A feasible central limit theory for realised volatility under leverage

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Abstract

In this note we show that the feasible central limit theory for realised volatility and realised covariation recently developed by Barndorff-Nielsen and Shephard applies under arbitrary diffusion based leverage effects. Results from a simulation experiment suggest that the feasible version of the limit theory performs well in practice.

Keywords: Euler approximation; Functional central limit theory; Quadratic variation; Realised volatility; Stochastic volatility.

1 Introduction

The econometric literature on time-varying volatility in financial markets has recently been revolutionalised by the harnessing of high frequency data. This has produced an order of magnitude improvement in the predictive power of time series volatility models, as demonstrated in careful empirical work carried out by Andersen, Bollerslev, Diebold, and Labys (2001), Andersen, Bollerslev, Diebold, and Ebens (2001) and Andersen, Bollerslev, Diebold, and Labys (2003). The theory behind these forecasting methods is the empirical estimation of the quadratic variation process, based on, for example, 5 or 10 minute returns. This estimator is called the realised quadratic variation process, while its daily increments are often called realised variances or volatilities in the econometric literature.

In some recent papers Barndorff-Nielsen and Shephard (2002) and Barndorff-Nielsen and Shephard (2004a) have developed a self-contained feasible central limit theory for the realised quadratic variation process, so providing an insight into the accuracy of forecasted volatility in practice. By feasible we mean that the asymptotic standard error can be estimated from the data directly and so this limit theory can be used to, for example, construct confidence intervals. A key assumption used in these proofs has been to rule out the leverage effect, which has limited the rigorous application of these techniques to exchange rate data, where the no leverage assumption is empirically reasonable. Barndorff-Nielsen and Shephard (2004b) discuss simulations which suggest that the central limit theories should not be effected by dropping this assumption while Meddahi (2002) showed that the effect of leverage on the unconditional mean square error of the realised variance is asymptotically negligible in a tractable class of diffusion based volatility models. Andersen, Bollerslev, and Meddahi (2002) extend the results of Meddahi (2002) to more general volatility process. Corradi and Distaso (2003) and Andreou and Ghysels (2002) provide additional insights into this issue. Bandi and Russell (2003) discuss a functional central limit theory for the realised quadratic variation process, but their result cannot be used to construct confidence intervals for realised volatility as they do not show independence of the integrated quarticity process and the corresponding Brownian motion (these terms will be defined in a moment). However, taken together these results all point to the more general result of the feasible asymptotics working in the leverage case.

In this paper we will provide a rigorous derivation of a feasible central limit theory for the realised quadratic variation process in the presence of leverage. Our Proof relies on a very elegant recent result by Jacod and Protter (1998, Theorem 5.5) who derived an infeasible limit theory for Euler approximations to a class of continuous semimartingales. An infeasible asymptotic result on the realised quadratic variation process was briefly mentioned in some unpublished work by Jacod (1994), who obtained it using more involved methods. Having established the mathematical results, we then report simulations to show how well our feasible theory works in practice. We will see leverage effects have basically no impact on the finite sample performance of the feasible central limit theory developed by Barndorff-Nielsen and Shephard (2002) and Barndorff-Nielsen and Shephard (2004a).

The structure of the paper is as follows. In Section 2 we recall various definitions of semimartingales and stochastic volatility. In Section 3 we give feasible limit theorems for the realised quadratic variation process and its multivariate generalisation. In Section 4 we report results from various simulation experiments we have conducted to assess the finite sample behaviour of this limit theory. Section 5 concludes.

2 Background

2.1 Definitions

In this Section we briefly review an econometric strategy for measuring and predicting the variability of, and codependence between, asset prices. This is based on high frequency returns which are transformed into statistics called realised covariation. These statistics are intimately related to quadratic variation.

We write the q-dimensional vector of log-prices of assets as Y. We will often estimate objects over a fixed interval of time which, for sake of concreteness, we usually think of as representing a day. We use $\hbar$ to denote this time interval.

Substantial gains can be obtained by working with the set of high frequency vector returns recorded over periods of length δ, where $\delta \ll \hbar$. We might think of δ as five minutes, for concreteness. On the i-th day, these high frequency returns will be written as

$$Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}, \quad j = 1, ..., n, \tag{1}$$

which occur between time $(i-1)\hbar$ and $i\hbar$. Here $n = \lfloor \hbar/\delta \rfloor$ indexes the sample size within each day, recalling that $\lfloor x \rfloor$ is the integer part of x.

The realised covariation matrix is then defined as

$$[Y_\delta]_i = \sum_{j=1}^{n} \left(Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}\right) \left(Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}\right)'. \tag{2}$$

We will write the k-th element of Y as $Y_{(k)}$.

The notation $[Y_\delta]_i$ is designed to reflect its connection to quadratic variation (which will become clearer in a moment), that this matrix is based on the Y process with returns measured over periods of length δ and computed on the i-th day. The realised covariation matrix has the k-th *realised variance* or realised quadratic variation

$$\left[Y_{(k)\delta}\right]_i = \sum_{j=1}^{n} \left(Y_{(k)(i-1)\hbar+\delta j} - Y_{(k)(i-1)\hbar+\delta(j-1)}\right)^2, \tag{3}$$

as its k,k-th element and the *realised covariance*

$$\left[Y_{(k)\delta}, Y_{(l)\delta}\right]_i = \sum_{j=1}^{n} \left(Y_{(k)(i-1)\hbar+\delta j} - Y_{(k)(i-1)\hbar+\delta(j-1)}\right) \left(Y_{(l)(i-1)\hbar+\delta j} - Y_{(l)(i-1)\hbar+\delta(j-1)}\right), \tag{4}$$

as its k,l-th and l,k-th elements. The corresponding

$$\sqrt{\sum_{j=1}^{n} \left(Y_{(k)(i-1)\hbar+\delta j} - Y_{(k)(i-1)\hbar+\delta(j-1)}\right)^2}, \tag{5}$$

is called the k-th *realised volatility*. The realised covariance matrix can be transformed into realised regressions and realised correlations between the Y variables. These were emphasised in the work of Andersen, Bollerslev, Diebold, and Labys (2003) and Barndorff-Nielsen and Shephard (2004a).

2.2 Quadratic variation and semimartingales

The probability limit of $[Y_\delta]_i$ is well known when we assume Y is a semimartingale ($\mathcal{SM}$) by using the theory of quadratic variation. Here we briefly review this textbook material before we go beyond it to develop the corresponding asymptotic distribution theory.

Recall if $Y \in \mathcal{SM}$ then it can be decomposed as

$$Y = A + M, \tag{6}$$

where A is a process with *finite variation* ($\mathcal{FV}$) paths and M is a *local martingale* ($\mathcal{M}_{loc}$). For all $Y \in \mathcal{SM}$ the *quadratic variation* (QV) or *covariation* process can be defined as

$$[Y]_t = \text{p}-\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)\left(Y_{t_{j+1}} - Y_{t_j}\right)', \tag{7}$$

for any sequence of partitions $t_0 = 0 < t_1 < ... < t_n = t$ with $\sup_j\{t_{j+1} - t_j\} \to 0$ for $n \to \infty$.

These general considerations from stochastic analysis have, in particular, the following implications for the theory of realised covariation. For all $Y \in \mathcal{SM}$ as $\delta \downarrow 0$

$$\begin{aligned} [Y_\delta]_i &\xrightarrow{p} [Y]_{\hbar i} - [Y]_{\hbar(i-1)} \\ &= [Y]_i, \end{aligned}$$

meaning realised covariation, $[Y_\delta]_i$, consistently estimates increments of QV, $[Y]_i$.

2.3 $\mathcal{SVSM}^c$ processes

We are unable to calculate a central limit theory for $[Y_\delta]_i$ in the general class of semimartingales. Instead we specialise. We will model M as a stochastic volatility process (e.g. Ghysels, Harvey, and Renault (1996))

$$M_t = \int_0^t \Theta_u \mathrm{d}W_u,$$

where the elements of Θ are assumed to be càdlàg and W is a vector Brownian motion. We need to additionally assume that (for all $t < \infty$) $\int_0^t \Sigma_{(kk)u}\mathrm{d}u < \infty$ for each k where $\Sigma_{(kl)}$ is the notation for the (k,l)-th element of the $\Sigma = \Theta\Theta'$ process. Arbitrage theory then implies that A must be absolutely continuous and so can be written as

$$A_t = \int_0^t a_u \mathrm{d}u,$$

where again the elements of a are assumed to be càdlàg. This means that Y must be continuous and we call this model a stochastic volatility semimartingale ($\mathcal{SVSM}^c$), noting that sometimes it is called a Brownian semimartingale in the probability literature.

M is said to have no leverage (e.g. Black (1976) and Nelson (1991)) if $\Theta \perp\!\!\!\perp W$ (i.e. Θ is stochastically independent from W). This condition is often appropriate for exchange rate data, but is inappropriate for the analysis of equity data. Barndorff-Nielsen and Shephard (2004a) use it to derive their limit theory. However, Monte Carlo results reported in Barndorff-Nielsen and Shephard (2004b) suggested the theory may well hold when the no leverage assumption is dropped. The next Section confirms this conjecture, while the Monte Carlo results will be presented in order to assess the finite sample performance of the theory.

3 Central limit for $\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right)$

3.1 Univariate case

Throughout our theoretical analysis we will derive the functional central limit theory for

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(Y_{j\delta} - Y_{(j-1)\delta}\right)\left(Y_{j\delta} - Y_{(j-1)\delta}\right)'.$$

We can then discretise the result to produce the desired result for the i-th realised covariation $[Y_\delta]_i$. We will present the univariate result first, since this has less notational clutter as we may drop the coordinate indices. The general case is treated in the next subsection.

Theorem 1 *Suppose that $Y \in \mathcal{SVSM}^c$ is one-dimensional and that (for all $t < \infty$) $\int_0^t a_u^2 \mathrm{d}u < \infty$, then as $\delta \downarrow 0$ so*

$$\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right) \to \sqrt{2}\int_0^t \Theta_u^2 \mathrm{d}B_u,$$

where B is a Brownian motion which is independent from Y and the convergence is in law stable as a process.

Proof. By Ito's lemma for continuous semimartingales

$$Y_t^2 = [Y]_t + 2\int_0^t Y_u \mathrm{d}Y_u,$$

then

$$\left(Y_{j\delta} - Y_{(j-1)\delta}\right)^2 = [Y]_{\delta j} - [Y]_{\delta(j-1)} + 2\int_{\delta(j-1)}^{\delta j}(Y_u - Y_{(j-1)\delta})\mathrm{d}Y_u.$$

This implies that

$$\begin{aligned}\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right) &= \delta^{-1/2}\sum_{j=1}^{\lfloor t/\delta \rfloor}\int_{\delta(j-1)}^{\delta j}(Y_u - Y_{(j-1)\delta})\mathrm{d}Y_u \\ &= 2\delta^{-1/2}\int_0^{\delta\lfloor t/\delta \rfloor}(Y_u - Y_{\delta\lfloor u/\delta \rfloor})\mathrm{d}Y_u.\end{aligned}$$

Jacod and Protter (1998, Theorem 5.5) show that for Y satisfying the conditions in Theorem 1 then

$$\delta^{-1/2}\int_0^t (Y_u - Y_{\delta\lfloor s/\delta \rfloor})\mathrm{d}Y_u \to \frac{1}{\sqrt{2}}\int_0^t \Theta_u^2 \mathrm{d}B_u,$$

where $B \perp\!\!\!\perp Y$ and the convergence is in law stable as a process. This implies

$$\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right) \to \sqrt{2}\int_0^t \Theta_u^2 \mathrm{d}B_u,$$

which is the result given in the Theorem.

□

The most important point of this Theorem is that $B \perp\!\!\!\perp Y$. The appearance of the additional Brownian motion B is striking. This means that Theorem 1 implies, for a single t, $\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right)$ has a mixed Gaussian limit with a random variance of $2\int_0^t \Theta_u^4 \mathrm{d}u$ and where the Gaussian variable is independent from Y_δ. This is precisely the result that Barndorff-Nielsen and Shephard (2004a) established under the no leverage assumption.

Theorem 1 is infeasible as we do not know a priori the value of $\int_0^t \Theta_u^4 \mathrm{d}u$. Barndorff-Nielsen and Shephard (2004a) showed that the process $2\int_0^t \Theta_u^4 \mathrm{d}u$ can be consistently estimated using a variety of statistics like

$$\frac{2}{3}\delta^{-1}\sum_{j=1}^{\lfloor t/\delta \rfloor}\left(Y_{j\delta} - Y_{(j-1)\delta}\right)^4.$$

This means that confidence intervals can be established for the increment to the QV process even under leverage. In particular

$$\frac{[Y_\delta]_t - [Y]_t}{\sqrt{\frac{2}{3}\sum_{j=1}^{\lfloor t/\delta \rfloor}\left(Y_{j\delta} - Y_{(j-1)\delta}\right)^4}} \xrightarrow{d} N(0,1),$$

while the delta method can be used to imply that

$$\frac{\log[Y_\delta]_t - \log[Y]_t}{\sqrt{\dfrac{\frac{2}{3}\sum_{j=1}^{\lfloor t/\delta \rfloor}\left(Y_{j\delta} - Y_{(j-1)\delta}\right)^4}{\left([Y_\delta]_t\right)^2}}} \xrightarrow{d} N(0,1).$$

Barndorff-Nielsen and Shephard (2004b) noted that

$$\frac{\int_0^t \Theta_u^4 \mathrm{d}u}{\left(\int_0^t \Theta_u^2 \mathrm{d}u\right)^2} \geq \frac{1}{t},$$

which means it make sense to use the following asymptotic distribution in practice

$$\frac{\log\left([Y_\delta]_t\right) - \log\left([Y]_t\right)}{\sqrt{\min\left(\dfrac{\frac{2}{3}\sum_{j=1}^{\lfloor t/\delta \rfloor}\left(Y_{j\delta} - Y_{(j-1)\delta}\right)^4}{\left([Y_\delta]_t\right)^2}, \frac{2}{t}\delta\right)}} \xrightarrow{d} N(0,1).$$

3.2 Multivariate case

Theorem 2 *Suppose that* $Y \in \mathcal{SVSM}^c$ *and that (for all* $t < \infty$*)* $\int_0^t a^2_{(k)u}\mathrm{d}u < \infty$ *for each* k*, then as* $\delta \downarrow 0$ *so*

$$\delta^{-1/2}\left([Y_\delta]_{(kl)t} - [Y]_{(kl)t}\right) \to \frac{1}{\sqrt{2}}\left(\sum_{b=1}^{q}\sum_{c=1}^{q}\int_0^t \left(\Theta_{(kb)u}\Theta_{(cl)u} + \Theta_{(lb)u}\Theta_{(ck)u}\right)\mathrm{d}B_{(bc)u}\right),$$

where B *is a* $q \times q$ *matrix of independent Brownian motions, independent of* Y *and the convergence is in law stable as a process.*

Proof. By Ito's lemma

$$Y_t Y_t' = [Y]_t + \int_0^t Y_u' \mathrm{d}Y_u + \int_0^t Y_u \mathrm{d}Y_u'$$

then

$$\begin{aligned} & \left(Y_{(k)j\delta} - Y_{(k)(j-1)\delta}\right)\left(Y_{(l)j\delta} - Y_{(l)(j-1)\delta}\right) \\ = \; & [Y]_{(kl)\delta j} - [Y]_{(kl)\delta(j-1)} + \int_{\delta(j-1)}^{\delta j}(Y_{(k)u} - Y_{(k)(j-1)\delta})\mathrm{d}Y_{(l)u} + \int_{\delta(j-1)}^{\delta j}(Y_{(l)u} - Y_{(l)(j-1)\delta})\mathrm{d}Y_{(k)u}, \end{aligned}$$

while

$$\delta^{-1/2}\sum_{j=1}^{\lfloor t/\delta \rfloor}\int_{\delta(j-1)}^{\delta j}(Y_{(k)u} - Y_{(k)(j-1)\delta})\mathrm{d}Y_{(l)u} = \delta^{-1/2}\int_0^{\lfloor t/\delta \rfloor}(Y_{(k)u} - Y_{(k)\delta\lfloor s/\delta \rfloor})\mathrm{d}Y_{(l)u}.$$

But Jacod and Protter (1998, Theorem 5.5) showed that

$$\delta^{-1/2}\int_0^{\lfloor t/\delta \rfloor}(Y_{(k)u} - Y_{(k)\delta\lfloor s/\delta \rfloor})\mathrm{d}Y_{(l)u} \to \frac{1}{\sqrt{2}}\sum_{b=1}^{q}\sum_{c=1}^{q}\int_0^t \Theta_{(kb)u}\Theta_{(cl)u}\mathrm{d}B_{(bc)u},$$

where the convergence is in law stable as a process. This implies that

$$\delta^{-1/2}\left([Y_\delta]_{(kl)t} - [Y]_{(kl)t}\right) \to \frac{1}{\sqrt{2}}\sum_{b=1}^{q}\sum_{c=1}^{q}\int_0^t \left(\Theta_{(kb)u}\Theta_{(cl)u} + \Theta_{(lb)u}\Theta_{(ck)u}\right)\mathrm{d}B_{(bc)u}.$$

□

Again the absolutely central feature of this result is that $B \perp\!\!\!\perp Y$, which means that $\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right)$ has a mixed Gaussian distribution, with a zero mean and asymptotic covariance Ω_t, which is a $q^2 \times q^2$ array with elements

$$\Omega_t = \left\{\int_0^t \{\Sigma_{(kk')u}\Sigma_{(ll')u} + \Sigma_{(kl')u}\Sigma_{(lk')u} + \Sigma_{(kl)u}\Sigma_{(k'l')u}\}\mathrm{d}u\right\}_{k,k',l,l'=1,\ldots,q}, \qquad (8)$$

where we recall that $\Sigma = \Theta\Theta'$. Barndorff-Nielsen and Shephard (2004a) showed how to use high frequency data to estimate the Ω_t process. We refer the reader to that paper for details.

4 How accurate is the feasible asymptotic distribution?

So far the asymptotic theory has been discussed in terms of the realised quadratic variation process. Before we start studying the finite sample properties of this theory, we will briefly spell out the form of the feasible asymptotic theory of realised variance, restricting attention to the one-dimensional case. The extension to the multivariate case is discussed in Barndorff-Nielsen and Shephard (2004a) (under the no leverage assumption). Recall that the realised variance on the i-th day is

$$[Y_\delta]_i = \sum_{j=1}^{n} \left(Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}\right)^2,$$

so we are interested in the daily raw t-statistics of the form

$$\frac{[Y_\delta]_i - [Y]_i}{\sqrt{\frac{2}{3}\sum_{j=1}^{n}\left(Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}\right)^4}} \xrightarrow{d} N(0,1), \tag{9}$$

and the corresponding log version

$$\frac{\log [Y_\delta]_i - \log [Y]_i}{\sqrt{\min\left(\frac{\frac{2}{3}\sum_{j=1}^{n}\left(Y_{(i-1)\hbar+\delta j} - Y_{(i-1)\hbar+\delta(j-1)}\right)^4}{\left([Y_\delta]_i\right)^2}, \frac{2}{\hbar}\delta\right)}} \xrightarrow{d} N(0,1), \tag{10}$$

where $n = \lfloor \hbar/\delta \rfloor$. In this Section we will take $\hbar = 1$ and see how well the normal distribution approximates the actual behaviour of the t-statistic for a variety of values of n.

To perform the simulation we will use the stochastic volatility model $Y_t = \int_0^t \sigma_u \mathrm{d}W_u$ and take the variance process σ^2 to be based on the sum of K independent CIR processes

$$\sigma_t^2 = \sum_{k=1}^{K} \sigma_t^{2(k)},$$

where $\sigma^{2(k)}$ follows the solution to the stochastic differential equation

$$\mathrm{d}\sigma_t^{2(k)} = -\lambda_k \left(\sigma_t^2 - \frac{w_k \nu}{\alpha}\right)\mathrm{d}t + \omega\sigma_t \mathrm{d}B_{\lambda_i t}^{(k)}, \tag{11}$$

which implies that $\sigma_t^{2(k)} \sim \Gamma(w_k\nu, \alpha)$, where we assume that $\{w_k \geq 0\}$ and $\sum_{k=1}^{K} w_k = 1$. This implies the mean of $\sigma_t^{2(k)}$ is $w_k\nu/\alpha$, while the variance is $w_k\nu/\alpha^2$.

The analysis is based on taking $\nu = 4$, $\alpha = 8$, $K = 2$, $w_1 = 0.8$, $\lambda_1 = 4$ and $\lambda_2 = 0.03$. This means that the second component in the variance has considerable memory, while the first component has very little indeed. To allow for the possibility of leverage we will assume

$$\mathrm{Cor}\left(b_{\lambda t}, w_t\right) = \rho t\sqrt{\lambda}.$$

The correlation parameter ρ indexes the leverage effect in the model and would be expected to be negative for equity data (e.g. Black (1976) and Nelson (1991)).

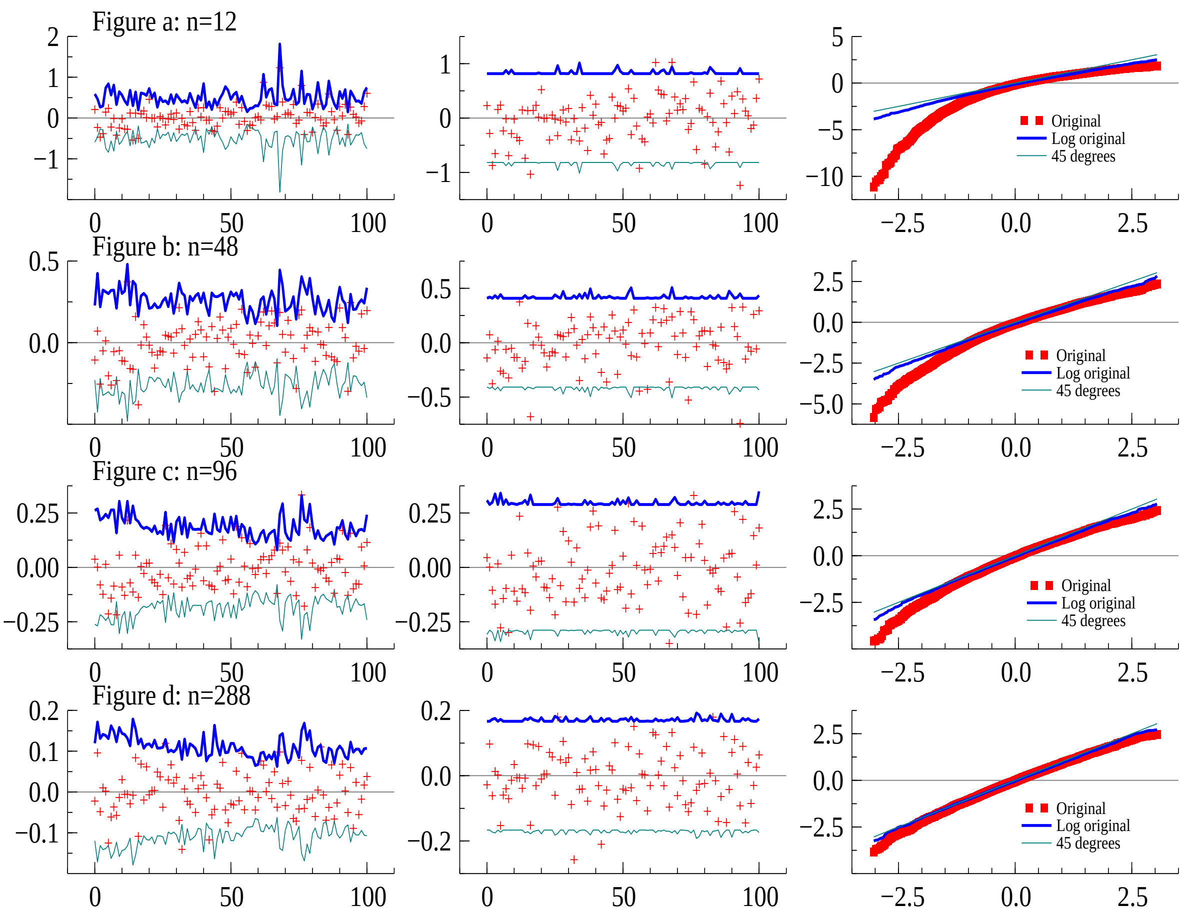

Figure 1: Case of no leverage, $\rho = 0$. *Comparison of finite sample behaviour for raw and log-based asymptotics. LHS: raw version, graphs* $[Y_\delta]_i - [Y]_i$ *against* i*, together with 95% CIs. Middle: log version, graphs* $\log [Y_\delta]_i - \log[Y]_i$ *against* i*, together with 95% CIs. RHS: QQ plots for performance of the raw and log versions of the theory (X-axis has the expected quantiles, Y-axis the observed). Based on* $n = 12$, 48, 96 *and* $n = 288$*, using 5,000 replications. Code:* `leverage.ox`.

Figure 1 shows results for the case where there is no leverage. This will provide a benchmark for the leverage case. On the left hand side of the picture we give a time series plot of the first 100 realised variance errors, showing $[Y_\delta]_i - [Y]_i$ graphed against i. Also given are 95% confidence intervals for these errors, which vary dramatically over i. They increase and decrease with the level of the volatility of the process. In the middle column the corresponding results for $\log [Y_\delta]_i - \log[Y]_i$ are graphed against i, again with 95% confidence intervals. These are much more stable through time. On the right hand side of the graph we have shown QQ plots of the t-statistic versions of the realised variance errors and their log versions. We can see that even if $n = 12$ the asymptotic theory works well in the log case, with small improvements being made as n increases. On the other hand the standardised untransformed errors are very far from being Gaussian even when $n = 96$.

The corresponding results for the extreme case of $\rho = -1$ are given in Figure 2. We should note that common random numbers have been used in the leverage and non-leverage cases. The broad conclusions from the Figure are really the same as for the non-leverage case. It is not really possible to see from the graphs any difference between these two cases.

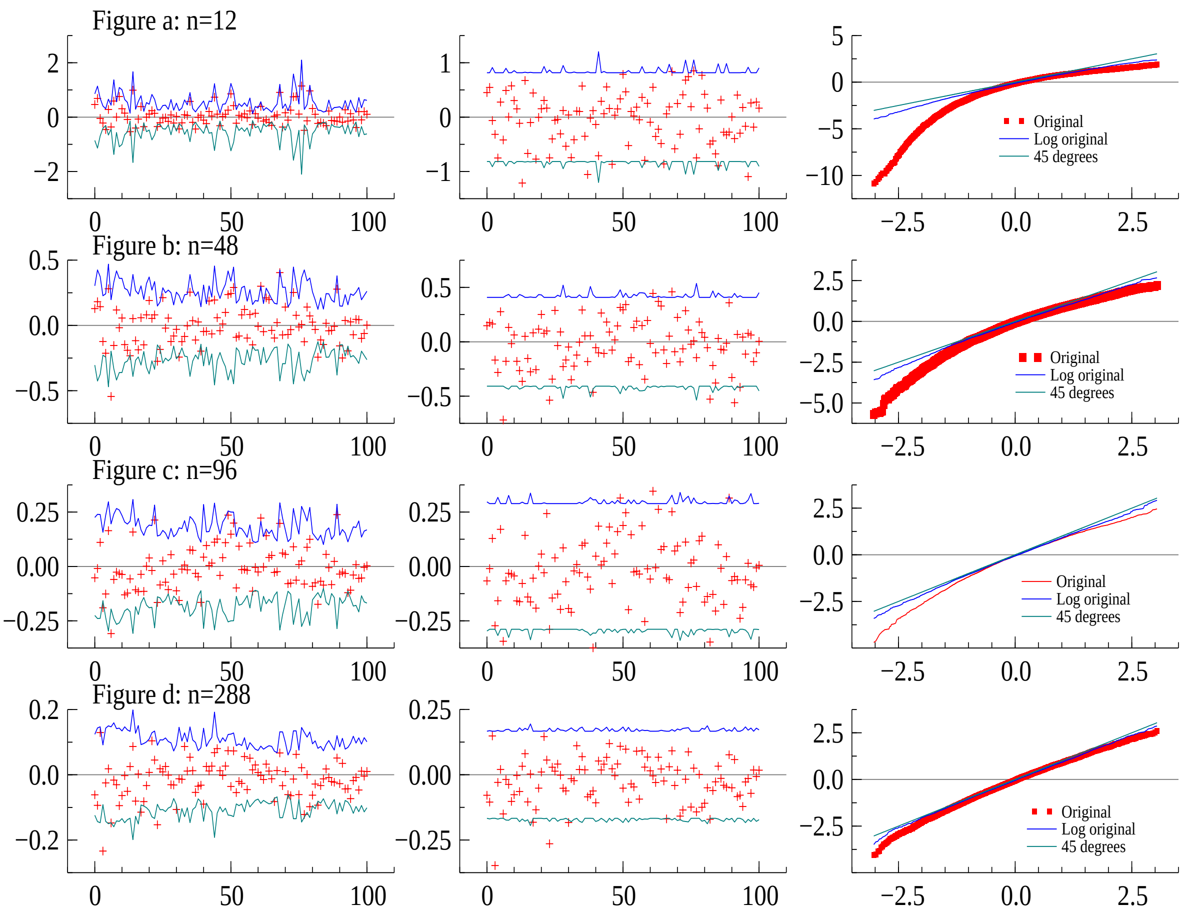

Figure 2: Case of perfect leverage, $\rho = -1$. *Comparison of finite sample behaviour for standard and log-based asymptotics. LHS: raw version, graphs* $[Y_\delta]_i - [Y]_i$ *against* i*, together with 95% CIs. Middle: log version, graphs* $\log [Y_\delta]_i - \log[Y]_i$ *against* i*, together with 95% CIs. RHS: QQ plots for performance of the raw and log versions of the theory (X-axis has the expected quantiles, Y-axis the observed). Based on* $n = 12$, 48, 96 *and* $n = 288$*, using 5,000 replications. Code:* `leverage.ox`.

These results are reinforced by Table 1 which shows the mean and standard error of the normalised statistics (9) and (10) for a wider variety of values of ρ. In the Table the former is called the raw t-statistics, the latter the log version. The Table shows that there is a negative bias in the raw t-statistic, which corresponds to the realised variance being too small and at the same time the corresponding denominator being too small. This bias is an order of magnitude smaller on the log-version of the statistic. In both cases the standard error of the normalised statistic is roughly one.

The Table also gives some results on the coverage performance of the asymptotic theory. This records the percentage of times the realised variance minus the actual variance is larger, in absolute value, than twice the feasible asymptotic standard error. Thus, if the asymptotic theory was exact then we would expect the coverage percentage to be 95. The results suggest that this is not a poor approximation for moderately large values of n.

n	No leverage, $\rho = 0$						Strong leverage, $\rho = -1$					
	Raw			Log			Raw			Log		
	Bias	S.E.	Cove	Bias	S.E.	Cove	Bias	S.E.	Cove	Bias	S.E.	Cove
12	-0.552	1.64	85.7	-0.207	1.02	93.8	-0.558	1.66	85.6	-0.205	1.03	93.6
48	-0.244	1.14	91.6	-0.116	0.99	94.8	-0.240	1.15	91.7	-0.110	1.00	94.4
96	-0.168	1.07	93.3	-0.084	0.99	94.9	-0.179	1.07	93.0	-0.093	1.00	94.6
288	-0.096	1.02	94.3	-0.051	0.99	95.0	-0.102	1.01	94.4	-0.057	0.98	95.0
n	Moderate leverage, $\rho = -1/3$						Quite strong leverage, $\rho = -2/3$					
12	-0.544	1.64	85.8	-0.201	1.02	93.7	-0.541	1.62	85.8	-0.199	1.03	93.9
48	-0.238	1.14	91.9	-0.110	1.00	94.7	-0.234	1.15	91.7	-0.106	1.00	94.7
96	-0.168	1.07	93.0	-0.083	1.00	94.9	-0.170	1.08	93.0	-0.084	1.00	94.6
288	-0.092	1.02	94.4	-0.046	0.99	95.1	-0.091	1.02	94.3	-0.046	0.99	95.0

Table 1: *Based on 20,000 simulations. Bias and standard error of the realised variance errors using the raw asymptotics and the log-based asymptotics. Simulations use a superposition of CIR variance models. Cove denotes estimated finite sample coverage using the asymptotic theory setting the nomimal level at 95.0. The Table deals with the no leverage ($\rho = 0$), moderate ($\rho = -1/3$), quite strong ($\rho = -2/3$) and strong leverage ($\rho = -1$) cases. File:* `leverage.ox`.

5 Conclusions

In this paper we have proved the conjecture that the feasible asymptotic theory for realised volatility and realised covariation, developed by Barndorff-Nielsen and Shephard (2002) and Barndorff-Nielsen and Shephard (2004a), is unaffected by the presence of leverage effects. This extension is made possible by harnessing an elegant result developed by Jacod and Protter (1998, Theorem 5.5). Simulation experiments suggest that the feasible theory has excellent finite sample behaviour.

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