of E. All Γ-orbits on T are also dense. The leaves of $\overline{\mathcal{F}_E}$ are finite coverings to T of which the covering degree k is independent of the leaves. If $k = 1$, then the whole T is a single dense GGS-Fatou component. If $k = 2$, then the whole T is a single ergodic GGS-Julia component. Otherwise, T is a recurrent GGS-Julia component.*
3) 3a) *The closures of Γ-orbits form a real codimension-one regular foliation. T is the union of semi-wandering GGS-Fatou components and ergodic GGS-Julia components.*
   3b) *The closures of Γ-orbits form a singular foliation in the sense of Molino [21]. The number of singular orbits is at most two. The complement of the singular orbits is the union of semi-wandering*

GGS-Fatou components and ergodic GGS-Julia components, and the singular orbits form the recurrent GGS-Julia component.

4) *All Γ-orbits are discrete. The union of Γ-orbits without holonomy is dense and is a single wandering GGS-Fatou component. The complement is the union of recurrent Julia components. Moreover, there is a Γ-invariant meromorphic function on T.*

The union of ergodic GGS-Julia components is open in the GGS-Julia set.

Proof. The classification follows from Theorems 4.18, 4.19 and 4.20. The first three cases correspond the cases where $\dim G_D = 3$, $\dim G_D = 2$ or $\dim G_D = 1$, respectively. Assume that $\dim G_D = 0$. Since the Lebesgue measure of the GGS-Julia set should be zero, only recurrent components are possible. The claim on the meromorphic function is a part of the following theorem due to Brunella-Nicolau and Haefliger. □

Theorem 4.25 (Brunella-Nicolau [5], Haefliger [15])**.** *Let (Γ, T) be a compactly generated pseudogroup of holomorphic transformations of a one-dimensional complex manifold T such that $\Gamma\backslash T$ is connected. Then either there is a finite number of closed orbits, or all orbits are closed and there is a non-constant Γ-invariant meromorphic function on T.*

5. Properties of the Julia set and Conformal measures

Throughout this section, we assume that $J(\Gamma) \neq \varnothing$. An important consequence of the above theorem of Brunella-Nicolau and Haefliger is as follows.

Proposition 5.1. *$J(\Gamma)$ contains at most finite number of discrete Γ-orbits.*

Proof. If there are infinite number of discrete Γ-orbits, then all Γ-orbits are discrete and $J(\Gamma) = \varnothing$. □

Remark 5.2. The number of discrete Γ-orbits are essentially bounded by the dimension of a certain cohomological space [15].

The Julia set can be characterized as follows (see also Remark 5.9).

Theorem 5.3. *Let $z \in T'$, then $z \in J(\Gamma')$ if and only if there are a sequence $\{z_n\}$ in T' and $\gamma_n \in \Gamma'_{z_n}$ such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} |\gamma'_n|_{z_n} = +\infty$. Here the case where $z_n = z$ for all n is allowed.*

Proof. Let $z \in T'$ and assume that there are a neighborhood U of z in T' and a real number $M > 2$ with the property that $|\gamma'|_w \leq M$ if $\gamma \in \Gamma$ is obtained by extending the germ of an element of Γ'_u, where $u \in U$ and $w \in U \cap \operatorname{dom}\gamma$. We will show that $z \in F(\Gamma')$ by modifying Ghys' lemma in [10]. First, there is a finite set of generators $\{\gamma_1, \dots, \gamma_m\}$ of Γ' because Γ is compactly generated. Let $\Gamma'(k)$ be the subset of Γ' which consists of elements of Γ' which can be realized by composing at most k generators. Then the germ of any element of Γ' is the germ of an element of $\Gamma'(k)$ for some k. Let $\delta_0 > 0$ be such that the germ of any generator γ_i at a point $w \in T'$ is extended to $D_w(\delta_0)$ as an element of Γ, and set $V = D_z(\delta_0/M)$. We may assume that $V \subset U$ by shrinking V if necessary. If $\gamma \in \Gamma'_u$, where $u \in V$, then γ is actually the germ of an element of $\Gamma'(k)$ for some k. If $k = 1$, then γ can be defined on V as an element of Γ because $V \subset D_u(\delta_0)$. Moreover, $|\gamma'|_w \leq M$ if $w \in V$ because $V \subset U$. Hence $\gamma(V) \subset D_{\gamma(u)}(\delta_0)$. Assume that γ can be defined on V as an element of Γ if γ is the germ of an element of $\Gamma'(k)$, and let γ be the germ of an element of $\Gamma'(k+1)$. Then, we can decompose γ as $\gamma = \gamma_i \circ \zeta$, where $\zeta \in \Gamma'(k)$. By the assumption, ζ is defined on V as an element of Γ and $\zeta(V) \subset D_{\zeta(u)}(\delta_0)$ because $|\zeta'|_w \leq M$ if $w \in V$. Therefore γ is also defined on V as an element of Γ, namely, V is a Fatou neighborhood which contains z.

It follows that there are sequences $\{z_n\}, \{u_n\}$ in T' which converge to z and a sequence $\{\gamma_n\}$ such that $\gamma_n \in \Gamma'_{u_n}$ and $|\gamma'_n|_{z_n}$ tends to the infinity, where z_n belongs to the domain of γ_n as an element of Γ. By passing to a subsequence, we may assume that $\{\gamma_n(z_n)\}$ converges to $z_0 \in \overline{T'} \subset T$. Choose an element γ of Γ such that $\gamma(z_0) \in T'$, then the pair $(\{z_n\}, \{\gamma \circ \gamma_n\})$ makes a sense for large n and is a desired one.

On the contrary assume that $z \in F(\Gamma')$, then there is a Fatou neighborhood, say U, of z. If $\gamma \in \Gamma'_w$, $w \in U$, then $|\gamma'|_z$ is bounded because Γ_U is a normal family. □

Remark 5.4. One cannot tell in general if the limit point $\gamma(z_0)$ belongs to the Fatou set or not.

Some notions for Kleinian groups and the Julia sets of mapping iterations will be useful. We begin with an analogy of the limit sets for Kleinian groups.

Definition 5.5. Let $\Lambda_0(\Gamma)$ and $\Lambda(\Gamma)$ be as follows. First,

$$\Lambda_0(\Gamma) = \{z \in J(\Gamma) \,|\, \exists x \in F(\Gamma), \exists \{\gamma_n\} \subset \Gamma_x \text{ such that } \gamma_n x \to z\},$$

and let $\Lambda(\Gamma) = \overline{\Lambda_0(\Gamma)}$. We call $\Lambda(\Gamma)$ the *limit set* of Γ.

It is evident that $\Lambda_0(\Gamma)$ and $\Lambda(\Gamma)$ are Γ-invariant sets.

Remark 5.6. We do not know any example of (Γ, T) such that $\partial F(\Gamma) \neq \Lambda(\Gamma) \setminus \operatorname{Int} J(\Gamma)$, where $\operatorname{Int} J(\Gamma)$ denotes the interior of $J(\Gamma)$.

The limit set of Γ and the limit sets of Kleinian groups have a common property as follows.

Lemma 5.7. *Suppose that x_1, x_2 belong to the same Fatou component, then $\overline{\varGamma_{x_1}x_1} \cap \partial F(\varGamma) = \overline{\varGamma_{x_2}x_2} \cap \partial F(\varGamma)$.*

Proof. By Lemma 2.20, there is an open neighborhood V of x_1 such that $\overline{\varGamma_{x_1}x_1} \cap \partial F(\varGamma) = \overline{\varGamma_y y} \cap \partial F(\varGamma)$ if $y \in V$. The claim follows since x_1 and x_2 belong to the same Fatou component. □

The following definition can be found in the theory of complex dynamical systems (see [31]) and also in the theory of Kleinian groups (see [29]).

Definition 5.8. A point $z \in J(\varGamma')$ is called *conical* if there exist $\theta > 0$ and an infinite sequence $\{\gamma_n\} \subset \varGamma_z$, $n \geq 1$, such that $\gamma_n(z) \in T'$, γ_n^{-1} is defined on $D_{\gamma_n(z)}(\theta) \subset T$ and $\lim\limits_{n\to\infty} |\gamma_n'|_z = +\infty$. The union of conical points are denoted by $J_c(\varGamma')$. A conical point is called *uniformly conical* if one can find a sequence $\{\gamma_n\}$ such that

$$\varlimsup_{n\to\infty} \frac{\left|\gamma_{n+1}'\right|_z}{\left|\gamma_n'\right|_z} < +\infty.$$

The union of uniformly conical points are denoted by $J_{uc}(\varGamma')$. If $(\varGamma, T)$ is the holonomy pseudogroup of a foliation $\mathcal{F}$, then (uniformly) conical leaves are defined in an obvious way.

$J_c(\varGamma')$ and $J_{uc}(\varGamma')$ are $\varGamma'$-invariant but not necessarily closed in general. See Example 8.3.

Remark 5.9. The condition that z is conical implies that Theorem 5.3 holds in a strong form, namely, the sequence $\{z_n\}$ can be chosen so that $z_n = z$, and the elements γ_n fulfill an extra condition on their targets.

Existence of a conical point implies existence of hyperbolic fixed points.

Lemma 5.10. *If $x \in J_c(\varGamma')$, then there are a neighborhood D of x and a sequence $\{\gamma_n\}$ of elements of $\varGamma_x'$ with the following properties:*

1) *$(\gamma_n)^{\circ m}$ is defined on D for any positive integers n and m,*
2) *for each n, γ_n has a hyperbolic fixed point z_n in D and $(\gamma_n)^{\circ m}$ uniformly converges to the constant mapping z_n as m tends to the infinity,*
3) *$\{\gamma_n\}$ uniformly converges to the constant mapping x as n tends to the infinity.*

Moreover, there is a $\varGamma'$-orbit of a hyperbolic fixed point which converges to x. Here the constant sequence equal to x is allowed.

Proof. Let θ and $\{\gamma_n\}$ be as in Definition 5.8. Set $x_n = \gamma_n(x)$, then we may assume that x_n converges to $y \in \overline{T'}$. We may also assume that γ_n^{-1} is defined on $D_y(\theta/2)$ for any n and that $\{\gamma_n^{-1}\}$ uniformly converges to the constant mapping x on $D_y(\theta/2)$. Let D be a disc contained in $\gamma_1^{-1}(D_y(\theta/2)) \cap T'$ and set $\zeta_n = \gamma_n^{-1}\gamma_1$. Then ζ_n is defined on D, and $\overline{\zeta_n(D)} \subset D$ for large n because $\{\zeta_n\}$ uniformly converges to x. Each ζ_n has a fixed point, say z_n, on D. It is clear that $(\zeta_n)^{\circ m}$ can be defined on D for all m and that $\{(\zeta_n)^{\circ m}\}$ converges to z_n. Fix now a fixed point z_n, then $\{\zeta_m(z_n)\}$ converges to x because $\{\zeta_m\}$ converges to x. □

Let $\mathrm{Hyp}(\Gamma)$ be the union of hyperbolic fixed points.

Corollary 5.11. $\overline{\mathrm{Hyp}(\Gamma')} \supset J_c(\Gamma') \supset J_{uc}(\Gamma') \supset \mathrm{Hyp}(\Gamma')$. *Hence if $J_c(\Gamma')$ is dense in $J(\Gamma')$, then* $\mathrm{Hyp}(\Gamma')$ *is dense in $J(\Gamma')$. Moreover, if F is a Fatou component and if $J_c(\Gamma') \cap \partial F$ is dense in ∂F, then* $\mathrm{Hyp}(\Gamma') \cap \partial F$ *is dense in ∂F.*

Proof. The first claim follows from the fact that hyperbolic fixed points are uniformly conical. If F is a Fatou component and if $x \in J_c(\Gamma') \cap \partial F$, then there are a neighborhood D of x and elements $\{\gamma_n\}$ of Γ' as in Lemma 5.10. Recall that each γ_n has a hyperbolic fixed point z_n in D. We have $z_n \in \partial F \cap \mathrm{Hyp}(\Gamma')$ because $\lim_{m\to\infty} (\gamma_n)^{\circ m} x = z_n$. On the other hand, $\lim_{m\to\infty} \gamma_m(z_n) = x$ so that $\mathrm{Hyp}(\Gamma') \cap \partial F$ is dense in ∂F. □

Remark 5.12. Let $\mathcal{F}$ be a transversally holomorphic foliation of a closed manifold. A recent result of Deroin and Kleptsyn [7] shows that $\mathrm{Hyp}(\Gamma)$ is non-empty if $\mathcal{F}$ admits no holonomy invariant measures.

If $F_i \subset F(\Gamma')$ is a Fatou component, then we denote by Λ_{F_i} the limit points of Γ'-orbits in F_i, namely, we set

$$\Lambda_{F_i} = \left\{ x \in \partial F_i \,\middle|\, \exists z \in F_i,\ \exists \{\gamma_n\} \subset \Gamma'_z \text{ s.t. } x = \lim_{n\to\infty} \gamma_n(z) \right\} \subset \Lambda_0(\Gamma').$$

Note that the choice of z is irrelevant by Lemma 5.7 and $|\gamma_n'|_z \to 0$ by Lemma 2.20. Each Λ_{F_i} is closed and $\Lambda_0(\Gamma') = \bigcup_i \Lambda_{F_i}$ holds. Under these notations, we have the following

Corollary 5.13. $J_c(\Gamma') \cap \partial F_i \subset \Lambda_{F_i}$ *and* $J_c(\Gamma') \cap \partial F(\Gamma') = \bigcup J_c(\Gamma') \cap \partial F_i$. *Consequently,* $J_c(\Gamma') \cap \partial F(\Gamma') \subset \Lambda_0(\Gamma')$.

Proof. Let $x \in J_c(\Gamma') \cap \partial F_i$. If $\{\gamma_n\}$ and D are as in Lemma 5.10, then $F_i \cap D$ is non-empty and $\lim_{n\to\infty} \gamma_n(z) = x$ for any $z \in F_i \cap D$. Hence $x \in \Lambda_{F_i}$. In order to show the second claim, let $x \in J_c(\Gamma') \cap \partial F(\Gamma')$. Then z as above can be chosen in $F(\Gamma') \cap D$. The point z belongs to some F_k so that $x \in J_c(\Gamma') \cap \partial F_k$. This proves the second claim. □

The equality $J_c(\Gamma') \cap \partial F(\Gamma') = \Lambda_0(\Gamma')$ does not hold in general. For example, if $J(\Gamma')$ consists of a single parabolic fixed point which is not hyperbolic, then $J_c(\Gamma') \cap \partial F(\Gamma') = \varnothing$ but $\Lambda_0(\Gamma') = J(\Gamma')$.

A well-known fact for the Julia sets of mapping iterations holds in the following weak form. Note that $\Lambda_{F_i} \neq \varnothing$ if $J_c(\Gamma') \cap \partial F_i \neq \varnothing$ by Corollary 5.13.

Proposition 5.14. *Let F be a Fatou component and suppose that $\Lambda_F \neq \varnothing$. Then $F = \Gamma'(U \cap F)$ for any neighborhood U of any point of Λ_F. If $\Lambda_{F_i} \neq \varnothing$ for every Fatou component F_i of Γ', then $T' = \Gamma'(U)$ for any neighborhood U of $J(\Gamma')$.*

Proof. Let F be a Fatou component with $\Lambda_F \neq \varnothing$. Let $z \in \Lambda_F$ and let U be any neighborhood of z. If $x \in F$, then we can choose a sequence in $\Gamma'_x x$ which converges to z by Lemma 5.7. Hence $\gamma x \in U \cap F$ for some $\gamma \in \Gamma'_x$. The second claim follows from the first one. □

Conformal measures are one of the most important tools in the study of Kleinian groups and Julia sets for mapping iterations. There are some difficulties when considering a direct analogue, for example, it is clear that the Julia set in Example 3.11 admits an invariant measure. Indeed, any atomic measure supported on $\{0\} \cup \{\infty\}$ is invariant. However, the standard construction using the Poincaré series does not work. Indeed, $\sum_{\gamma \in \Gamma_x} |\gamma'|_x^s$ does not converge for any $x \in F(\Gamma)$ and $s \in \mathbb{R}$. In addition, the set $\{\gamma(x)\}_{\gamma \in \Gamma_x}$ is not discrete in $F(\Gamma)$. We would like to find a construction which is also valid in such a case.

We will introduce an additional notion.

Definition 5.15. Let $g = \{g_i^2 dz_i \otimes d\bar{z}_i\}$ be a Hermitian metric on $F(\Gamma')$ and let O be an open subset of $F(\Gamma')$. We say g diverges at ∂O (resp. converges to 0 at ∂O) if $\lim_{n\to\infty} g_i(x_n) = +\infty$ (resp. $\lim_{n\to\infty} g_i(x_n) = 0$) for any i and any sequence $x_n \in O \cap T'_i$ with $\lim_{n\to\infty} x_n \in \partial O$.

If g is complete, then g diverges at ∂F for each Fatou component F.

We assume the following in the rest of this section.

Assumption 5.16. 1) *$F(\Gamma)$ is non-empty, and*
2) *g is a continuous invariant Hermitian metric on $F(\Gamma)$ which diverges at $\partial F(\Gamma)$ in the sense of Definition 5.15.*

There exist metrics which satisfy the above assumption by Theorem 4.21. Let dm be the 2-dimensional volume induced by g. The restriction of dm to $F_i = T_i \cap F(\Gamma)$ is denoted by dm_i. Let g_i be the positive function on F_i such that $dm_i = g_i^2 |dz_i|^2$. We extend g_i to T_i by setting $g_i = +\infty$ on the Julia set. Note that the function $1/g_i$ is continuous and bounded on T_i.

Set $F_i' = F(\varGamma') \cap T_i' = F(\varGamma) \cap T_i'$.

Definition 5.17. Let $(\varGamma, T)$ and g_i be as above. Let $(\varGamma', T')$ be a reduction and set

$$S_g(s) = \sum_i \int_{F_i'} g_i^{-s+2} |dz_i|^2 = \sum_i \int_{F_i'} g_i^{-s} dm_i.$$

The number $\delta(\varGamma, g) = \inf \{s \in \mathbb{R} \mid S_g(s) < +\infty\}$ is called the critical exponent of $J(\varGamma)$ with respect to g. The number $\delta(\varGamma) = \inf_g \delta(\varGamma, g)$ is called the critical exponent of $J(\varGamma)$, where g runs through invariant metrics which satisfy Assumption 5.16. If $(\varGamma, T)$ is the holonomy pseudogroup of a foliation $\mathcal{F}$, then the critical exponents $\delta(\mathcal{F}, g)$ and $\delta(\mathcal{F})$ are defined in the natural way.

Note that the integral remains the same even if we replace F_i' with $F(\varGamma) \cap \overline{T_i'}$.

Lemma 5.18.

1) *The critical exponents are independent of the choice of reductions.*
2) *If $s > \delta(\varGamma, g)$, then $S_g(s) < +\infty$. Moreover, we may assume that*

$$\sum_i \int_{F_i} g_i^{-s+2} |dz_i|^2 < +\infty$$

for $s > \delta(\varGamma, g)$.

3) $\delta(\varGamma, g) \le 2$.
4) $\delta(\varGamma, g) \ge 0$ *if the area of $F(\varGamma')$ with respect to g is infinite in the sense that*

$$\sum_i \int_{F_i'} dm_i = +\infty.$$

5) *The critical exponent depends only on the equivalence class of g in the sense of Definition 3.6. (Note that equivalence class is considered on $F(\varGamma')$.)*
6) *The critical exponent is independent of the choice of invariant Hermitian metrics if $\varGamma'|_{F(\varGamma')} \backslash F(\varGamma')$ is compact.*

Proof. The first claim in 2) is a consequence of Assumption 5.16. The second holds by replacing the pair $((T, \varGamma), (T', \varGamma'))$ with $((T', \varGamma'), (T'', \varGamma''))$. 3) is evident from the fact that T' is relatively compact. 1, 4) and 5) are clear. 6) follows from 5). □

Remark 5.19. It is not obvious from the definition that $\delta(\varGamma, g) > -\infty$. We will show that $\delta(\varGamma, g) \ge 0$ under a condition on $\varGamma$ (Corollary 5.26).

Remark 5.20. Fix a point $x \in F_i$ and let $\gamma \in \varGamma_x'$. We denote by i_γ the index such that $\gamma(x) \in T_{i_\gamma}'$. Since $dm_i = g_i^2 |dz_i|^2$ is invariant under $\varGamma'$, we have

$|\gamma'|_x\, g_{i_\gamma}(\gamma(x)) = g_i(x)$. Hence, quite roughly speaking, the sum $\sum\limits_{\gamma \in \Gamma'_x} \frac{1}{g_{i_\gamma}(\gamma(x))^s}$ can be regarded as the Poincaré series of Γ'. The above integration is obtained by replacing the sum with the integration with respect to dm.

Definition 5.21. A Borel measure μ on $\overline{T'}$ (resp. T') is called a δ-conformal measure if $\mu(\gamma(A)) = \int_A |\gamma'|_x^\delta\, d\mu(x)$ holds for any Borel subset A of $\overline{T'}$ (resp. T') and any element $\gamma \in \Gamma$ (resp. Γ') defined on A. Let $\mathcal{M}_\delta(\overline{T'})$ and $\mathcal{M}_\delta(T')$ be set of δ-conformal Radon probability measures on $\overline{T'}$ and T', respectively. We equip $\mathcal{M}_\delta(\overline{T'})$ with the weak-$*$ topology.

Under our assumptions, a δ-conformal measure is in fact a Radon measure if it is Borel regular. We will consider only Radon measures in what follows.

Lemma 5.22. *There is a bijection between $\mathcal{M}_\delta(T')$ and $\mathcal{M}_\delta(\overline{T'})$.*

Proof. If $\mu \in \mathcal{M}_\delta(\overline{T'})$, then $\operatorname{supp}\mu$ cannot be contained in $\partial\overline{T'}$ because (Γ, T) is compactly generated. Indeed, if $x \in \operatorname{supp}\mu \cap \partial\overline{T'}$, then there are an element γ of Γ and an open set U of T such that γ is defined on U, $\mu(U) \neq 0$ and $\gamma(U) \subset T'$. If V is a neighborhood of x in $\overline{T'}$ such that $\overline{V} \subset U$, then V is measurable and $\mu(\gamma(V)) \geq C\, |\gamma'|_x^\delta\, \mu(V)$ for some $C > 0$ by the δ-conformality of μ. We may still assume that $\mu(V) > 0$ so that $\gamma x \in \operatorname{supp}\mu$. Hence we can define $r\colon \mathcal{M}_\delta(\overline{T'}) \to \mathcal{M}_\delta(T')$ by setting $r(\mu) = \frac{1}{\mu(T')}\mu|_{T'}$. Conversely, let $e\colon \mathcal{M}_\delta(T') \to \mathcal{M}_\delta(\overline{T'})$ be as follows. Let $\nu \in \mathcal{M}_\delta(T')$ and let $A \subset \overline{T'}$ be a Borel subset. If $A \subset T'$, then set $\widetilde{\mu}(A) = \nu(A)$. Otherwise, let $A = A_1 \cup \cdots \cup A_r$ be a decomposition of A into disjoint Borel subsets such that an element γ_i of Γ is defined on A_i and $\gamma_i(A_i) \subset T'$. Set then $\widetilde{\mu}(A) = \sum\limits_{i=1}^{r} \int_{\gamma_i(A_i)} |\zeta_i'|_y^\delta\, d\nu(y)$, where $\zeta_i = \gamma_i^{-1}$. It is easy to verify that $\widetilde{\mu}$ is well-defined. Let $e(\nu) = \mu$, where $\mu = \frac{1}{\widetilde{\mu}(\overline{T'})}\widetilde{\mu}$. By the construction, $r \circ e$ is the identity on $\mathcal{M}_\delta(T')$. If $\mu_1, \mu_2 \in \mathcal{M}_\delta(\overline{T'})$ and if $r(\mu_1) = r(\mu_2)$, then $\frac{1}{\mu_1(T')}\mu_1|_{T'} = \frac{1}{\mu_2(T')}\mu_2|_{T'}$ holds in $\mathcal{M}_\delta(T')$. Let A be a Borel subset of $\overline{T'}$ and let $A = A_1 \cup \cdots \cup A_r$ be a decomposition of A as above. By the δ-conformality of μ_1 and μ_2, we have

$$\mu_1(A) = \sum_{i=1}^{r} \int_{\gamma_i(A_i)} |\zeta_i'|_y^\delta\, d\mu_1(y) = \sum_{i=1}^{r} \int_{\gamma_i(A_i)} \frac{\mu_1(T')}{\mu_2(T')} |\zeta_i'|_y^\delta\, d\mu_2(y) = \frac{\mu_1(T')}{\mu_2(T')}\mu_2(A).$$

Letting $A = \overline{T'}$ we see that $\mu_1(T') = \mu_2(T')$ and therefore $\mu_1 = \mu_2$. □

We topologize $\mathcal{M}_\delta(T')$ via the above identification, then $\mathcal{M}_\delta(T')$ become compact.

Proposition 5.23. *Assume that $F(\Gamma)$ is non-empty and let $\delta = \delta(\Gamma, g)$ be the critical exponent of $J(\Gamma)$ with respect to an invariant metric g. Assume in* *addition that $\delta > -\infty$, then, there is a δ-conformal Radon measure supported on $\partial F(\Gamma) \subset J(\Gamma)$ under Assumption 5.16.*

The following proof is an adaptation of a proof of a corresponding result for the limit sets of Kleinian groups and the Julia sets of mapping iterations found respectively in [24] and [23]. We work on (Γ', T').

Proof. First assume that $\lim_{s \searrow \delta} S_g(s) = +\infty$. Let $C(\overline{T'})$ be the set of continuous functions on $\overline{T'}$. Consider the functional

$$\varphi_s(f) = \frac{\sum_{i \in I} \int_{F_i'} f(x) g_i(x)^{-s+2} \left|dz_i\right|^2}{S_g(s)}, \text{ where } f \in C(\overline{T'})$$

and let μ_s be the probability measure on $\overline{T'}$ obtained by the Riesz representation theorem. Let μ_δ be a weak limit of $\{\mu_s\}$ as s tends to δ from above.

Claim 1. μ_δ is supported on $\partial F(\Gamma) \cap \overline{T'}$.

Indeed, let $x \in F(\Gamma) \cap \overline{T'}$ and let U be a Fatou neighborhood of x in $F(\Gamma)$. Then, g_i is bounded from above on U so that $\lim_{s \searrow \delta} \mu_s(U') = 0$, where $U' = U \cap \overline{T'}$. Since $\varliminf_{s \searrow \delta} \mu_s(U') \geq \mu_\delta(U')$, we have $\mu_\delta(U') = 0$. One can show that $\operatorname{Int} J(\Gamma) \cap \operatorname{supp} \mu_\delta = \varnothing$ by a similar argument.

Claim 2. μ_δ is δ-conformal.

Let $x \in \overline{T_i'}$ and let $\epsilon > 0$. By the Koebe theorem, there is a neighborhood U of x in $F(\Gamma) \cap \overline{T'}$ such that if $\gamma \in \Gamma$ is defined on U, then $\left| \frac{|\gamma'|_y}{|\gamma'|_x} - 1 \right| < \epsilon$ holds for any $y \in U$. On the other hand, by the definition of μ_s, we have $|\gamma'|_x^s (1-\epsilon)\mu_s(U) \leq \mu_s(\gamma(U)) \leq |\gamma'|_x^s (1+\epsilon)\mu_s(U)$. First take the limit as $s \searrow \delta$, and then $\epsilon \to 0$, we see that μ_δ is a δ-conformal measure on $\overline{T'}$. Replacing μ_δ with $r(\mu_\delta)$, where r is defined in Lemma 5.22, we obtain a δ-conformal measure on T'.

If $S(s)$ converges as s tends to δ, then we will apply Patterson's construction as follows (cf. [24, p.47]). Let $\{\epsilon_n\}$ be a sequence of positive numbers decreasing to zero. We will define a sequence $\{X_n\}$, with $X_n \to \infty$, and an increasing function h on $[0, +\infty)$ inductively. Let $X_0 = 0$, $X_1 = 1$ and set $h(x) = 1$ on $[0, 1]$. If h is defined on $[0, X_n]$, then choose X_{n+1} so that

$$\frac{h(X_n)}{{X_n}^{\epsilon_n}} \sum_{i \in I} \int_{X_n < g_i \leq X_{n+1}} g_i^{-\delta+2+\epsilon_n} \left|dz_i\right|^2 \geq 1.$$

This is possible because $S_g(\delta - \epsilon_n) = +\infty$. Set now

$$h(x) = h(X_n) \left(\frac{x}{X_n} \right)^{\epsilon_n} \quad \text{for } x \in [X_n, X_{n+1}],$$

then h is increasing. Define $S^*_g(s)$ by

$$S^*_g(s) = \sum_{i\in I} \int_{F'_i} h(g_i) g_i^{-s+2} |dz_i|^2,$$

then $S^*_g(\delta)$ diverges because the inequality

$$\begin{aligned}\sum_{i\in I} \int_{F'_i} h(g_i) g_i^{-\delta+2} |dz_i|^2 &= \sum_{i\in I}\sum_{n=0}^{\infty} \int_{g_i\in(X_n,X_{n+1}]} h(X_n)\left(\frac{g_i}{X_n}\right)^{\epsilon_n} g_i^{-s+2} |dz_i|^2 \\ &\geq \sum_{i\in I}\sum_{n=0}^{\infty} 1\end{aligned}$$

holds. For any $\epsilon > 0$, there is a real number r_0 such that $h(rt) \leq t^\epsilon h(r)$ holds for $r > r_0$ and $t > 1$. Indeed, $\log h(x) = \epsilon_n(\log x - \log X_n) + \log h(X_n)$ so that if $\epsilon_n < \epsilon$ and $r > X_n$, then $\log h(rt) = \epsilon_n(\log t + \log r - \log X_n) + \log h(X_n) \leq \epsilon \log t + \log h(r)$ for $t > 1$. Finally we show that $S^*(s)$ converges if $s > \delta$. Choose $\epsilon > 0$ so that $\delta + \epsilon < s$ and fix an $r > 1$ such that $h(rt) \leq t^\epsilon h(r)$ holds for $t > 1$. Since h is increasing, $\frac{h(g_i)}{h(r)} \leq \frac{h(rg_i)}{h(r)} \leq g_i^\epsilon$ if $g_i > 1$. Setting $C = h(r)$, we have $h(g_i)g_i^{-s+2} \leq C g_i^{-\delta+2}$ for $g_i > 1$. Consequently, $S^*_g(s)$ converges if $s > \delta$. Repeating the construction after replacing $S_g(s)$ with $S^*_g(s)$, a δ-conformal measure can be also obtained in this case. □

The following fact is well-known.

Lemma 5.24. *Let μ_δ be a δ-conformal measure and let $\operatorname{supp}\mu_\delta$ be its support. Assume that $z \in J_c(\Gamma') \cap \operatorname{supp}\mu_\delta$, then there is a positive constant C which depends on θ and μ_δ, and a sequence $\{r_n\}$ of positive numbers which converges to zero such that*

$$C^{-1} \leq \frac{\mu_\delta(D_z(r_n))}{r_n^\delta} \leq C.$$

Proof. Let $\{\gamma_n\}$ be as in Definition 5.8. Let $z_n = \gamma_n(z)$, $D_n = D_{z_n}(\theta)$ and let $\rho_n = \dfrac{\theta}{4|\gamma'_n|_z}$. Then $\gamma_n^{-1}(D_n)$ contains $D_z(\rho_n)$ by the Koebe distortion theorem. On the other hand, again by the Koebe distortion theorem, there is a constant $C_1 > 0$ independent of mappings such that $\left|\dfrac{|\gamma'_n|_{z_n}}{|\gamma'_n|_x} - 1\right| < C_1$ if $x \in D_{z_n}(\theta/2)$. Hence we have

$$\mu_\delta(D_z(\rho_n/2)) \leq \mu_\delta(\gamma_n^{-1}(D_{z_n}(\theta/2))) \leq (1+C_1)|\gamma'_n|_z^{-\delta} \mu_\delta(D_{z_n}(\theta/2)) \leq (1+C_1)|\gamma'_n|_z^{-\delta}.$$

On the other hand, set $\sigma_n = \min\left\{\dfrac{\rho_n}{2(1+C_1)|\gamma'_n|_z}, \dfrac{\theta}{2}\right\}$, then $\gamma_n^{-1}(D_{z_n}(\sigma_n)) \subset D_z(\rho_n/2)$. Hence we have

$$\mu_\delta(D_z(\rho_n/2)) \geq \mu_\delta(\gamma_n^{-1}(D_{z_n}(\sigma_n))) \geq (1+C_1)^{-1}|\gamma'_n|_z^{-\delta} \mu_\delta(D_{z_n}(\sigma_n)).$$

The proof is completed if we show the following:

Claim. For any $r > 0$, there is an $m > 0$ such that $\mu_\delta(D_x(r)) > m$ for any $x \in \operatorname{supp} \mu_\delta$.

Indeed, if not, then there is a sequence $\{x_n\} \subset \operatorname{supp} \mu_\delta$ such that $\mu_\delta(D_{x_n}(r)) \leq \frac{1}{n}$. We may assume that x_n converges to a point $x \in \overline{T'}$. Then, $\mu_\delta(D_x(r')) = 0$ if $r' < r/2$. On the other hand, there is an element $\gamma \in \Gamma$ such that $\gamma(x) \in T'$ because (Γ, T) is compactly generated. We may assume that γ is defined on $D_x(r')$, and then $\mu_\delta(\gamma(D_x(r'))) = 0$. This is a contradiction.

This completes the proof of the claim and the lemma. □

Assume that $\mathcal{M}_\delta(\overline{T'})$ is non-empty and let $\mathcal{E}_\delta(\overline{T'})$ be the set of extremal elements of $\mathcal{M}_\delta(\overline{T'})$. It can be shown by modifying Proposition 4.1.6 of [32] that $\mu \in \mathcal{E}_\delta(\overline{T'})$ if and only if μ is ergodic, where an element $\mu \in \mathcal{M}_\delta(\overline{T'})$ is said to be *ergodic* if either $\mu(A) = 0$ or $\mu(A) = 1$ if A is a $\Gamma|_{\overline{T'}}$-invariant measurable set. Ergodic measures on T' are also defined by replacing $\overline{T'}$ with T'. By the Choquet representation theorem [25], given an element $\mu \in \mathcal{M}_\delta(\overline{T'})$, there is a unique Borel probability measure τ_μ on $\mathcal{E}_\delta(\overline{T'})$ such that $\mu = \int_{\mathcal{E}_\delta(\overline{T'})} m d\tau_\mu(m)$.

Lemma 5.25. *Ergodic measures in $\mathcal{M}_\delta(\overline{T'})$ correspond to ergodic measures in $\mathcal{M}_\delta(T')$ under the mappings r and e in Lemma 5.22.*

Proof. The claim for r is easy to verify. To show the converse, let $\nu \in \mathcal{M}_\delta(T')$ and suppose that there is a $\Gamma|_{\overline{T'}}$-invariant measurable subset A of $\overline{T'}$ such that $0 < e(\nu)(A) < 1$. Since $A \cap T'$ is Γ'-invariant and measurable, either $\nu(A \cap T') = 0$ or $\nu(A \cap T') = 1$. If $\nu(A \cap T') = 0$, then $e(\nu)(A \cap T') = 0$ and $e(\nu)(A \setminus T') > 0$. By the δ-conformality, there is a measurable subset A' of $A \setminus T'$ and an element γ of Γ defined on a neighborhood of A' such that $\gamma(A') \subset T'$ and that $e(\nu)(\gamma(A')) > 0$. This is a contradiction because A is $\Gamma|_{\overline{T'}}$-invariant so that $e(\nu)(A \cap T') \geq e(\nu)(\gamma(A'))$. Hence $\nu(A \cap T') = 1$. Set $B = \overline{T'} \setminus A$, then $0 < e(\nu)(B) < 1$ holds so that $\nu(B \cap T') = 1$ by the same reason. This is impossible and the proof is completed. □

After identifying $\mathcal{M}_\delta(T')$ with $\mathcal{M}_\delta(\overline{T'})$ and passing to the reduction, we denote $\mathcal{M}_\delta(T')$ by $\mathcal{M}_\delta(T)$.

There is the following analogue to the Julia sets of rational mappings. The proof is a modification of a standard argument [23], [31]. For a Borel subset A of T, the Hausdorff dimension of A is denoted by $\dim_H(A)$ and the δ-dimensional Hausdorff measure of A is denoted by $H_\delta(A)$.

Corollary 5.26. *Let μ_δ be a δ-conformal measure and suppose that $\mu_\delta(J_{uc}(\Gamma)) \neq 0$. Then $\delta = \dim_H(J_{uc}(\Gamma) \cap \operatorname{supp} \mu_\delta)$. More precisely, there is a $C > 0$ such that $C^{-1}\mu_\delta(A) \leq H_\delta(A) \leq C\mu_\delta(A)$ holds for any Borel subset A of $J_{uc}(\Gamma) \cap \operatorname{supp} \mu_\delta$*

with $\mu_\delta(A) > 0$. *In addition,* $\mu_\delta(A) = 0$ *if* $H_\delta(A) = 0$. *Finally if* $\mu_\delta \in \mathcal{E}_\delta(T)$, *then there is a* $c > 0$ *such that* $\mu_\delta = cH_\delta$.

Note that if $\delta = \delta(\Gamma, g)$ in the above corollary, where g is an invariant metric g satisfying Assumption 5.16, it follows that $\delta(\Gamma, g) \geq 0$.

Proof. Fix an invariant metric g and denote $\delta(\Gamma', g)$ by δ. Let $x \in J_{uc}(\Gamma) \cap \operatorname{supp} \mu_\delta \subset \partial F(\Gamma')$. Let $\{\gamma_n\}$ and θ be as in Definition 5.8. We may assume that $\{|\gamma_n'|_x\}$ is strictly increasing. On the other hand, since x is uniformly conical, there is a real number $\alpha > 1$ such that $\dfrac{\left|\gamma_{n+1}'\right|_x}{|\gamma_n'|_x} < \alpha$. We will show that there is a positive real number C such that

$$\forall r \in (0, 1],\ \exists n \text{ s.t. } -C < |\gamma_n'|_x + \log r - \log\theta < C.$$

Indeed, set $C = \max\{\log |\gamma_1|_x, \log\alpha\}$, then there is an integer n such that $\log|\gamma_n'|_x - \log|\gamma_1'|_x + \log r - \log\theta \leq 0 < \log\left|\gamma_{n+1}'\right|_x - \log|\gamma_1'|_x + \log r - \log\theta$. Since $\log\left|\gamma_{n+1}'\right|_x - \log|\gamma_n'|_x < \log\alpha$, the inequalities $\log|\gamma_n'|_x + \log r - \log\theta < \log|\gamma_1'|_x < C$ and $\log|\gamma_n'|_x + \log r - \log\theta > \log\left|\gamma_{n+1}'\right|_x - \log\alpha + \log r - \log\theta > \log|\gamma_1'|_x - \log\alpha > -C$ hold. Therefore, there is a $C_1 > 1$ such that for a given $r \in (0, 1]$, there is an n such that $C_1^{-1} < r\,|\gamma_n'|_x\,\theta^{-1} < C_1$. By repeating the same argument in the proof of Lemma 5.24, we have

$$C_2^{-1} \leq \frac{\mu_\delta(D_x(r))}{r^\delta} \leq C_2$$

for a suitable $C_2 > 1$ independent of x.

We will compare μ_δ with the Hausdorff measure by following [29] (see also [24, Theorems 4.4.2 and 4.6.3]). Let $A \subset J_{uc}(\Gamma)$ be a Borel subset and set $A' = A \cap \operatorname{supp}\mu_\delta$. Let $\{D_i\}$ be any cover of A' by open balls centered at points in A' with $r_i < 1$, where r_i denotes the radius of D_i. Then, $\mu_\delta(A) = \mu_\delta(A') \leq C_2 \sum r_i^\delta$ so that $\mu_\delta(A) \leq C_2 H_\delta(A') \leq C_2 H_\delta(A)$.

Next we assume that $A \subset J_{uc}(\Gamma) \cap \operatorname{supp}\mu_\delta$ and $\mu_\delta(A) > 0$. We will show that $H_\delta(A) \leq C\mu_\delta(A)$ for some C which is independent of A. First we show the claim when $A = J_{uc}(\Gamma) \cap \operatorname{supp}\mu_\delta$. Fix a positive real number ϵ less than 1 and let $\{D_1, D_2, \ldots\}$ be an at most countable family of open balls which covers A such that the center of D_i belongs to $A \setminus (D_1 \cup \cdots \cup D_{i-1})$ and that $r_i \geq r_{i+1}$ and $r_1 \leq \epsilon$, where r_i is the radius of D_i. Let D_i' be the open ball concentric with D_i and of radius $r_i/2$. Let $\Omega = \bigcup_i D_i'$, then Ω is a disjoint union so that

$$\sum_i r_i^\delta = 2^\delta \sum_i \left(\frac{r_i}{2}\right)^\delta \leq 2^\delta C_2 \sum_i \mu_\delta(D_i') = 2^\delta C_2 \mu_\delta(\Omega) \leq 2^\delta C_2 \mu_\delta(A).$$

By taking the limit with respect to ϵ, we obtain $H_\delta(A) \leq 2^\delta C_2 \mu_\delta(A)$ if $A = J_{uc}(\Gamma) \cap \operatorname{supp}\mu_\delta$. In particular H_δ is finite on $J_{uc}(\Gamma) \cap \operatorname{supp}\mu_\delta$. Let A be a

Borel subset of $J_{uc}(\Gamma) \cap \operatorname{supp} \mu_\delta$ with $\mu_\delta(A) > 0$. Then, μ-almost every point of A is a density point, namely,

$$\lim_{t \to 0} \frac{\mu_\delta(D_a(t) \cap A)}{\mu_\delta(D_a(t))} = 1$$

holds for μ_δ-a.e. a (recall that μ_δ is a Radon measure). For any $\alpha > 0$, there are a measurable subset A' of A with $\mu_\delta(A \setminus A') < \alpha$, $H_\delta(A \setminus A') < \alpha$ and a $t_0 > 0$ such that

$$\frac{\mu_\delta(D_a(t) \cap A)}{\mu_\delta(D_a(t))} \geq 1 - \alpha$$

for all $a \in A'$ and $t < t_0$. Let $0 < \epsilon < \min\{1, t_0\}$. By repeating the same argument as above replacing A with A', we obtain

$$\sum_i r_i^\delta \leq 2^\delta C_2 \sum_i \mu_\delta(D_i') \leq \frac{2^\delta C_2}{1-\alpha} \sum_i \mu_\delta(D_i' \cap A) \leq \frac{2^\delta C_2}{1-\alpha} \mu_\delta(A).$$

Hence by taking the limit with respect to ϵ, we have $H_\delta(A') \leq \frac{2^\delta C_2}{1-\alpha} \mu_\delta(A)$. Therefore, we have $H_\delta(A) \leq 2^\delta C_2 \mu_\delta(A)$ by taking the limit with respect to α.

Finally assume that $\mu_\delta \in \mathcal{E}_\delta(T)$, then μ_δ is ergodic. Set $M = \mu_\delta + H_\delta$, then μ_δ is absolutely continuous with respect to M. Let $f = \frac{d\mu_\delta}{dM}$ be the Radon-Nikodym derivative. Then it is easy to see that f is μ_δ-measurable and invariant under Γ. By the ergodicity, f is constant which is neither 0 nor 1 by the inequality just established. This completes the proof. □

6. Characteristic classes

The arguments in [3] depend only on the fact that foliations restricted to the Fatou sets are transversally Hermitian. Hence they are also valid for the decomposition in the present paper, and the Godbillon-Vey class and the Bott class can be localized to the Julia set. The proof is completely the same as in [3] so that we will give only a sketch.

Theorem 6.1. *Let $(M, \mathcal{F})$ be a transversally holomorphic foliation of complex codimension one, of a closed manifold.*

1) *The Godbillon measure in the sense of Heitsch-Hurder* [16] *is supported on the Julia set.*
2) *The residue of the imaginary part of the Bott class* [3] *at the Julia set is well-defined.*

Sketch of the proof. Fix an invariant Hermitian metric g on $Q(\mathcal{F})|_{F(\mathcal{F})}$, where $Q(\mathcal{F})$ denotes the complex normal bundle of $\mathcal{F}$. If U is a neighborhood of $J(\mathcal{F})$ (which is not necessarily saturated), then there is a Hermitian metric h on $Q(\mathcal{F})$ which coincides with g on a neighborhood, say V, of $F(\mathcal{F}) \setminus U$.

We can find a Bott connection ∇^b which is a unitary connection for h on $M \setminus V'$, where V' is an open set slightly smaller than V. If we denote by ∇^u a unitary connection for h, then, representatives of Godbillon-Vey class and the imaginary part of the Bott class obtained by using ∇^b and ∇^u vanish on V'. □

We have the following weak version of Duminy's theorem [9] (see also [16]).

Corollary 6.2. *Let $(M, \mathcal{F})$ be a transversally holomorphic foliation of complex codimension one, of a closed manifold.*

1) *The Godbillon-Vey class vanishes if the Julia set is empty.*
2) *The imaginary part of the Bott class vanishes if the Julia set is empty.*

Remark 6.3. The first claim follows also from the second claim, because the Godbillon-Vey class is equal to the product of the imaginary part of the Bott class and the first Chern class of complex normal bundle [2].

Remark 6.4. $J(\mathcal{F}) \neq \varnothing$ implies that there is either a leaf with a hyperbolic holonomy or a leaf to which a series of expanding local holonomy converges by Theorem 5.3. If one happens to know that $J_c(\mathcal{F}) \neq \varnothing$, then there is really a hyperbolic holonomy by Lemma 5.10. On the other hand, it is known that the support of the Godbillon measure contains leaves of exponential growth if it is non-empty [18]. Theorem 6.1 implies that $J(\mathcal{F})$ contains leaves of exponential growth if the Godbillon-Vey class of $\mathcal{F}$ is non-trivial.

The real part of the Bott class can be non-trivial even if the Julia set is empty.

Example 6.5. Let (z_0, z_1) be the standard coordinates of $\mathbb{C}^2$ and let $X = z_0\frac{\partial}{\partial z_0} + \lambda z_1\frac{\partial}{\partial z_1}$, where $\lambda \in \mathbb{C} \setminus \{t \in \mathbb{R} \,|\, t \leq 0\}$. It is known that the integral curves of X induces a transversally holomorphic foliation $\mathcal{F}$ of $S^3 \subset \mathbb{C}^2$ and the Bott class $B(\mathcal{F})$ of $\mathcal{F}$ is given by $B(\mathcal{F}) = \lambda + \frac{1}{\lambda} \in H^3(S^3; \mathbb{C}/\mathbb{Z})$. If $\lambda \in \mathbb{R}$, then $\mathcal{F}$ is transversally Hermitian but the real part of $B(\mathcal{F})$ is non-zero.

There is another kind of such examples which is essentially due to Bott and Heitsch [4].

Example 6.6. Let k be an integer greater than 2 and realize $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ as $\{t \in \mathbb{C} \,|\, t^m = 1\}$. Define a $\mathbb{Z}_m$-action on $S^{2k-1} \times \mathbb{C}P^1$ by $t(x, [z_0 : z_1]) = (tx, [t^{-1}z_0 : z_1])$. Let $M = (S^{2k-1} \times \mathbb{C}P^1)/\mathbb{Z}_m$, then M fibers over the Lens space $L(m; 1) = S^{2k-1}/\mathbb{Z}_m$ with projection p. (M, p) is a foliated fiber bundle in the sense that M is equipped with a foliation $\mathcal{F}$ with leaves $(S^{2k-1} \times \{[z_0 : z_1]\})/\mathbb{Z}_m$. If we set $U = (S^{2k-1} \times \mathbb{C})/\mathbb{Z}_m$, where $\mathbb{C} = \{[z : 1]\} \subset \mathbb{C}P^1$, then

$U \subset M$ and U is a line bundle over $L(m;1)$. Let $\mathcal{F}_U$ be the restriction of $\mathcal{F}$ to U, then the line bundle is isomorphic to the complex normal bundle of $\mathcal{F}_U$ pulled back by the inclusion of $L(m;1) \times \{[0:1]\}$ into M. On the other hand, since $p^* \colon H^*(U;\mathbb{Z}) \cong H^*(L(m;1);\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is injective, it suffices to see that $c_1(U)^2$ has a torsion part. The mapping $[x,z] \to ([x], zx)$ is an embedding of U to $L(m;1) \times \mathbb{C}^k$, where the bracket means the equivalence class. It follows that U is the pull-back of the tautological bundle over $\mathbb{C}P^{k-1}$ by the natural projection, which we denote by π. As π^* is the projection from $\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ in degree 4, $c_1(U)^2$ is its generator. On the other hand, the foliation is clearly transversally Hermitian and therefore the Julia set is empty.

7. THE TRANSVERSAL KOBAYASHI METRIC

The invariant metric constructed in Section 3 is not canonical although the Fatou-Julia decomposition has naturality (Lemma 2.12). A canonical (pseudo-)metric can be constructed by modifying the construction of the Kobayashi metric. By integrating the Kobayashi metric, the transversal Kobayashi distance is obtained. The transversal Kobayashi distance was studied by Duchamp and Kalka [8]. Here we discuss some properties of the transversal Kobayashi metric.

Let (Γ, T) be a (not necessarily compactly generated) pseudogroup of local biholomorphic diffeomorphisms of $\mathbb{C}^q$ and we denote by TT the holomorphic tangent bundle of T.

Definition 7.1 (cf. [14])**.** Let X be a 1-dimensional complex manifold. A *holomorphic* 1*-cocycle valued in* Γ defined on X is a triplet $(\{\varphi_i\}, \{U_i\}, \{\gamma_{ji}\})$ as follows:

1) $\{U_i\}$ is an open covering of X,
2) each φ_i is a holomorphic map from U_i to a component of T,
3) if $U_i \cap U_j \neq \varnothing$, then there is an element γ_{ji} of Γ such that $\varphi_j = \gamma_{ji} \circ \varphi_i$ on $U_i \cap U_j$, moreover, $\gamma_{ii} = \mathrm{id}$, and
4) $\gamma_{ik}\gamma_{kj}\gamma_{ji} = \mathrm{id}$ if $U_i \cap U_j \cap U_k \neq \varnothing$.

Holomorphic 1-cocycles valued in Γ defined on X correspond to transversally holomorphic mappings from X to a foliated manifold.

Definition 7.2. For $(x,v) \in TT$, denote by $\Omega(x,v)_R$ the set of holomorphic 1-cocycles valued in Γ defined on $D_0(R)$ such that $\varphi(0) = x$ and $\varphi_{*0}e_0 = v$, where e_0 is a unit vector at the origin with respect to the standard Hermitian metric on $\mathbb{C}$. It is clear that $\Omega(x,v)_R$ is non-empty if R is small enough. Set then

$$K_T(x,v) = \inf_{\Omega(x,v)_R \neq \varnothing} \frac{1}{R}.$$

It is immediate that $K_T(x,0)=0$.

Actually $D_0(R)$ is considered as the Poincaré disc of radius R and centered at the origin, equipped with the metric $\frac{R^2dz^2}{(R^2-|z|^2)^2}$. The same function can be obtained even if 1-cocycles such that $\varphi(p)=x$ for some $p \in D_0(R)$ and $\varphi_{*p}(e_p)=v$ are considered in the definition if $\frac{1}{R}$ is replaced with $\frac{R}{R^2-|p|^2}$.

We recall some fundamental properties [26].

Lemma 7.3 ([26])**.** *$K_T(x,\alpha v) = |\alpha|\, K_T(x,v)$ for any $(x,v) \in TT$ and $\alpha \in \mathbb{C}$.*

Proof. Let $(\{\varphi_i\},\{U_i\},\{\gamma_{ji}\}) \in \Omega(x,v)_R$. Then the cocycle $(\{\psi_i\},\{V_i\},\{\gamma_{ji}\})$, where $\psi_i(z)=\varphi_i(\alpha z)$ and $V_i = \frac{1}{|\alpha|}U_i$, belongs to $\Omega(x,\alpha v)_{|\alpha|^{-1}R}$. □

Lemma 7.4. *The function K_T is Γ-invariant in the sense that $K_T(\gamma x, \gamma_{*x}v) = K_T(x,v)$ for any $\gamma \in \Gamma_x$.*

Proof. Let $(\{\varphi_i\},\{U_i\},\{\gamma_{ji}\}) \in \Omega(x,v)_R$. Assume that $\varphi_0(0)=x$ and $(\varphi_0)_{*0}e_0 = v$. Let W be an open neighborhood of x of which the closure is contained in $\operatorname{dom}\gamma$, and let $V_\infty = \varphi_0^{-1}(\operatorname{dom}\gamma)$ and $V' = \varphi^{-1}(W)$. We define a 1-cocycle ψ as follows. If we set $V_i = U_i \setminus \overline{V'}$, then $\{V_i\} \cup \{V_\infty\}$ is an open covering of $D_0(R)$. Let ψ_i be the restriction of φ_i and let $\psi_\infty = \gamma \circ \varphi_0$ on V_∞. Noticing that $V_i \cap V_\infty \subset U_i \cap U_0$, set $\gamma_{\infty i} = \gamma \circ \gamma_{0i}$ and $\gamma_{i\infty} = \gamma_{i0} \circ \gamma^{-1}$ if $V_i \cap V_\infty \neq \varnothing$. It is easy to see that $(\{\psi_i\}\cup\{\psi_\infty\},\{V_i\}\cup\{V_\infty\},\{\gamma_{ji}\}\cup\{\gamma_{ab}\})$, where $a=\infty$ or $b=\infty$, is a holomorphic 1-cocycle which belongs to $\Omega(\gamma x, \gamma_{*x}v)_R$. □

There is a following property as usual.

Proposition 7.5. *The function K_T is upper semicontinuous.*

Proof. We need the Royden lemma [26], [27] if the dimension of T is greater than one. Here we give an elementary proof in one-dimensional case. We may assume that $T \subset \mathbb{C}$ and denote φ_* by φ'. Given a positive real number $\epsilon > 0$, choose $\delta > 0$ so that $\frac{1}{R(1-\delta)} < 1/R + \epsilon$ holds. If $(x,v) \in TT$, then there is a 1-cocycle in $\Omega(x,v)_R$ such that $\varphi_0(0)=x$, $\varphi_0'(0)=v$ and $F_T(x,v)+\epsilon > 1/R$. If (y,w) is close enough to (x,v), then $y \in U_0$ so that $y=\varphi_0(p)$ for some $p \in U_0$. Moreover, $\lambda\varphi_0'(p)=w$ holds for some $\lambda \in \mathbb{C}$ close enough to 1. By composing with a Möbius transformation of $D_0(R)$, we can find a holomorphic 1-cocycle ψ defined on $D_0(R)$ such that $\psi(0)=y$ and $\lambda\psi'(0)=w$, where $||\lambda|-1| < \delta$. It follows that $K_T(y,w) \leq \frac{1}{R(1-\delta)} < 1/R + \epsilon < K_T(x,v) + 2\epsilon$. □

By integrating K_T, a locally defined (pseudo-) distance function d_T on T can be obtained. It is easy to see that d_T is continuous.

Remark 7.6. The locally defined distance d_T is distinct from the Kobayashi distance in general. In order to obtain the Kobayashi distance, we need the infimum of the length of Γ-paths with respect to K_T.

Definition 7.7. (Γ, T) is said to be *Kobayashi hyperbolic* if d_T is locally a distance.

The Kobayashi hyperbolicity is invariant under equivalence of pseudogroups. If (Γ, T) is Kobayashi hyperbolic, then d_T induces a metric on each component of T_i. Moreover, d_T induces the same topology on T as an open subset of $\mathbb{C}$.

Remark 7.8. It is not difficult to see that the set $\{x \in T \mid K_T(x, v) = 0 \text{ for any } v \in T_xT\}$ is open. If it is also closed and $K_T(x, v) \neq 0$ for some (x, v), then (Γ, T) is Kobayashi hyperbolic.

Theorem 7.9. *Let (Γ, T) be a compactly generated pseudogroup of local biholomorphic diffeomorphisms of $\mathbb{C}$. If (Γ, T) is Kobayashi hyperbolic, then (Γ, T) is C^ω-Hermitian and the conclusion of Theorem 4.24 holds.*

Proof. We proceed as in the proof of Lemma 2.16 and retain the notation. First we show that for $\forall \epsilon > 0$, $\exists \delta > 0$ such that $D'_y(\delta) \subset D_y(\epsilon)$ for any $y \in \overline{T'}$, where $D'_y(\delta)$ denotes the open δ-ball centered at y with respect to d_T. If not, there is an $\epsilon > 0$ and a sequence $\{y_n\}$ such that $D'_{y_n}(1/n) \not\subset D_{y_n}(\epsilon)$. We may assume that $\{y_n\}$ converges to a point y in $\overline{T'}$. Note that $d_T(y, y_n)$ converges to 0. If $\epsilon_1 > 0$, then $D'_y(\epsilon_1) \supset D'_{y_n}(1/n)$ provided $d_T(y, y_n) + 1/n < \epsilon_1$. Hence $D'_y(\epsilon_1) \not\subset D_y(\epsilon/2)$ for any $\epsilon_1 > 0$. This is a contradiction.

Let $x \in T'$ and let δ' be such that $D'_y(\delta') \subset D_y(\delta/2)$ for any $y \in \overline{T'}$, where δ is chosen as in the proof of Lemma 2.16. Let δ'' be such that $D_x(\delta'') \subset D'_x(\delta') \cap T'$. Assume that the germ at $z \in D_x(\delta'')$ of any element of $\Gamma'(k)$ is defined on $D_x(\delta'')$ as an element of Γ, then $d_T(\gamma x, \gamma y) < \delta'$. It follows that $\gamma y \in D'_{\gamma x}(\delta') \subset D_{\gamma x}(\delta/2)$. Therefore $\gamma D_x(\delta'') \subset D_{\gamma x}(\delta/2) \subset D_{\gamma z}(\delta)$ and $\gamma_i \gamma$ is defined on $D_x(\delta'')$ as an element of Γ. Therefore, $D_x(\delta'')$ is a Fatou neighborhood and consequently $T = F(\Gamma)$. □

Remark 7.10. The proof of Theorem 7.9 requires only that the d_T induces the same topology as the original one on T, not that it is induced by K_T.

The above corollary can be regarded as a variant of the following result of Duchamp-Kalka [8, Lemma 3.6 and Theorem 3.7].

Theorem 7.11 ([8])**.** *Let $(M, \mathcal{F})$ be a transversally holomorphic foliation of complex codimension q. If the transversal Kobayashi distance distinguishes* *distinct leaves, then the the leaf space $M/\mathcal{F}$ is Hausdorff. If moreover M is closed, then $\mathcal{F}$ is a (generalized) Seifert fibration.*

8. Examples

Example 8.1. Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates of $\mathbb{C}P^2$ and let $U_i = \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid z_i \neq 0\}$, and let (u_1, u_2), (v_1, v_2) and (w_1, w_2) be the inhomogeneous coordinates on U_0, U_1 and U_2, respectively. Let X_i be a vector field on respective U_i given by the formula

$$\begin{aligned} X_0 &= \lambda_1 u_1 \frac{\partial}{\partial u_1} + \lambda_2 u_2 \frac{\partial}{\partial u_2}, \\ X_1 &= -\lambda_1 v_1 \frac{\partial}{\partial v_1} + (-\lambda_1 + \lambda_2) v_2 \frac{\partial}{\partial v_2}, \\ X_2 &= -\lambda_2 w_1 \frac{\partial}{\partial w_1} + (\lambda_1 - \lambda_2) w_2 \frac{\partial}{\partial w_2}, \end{aligned}$$

We assume that $\lambda_1\lambda_2 \neq 0$, $\lambda_1 \neq \lambda_2$ and $\lambda_1/\lambda_2 \notin \mathbb{R}$, then the (singular) foliation $\mathcal{F}$ of $\mathbb{C}P^2$ induced from these vector fields has three singularities $p_1 = [0 : 0 : 1]$, $p_2 = [0 : 1 : 0]$ and $p_3 = [1 : 0 : 0]$. If we set $L_i = \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 \mid z_i = 0\}$, then $\mathcal{F}$ is Hermitian when restricted to $\mathbb{C}P^2 \setminus L$, where $L = L_0 \cup L_1 \cup L_2$. Indeed, choose $\mu_1, \mu_2 \in \mathbb{C}$ such that $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$, $\mu_1 \neq \mu_2$ and let $Y = \mu_1 u_1 \frac{\partial}{\partial u_1} + \mu_2 u_2 \frac{\partial}{\partial u_2}$. Then Y induces a foliated section of $Q(\mathcal{F})$ on $\mathbb{C}P^2 \setminus L$. Hence by requiring the length of Y to be 1, a transverse invariant Hermitian metric, say h, is obtained. Since Y and X_i are linearly dependent on L, the metric h diverges at L in the sense of Definition 5.15.

Let D_i be a small round ball centered at p_i and let $S_i \approx S^3$ be its boundary. The condition $\lambda_1/\lambda_2 \notin \mathbb{R}$ implies that $\mathcal{F}$ is transversal to S_i. Let $M = \mathbb{C}P^2 \setminus (D_1 \cup D_2 \cup D_3)$ and let M_3 be its double. Then M_3 naturally inherits a transversally holomorphic foliation $\mathcal{F}_3$ induced from $\mathcal{F}$. The foliation $\mathcal{F}_3$ has three compact leaves $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$, namely, the leaves induced from L_0, L_1 and L_2. The above description shows that $F(\mathcal{F}_3) = M_3 \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2)$. The residue of the Bott class at $J(\mathcal{F}_3)$ is calculated in [3].

The number of the Julia components can be arbitrarily large. Let M' be a copy of M and let $\partial M' = S_1' \cup S_2' \cup S_3'$. Let M_1 be the manifold with boundary obtained by gluing M with M' along S_1 and S_1', and S_2 and S_2'. Then $\partial M_1 = S_3 \cup S_3'$. If we denote by $\mathcal{F}_4$ the natural foliation of the double M_4 of M_1, then $J(\mathcal{F}_4)$ consists of 4 connected components. In general, let $N_1, \dots, N_{r-2}$ be copies of M_1 and let M_r be the manifold obtained by gluing them. Let $\mathcal{F}_r$ be the naturally induced foliation of M_r. Then $J(\mathcal{F}_r)$ consists of

r connected components. The Julia sets of foliations in this example consist of conical leaves and the critical exponents are equal to zero.

There is another description of the above example.

Example 8.2. Let $\{U_0, U_1, U_2\}$ be as in the previous example. We blow up $\mathbb{C}P^2$ at the origin of U_0, namely, let $\widetilde{\mathbb{C}^2} = \{((u_1, u_2), [t_1 : t_2]) \mid t_1u_2 - t_2u_1 = 0\}$ be $\mathbb{C}^2$ blown up at the origin and replace U_0 by $\widetilde{\mathbb{C}^2}$. Denote by $\widetilde{\mathbb{C}P^2}$ the resulting manifold.

Consider again the vector field X_0 on $\mathbb{C}^2$, then X_0 can be lifted to $\widetilde{\mathbb{C}P^2}$ as follows. Let $V_i = \left\{((u_1, u_2), [t_1 : t_2]) \in \widetilde{\mathbb{C}^2} \,\middle|\, t_i \neq 0\right\}$ $(i = 1, 2)$ and let $\varphi_i \colon V_i \to \mathbb{C}^2$ be as follows, namely, define φ_1 by $\varphi_1(((u_1, u_2), [t_1 : t_2])) = (u_1, t_2/t_1)$ and φ_2 by $\varphi_2(((u_1, u_2), [t_1 : t_2])) = (u_2, t_1/t_2)$, respectively. Let $(Z_1, Z_2) = \varphi_1(((u_1, u_2), [t_1 : t_2]))$ and $(W_1, W_2) = \varphi_2(((u_1, u_2), [t_1 : t_2]))$. Define vector fields $\widetilde{X}_0$ on $\widetilde{\mathbb{C}}^2$ by the property

$$\begin{aligned}\varphi_{1*}\widetilde{X}_0 &= \lambda_1 Z_1 \frac{\partial}{\partial Z_1} + (\lambda_2 - \lambda_1) Z_2 \frac{\partial}{\partial Z_2},\\ \varphi_{2*}\widetilde{X}_0 &= \lambda_2 W_1 \frac{\partial}{\partial W_1} + (\lambda_1 - \lambda_2) W_2 \frac{\partial}{\partial W_2}.\end{aligned}$$

It is easy to see that $\widetilde{X}_0$ is well-defined and it coincides with X_0 on $\mathbb{C}^2 \setminus \{0\} = \mathbb{C}^2 \setminus E$, where E denotes the exceptional fiber. Thus obtained foliation of $\widetilde{\mathbb{C}P^2}$ has 4 singularities. The leaves induced from L_1, L_2, L_3 and the exceptional fiber E are separatrices. By imitating the previous construction, one can obtain a (non-singular) foliation of which the Julia set consists of 4 components. Then by continuing cut and paste procedures or taking blow-ups, foliations with arbitrary number (greater than 3) of Julia components can be obtained.

We will examine some examples in [11].

Example 8.3 ([11, Example 8.4])**.** Let Γ be a Kleinian group and let $\mathbb{C}P^1 = \Omega(\Gamma) \sqcup \Lambda(\Gamma)$ be the decomposition into the domain of discontinuity and the limit set. Let $\mathcal{F}$ be a suspension of this action. Then $F(\mathcal{F})$ corresponds to $\Omega(\Gamma)$ and the $J(\mathcal{F})$ corresponds to $\Lambda(\Gamma)$. Indeed, one can repeat the same argument as in the proof of Proposition 4.2 after introducing the Poincaré metric on each component of Ω. Note that if we begin with a Kleinian group Γ such that its conical limit sets $\Lambda_c(\Gamma)$ is not the same as the limit set $\Lambda(\Gamma)$, then we can obtain a foliation such that $J_c(\mathit{\Gamma})$ is not closed. Note also that if Γ is not torsion-free, then we have $F(\mathcal{F}) \supsetneq F_{\mathrm{GGS}}(\mathcal{F})$. On the other hand, if Γ is geometrically finite, then the conformal measure constructed in Section 5 coincides with the Patterson-Sullivan measure by the uniqueness [29]. Moreover, the critical exponent of $\mathcal{F}$ is equal to the critical exponent of the Poincaré series of Γ. The case where $\Gamma \subset \mathrm{Aff}(\mathbb{R})$ is non-discrete and non-abelian is important. In this case, $J(\mathcal{F}) = J_{\mathrm{GGS}}(\mathcal{F})$ and they correspond to $\mathbb{R} \cup \{\infty\}$. The Julia set consists of conical points, namely, we have $J_c(\mathit{\Gamma}) = J(\mathit{\Gamma})$. The critical exponent of $\mathcal{F}$ is equal to 1.

The same construction by suspension is also possible if Γ is non-discrete but finitely generated. If $\overline{\Gamma} = \mathrm{PSL}(2; \mathbb{C})$, where the closure is taken with respect to the Hausdorff topology, then $J(\mathcal{F})$ is the whole manifold.

Example 8.4 ([11, Example 8.10])**.** Example 8.3 can be modified using ramified covers. We adopt the notation in [11]. Let $h\colon \pi_1(B) \to \mathrm{Aff}(\mathbb{R}) \subset \mathrm{PSL}(2; \mathbb{C})$ be a homomorphism and form the suspension. Assume that the image is non-discrete and non-abelian and that the ambient manifold M is diffeomorphic to $B \times \mathbb{C}P^1$. If we denote by L the leaf which corresponds to $\infty \in \mathbb{C}P^1$, then the holonomy group of L consists of germs of mappings of the form $z \mapsto az/(1 + bz)$ with $a > 0$ and $b \in \mathbb{R}$, where ∞ is considered as the origin. Set $M' = B \times S^3$ and consider the mapping $M' \to M$ induced by the Hopf fibration $S^3 \to \mathbb{C}P^1$. By pulling-back, M' is equipped with a foliation with a compact leaf L which is equal to B times the fiber of the Hopf fibration. By construction, there is a non-trivial homomorphism from $\pi_1(M' \setminus L)$ onto $\mathbb{Z}$. Hence there is an n-fold covering M'_n of M' ramified along L for any $n > 0$. Let $\mathcal{F}_n$ be the foliation of M'_n by pull-back. Then $\mathcal{F}_n$ is naturally transversally holomorphic and has a compact leaf, say L_n, with holonomy group which consists of the germs of the mappings of the form $z \mapsto (az^n/(1 + bz^n))^{1/n}$. The Fatou-Julia decompositions also coincide in this case: $J(\mathcal{F}_n)$ is the pull-back of real line of $\mathbb{C}P^1$, which is locally the union of codimension-one submanifolds with singular locus L_n, while $F(\mathcal{F}_n)$ consists of two components which are pull-back of the upper and lower half spaces. The critical exponent of $\mathcal{F}_n$ is equal to 1.

Example 8.5 ([11, Example 8.6])**.** There is a foliation which is transversally Hermitian but of which the GGS-Julia set is the whole manifold. On the other hand, the Julia set in our sense is empty by Lemma 2.16. In particular, $F(\mathcal{F}) \supsetneq F_{\mathrm{GGS}}(\mathcal{F})$.

Example 8.6 ([11, Example 8.9])**.** There is a foliation of a connected manifold of which the GGS-Julia set has non-empty interior without being the whole manifold. It is constructed by inserting a certain foliation ([11, Example 8.7]) into Example 8.3 which has two GGS-Fatou components corresponding to the upper and the lower half spaces. Then, one of the GGS-Fatou components is changed into a GGS-Julia component so that this GGS-Julia component has non-empty interior without being the whole manifold. The Fatou-Julia decomposition of the original foliation is the same as ours. On the other hand, the modified foliation is still transversally Hermitian on the modified part. It is easy to see that the new GGS-Julia component is still a Fatou component in our sense so that the interior of the Julia set is empty.

In fact, the author does not know if there is an example of a compactly generated pseudogroup (Γ, T) such that $\Gamma \backslash T$ is connected and that the Julia set $J(\Gamma)$ has non-empty interior without being equal to T.

We will present some other examples.

Example 8.7. Let Γ be a lattice in $\mathrm{SL}(2;\mathbb{C})$ such that $M = \Gamma\backslash\mathrm{SL}(2;\mathbb{C})/\mathrm{U}(1)$ is a closed manifold, where $\mathrm{U}(1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, |a| = 1 \right\}$. Let $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C}^*,\ b \in \mathbb{C} \right\}$, and let $\widetilde{\mathcal{F}}$ be the foliation of $\mathrm{SL}(2;\mathbb{C})/\mathrm{U}(1)$ with leaves $gH/\mathrm{U}(1)$, $g \in \mathrm{SL}(2;\mathbb{C})$. There is a foliation $\mathcal{F}$ of M naturally induced from $\widetilde{\mathcal{F}}$. It is easy to see that $J(\mathcal{F}) = M$, on the other hand, it is known that the Godbillon-Vey class of $\mathcal{F}$ is non-trivial [2].

There are foliations of which the Julia set is the whole manifold as in Examples 8.3 and 8.7. There is another kind of such examples.

Example 8.8. Let $T = (\mathbb{C} \setminus \{0\})/\langle\gamma\rangle$, where $\langle\gamma\rangle$ denotes the group generated by the mapping $\gamma(z) = 2z$. We denote again by z the point in T represented by z by abuse of notation. Let $\xi\colon T \to T$ be $\xi(z) = z^2$. The mapping ξ is not a diffeomorphism but there is an open covering $\{O_i\}$ of T such that the each restriction ξ_i of ξ to O_i is a diffeomorphism onto its image. It is easy to see that the pseudogroup Γ generated by ξ_i's acting on T is compactly generated. It is also easy to see that $J(\Gamma) = T$. Γ can be realized as the holonomy pseudogroup of a transversally holomorphic foliation by modifying Hirsch'es construction [17]. The following construction is due to S. Matsumoto [20]. Let $T' = \mathbb{C}/\mathbb{Z}^2$ and let φ be the automorphism of T' given by $\varphi(z) = 2z$. Then (T', φ) is holomorphically conjugate to (T, ξ). Let D^3 be the closed unit ball in $\mathbb{R}^4$ and let $f\colon T' \to D^3$ be a smooth embedding into the interior of D^3. Define $g\colon T' \to T' \times D^3$ by $g(z) = (\varphi(z), f(z))$, then g is also an embedding. Let N be a closed tubular neighborhood of $g(T')$. Then ∂N is homeomorphic to $T' \times S^2$. Let $\mathcal{F}_1$ be the foliation of $T' \times D^3$ with leaves $\{\{z\} \times D^3\}$, where $z \in T'$. Then the leaves of restriction of $\mathcal{F}_1$ to ∂N are $\{\{z\} \times S^2\}$, $z \in T'$. By gluing ∂N and $\partial(T' \times D^3)$, we obtain a foliated manifold M equipped with a transversally holomorphic foliation $\mathcal{F}$. The holonomy pseudogroup of $\mathcal{F}$ is equivalent to Γ.

We do not know if there is a reasonable extension of the Fatou-Julia decomposition to not necessarily compactly generated pseudogroups. Indeed, it is easy to obtain non-compactly generated pseudogroups such that they are equivalent but the Julia sets do not correspond under the equivalence. In terms of foliations, this implies that the Fatou-Julia decomposition of a foliation of a non-compact manifold depends on the choice of the realization of the holonomy pseudogroup.

If (Γ, T) is not compactly generated, we tentatively say that $U \subset T$ is a Fatou neighborhood if any germ $\gamma_u \in \Gamma_u$, $u \in U$, extends to an element of Γ defined on U, and let $\widetilde{F}(\Gamma)$ be the union of Fatou neighborhood. The Julia set in this sense can have non-empty interior without being the whole space.

Example 8.9. Let (Γ, T) be as in Example 8.8 and let $S = \{z \in \mathbb{C} \mid |z| < 1 + \epsilon\}$, where ϵ is a small positive real number. Let $O' = \{z \in \mathbb{C} \mid 1 < |z| < 1 + \epsilon\}$ and let $\eta \colon O' \to T$ be the mapping naturally induced by the inclusion of O' into $\mathbb{C}$. If we denote by Γ_1 the pseudogroup generated by Γ and η, and set $T_1 = T \sqcup S$, then $\widetilde{J}(\Gamma_1) = T_1$. The pseudogroup Γ_1 is however not compactly generated.

Example 8.10. Let $D_{5+\epsilon}(0)$ be a disc of radius $5 + \epsilon$ centered at 0 and let $T = T_1 \sqcup T_2$, where $T_1 = T_2 = D_{5+\epsilon}(0)$. We denote the natural coordinates on T_1 and T_2 by z and w, respectively. Let Γ be the pseudogroup generated by γ_0, γ_1 and γ_2 defined as follows. First set

$$S_i = \{z \in T_i \mid 25/(5 + \epsilon) < |z| < 5 + \epsilon\}, \ i = 1, 2,$$

and define $\gamma_0 \colon S_1 \to S_2$ by $\gamma_0(z) = 25/z$. Second, let

$$O_1 = \left\{ re^{\sqrt{-1}t} \in T_1 \,\middle|\, 1 < r < 2, \ |t| < \delta \right\},$$

where δ is chosen so small that $\gamma_1 \colon O_1 \to T_1$ defined by $\gamma_1(z) = z^2$ is a diffeomorphism onto its image. Finally set

$$O_2 = \left\{ re^{\sqrt{-1}t} \in T_1 \,\middle|\, 2 < r < 4, \ |t| < \delta \right\},$$

and define $\gamma_2 \colon O_1 \to O_2$ by $\gamma_2(z) = 2z$.

It is easy to see that the pseudogroup Γ is not compactly generated, and $\widetilde{J}(\Gamma) = [1, 4] \cup \bigcup I_k \cup \bigcup A_l$, where $I_k = \left\{ e^{2^{k-1}\sqrt{-1}\delta} t \,\middle|\, 1 \leq t \leq 4 \right\}$, $k = 0, 1, \ldots$, and $A_l = \left\{ 2^{i/l} e^{\sqrt{-1}t} \,\middle|\, i = 0, \ldots, 2l, 0 \leq t \leq 2\delta \right\}$. Adding an irrational rotation to Γ as a generator, one can obtain a pseudogroup Γ_1 such that $\widetilde{J}(\Gamma_1) = \{z \in T_1 \mid 1 \leq |z| \leq 4\}$. The pseudogroup Γ_1 is not compactly generated, either.

Finally we will mention semigroups. If f is a rational mapping from $\mathbb{C}P^1 \to \mathbb{C}P^1$, then it is well-known that the Julia set $J(f)$ of f is defined. It can be considered as the Julia set of the semigroup generated by f. It is natural to ask if there is a suitable notion which unifies such a kind of semigroups and compactly generated pseudogroups, and if it is possible to introduce the notion of Julia sets in a compatible way. We think that the answer is positive, and will discuss this problem in a forthcoming paper.

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