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A FEW REMARKS ON THE CAMASSA-HOLM EQUATION

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Abstract. In the present paper, we use some standard a priori estimates for linear transport equations to prove the existence and uniqueness of solutions for the Camassa-Holm equation with minimal regularity assumptions on the initial data. We also derive some explosion criteria and a sharp estimate from below for the existence time. We finally address the question of global existence for certain initial data. This yields, among other things, a different proof for the existence and uniqueness of Constantin and Molinet's global weak solutions (see [9]).

Introduction. In the past few years, a large amount of literature has been devoted to the following one-dimensional nonlinear dispersive equation:

$$\partial_t v - \partial^3_{txx} v + 2\kappa \partial_x v + 3v\partial_x v = 2\partial_x v \partial^2_{xx} v + v\partial^3_{xxx} v. \tag{0.1}$$

The model above, commonly called the Camassa-Holm equation, has been derived independently by A. Fokas and B. Fuchssteiner in [12], and by R. Camassa and D. Holm in [3] (see also [13] and [4]). Some generalizations of (0.1) with higher-order terms are also relevant (see e.g. [14]). In the above equation, the function $u = u(t,x)$ stands for the fluid velocity at time $t \geq 0$ in the x direction and κ is a nonnegative parameter. In the present work, x will be in $\mathbb{R}$ or in $\mathbb{T}$ where $\mathbb{T}$ denotes the circle $\mathbb{R}/\mathbb{Z}$, and most of our results shall apply indistinctly to both cases. We shall merely use the notation $x \in A$ to mean that x belongs to $\mathbb{R}$ or to $\mathbb{T}$.

Like the celebrated KdV equation,

$$\partial_t u + 6u\partial_x u + \partial^3_{xxx} u = 0,$$

the Camassa-Holm equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. It

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turns out that (0.1) is also relevant to describe the propagation of nonlinear waves in cylindrical hyperelastic rods (see [11]).

The reason for the success encountered by this new model lies in the fact that it has both solitary waves interacting like solitons (see e.g. [3], [4], [10] and [7]) and, in contrast to KdV, solutions which blow up in finite time as a result of the breaking of waves (see e.g. [7] and [8]).

Throughout the paper, we shall concentrate on the case $\kappa = 0$ (which actually is not restrictive since the change of variables $u(t,x) = v(t, x-\kappa t)+\kappa$ leads to (0.1) with $\kappa = 0$), and we shall address the question of the initial value problem for positive times. As soon as uniqueness holds, it is clear however that we have similar results for negative times. This is just a matter of changing the initial condition u_0 into $-u_0$.

At this point, one can wonder which regularity assumptions are relevant for initial data u_0 so that the initial value problem be well-posed in the sense of Hadamard (i.e., (CH) has a unique local solution in a suitable functional setting, and continuity with respect to the initial data holds). In the first mathematical works devoted to (CH), the initial data u_0 were taken in the nonhomogeneous Sobolev space H^3, and well-posedness was shown in $C([0,T]; H^3) \cap C^1([0,T]; H^2)$. It was stated besides that the solutions may blow up in finite time or be global (see e.g. [6], [7] and [8]).

The link of (0.1) with nonlinear transport equations was pointed out in [6]. Indeed, (0.1) may be rewritten as follows:

$$\partial_t u + u\partial_x u = P(D)(u^2 + \frac{1}{2}(\partial_x u)^2), \quad u_{|t=0} = u_0, \tag{CH}$$

with $P(D) = -\partial_x(1-\partial_x^2)^{-1}$. Obviously, the H^3 assumption on the initial data can be weakened. This was noticed before in [18] and [16] where local well-posedness is stated for initial data in the nonhomogeneous Sobolev spaces H^s with $s > 3/2$.

Actually, taking initial data in Sobolev spaces does not matter. What does matter is the choice of a Banach space E which is included in the space Lip of continuous, bounded functions with bounded derivatives. Indeed, under such an assumption on E, it is in general possible to get a priori estimates in $C([0,T]; E)$ for the solution of linear transport equations

$$\partial_t v + a\partial_x v = f, \quad v(0) \in E, \quad a, f \in C([0,T]; E).$$

Therefore, the existence of local solutions might be proved in a plethora of Banach spaces such that in addition $G : u \mapsto P(D)\big(u^2 + (\partial_x u)^2/2\big)$ maps E to E continuously.

We concentrate on the case where E is a nonhomogeneous Besov space $B^s_{p,r}$ (see their definition in section 1). We deliberately use these very spaces which are built on L^p to emphasize that using spaces built on L^2 (such as Sobolev spaces) is not so important. Note in passing that, since $B^s_{2,2} = H^s$, the results of [16] or [18] come up as a special case of our results.

In the Besov spaces framework, the condition $E \subset \mathrm{Lip}$ is equivalent to $s > 1 + 1/p$ (or $s \geq 1 + 1/p$ if $r = 1$), and it turns out that no further restrictions are needed for the continuity of the map G (except that the endpoint $r = 1$, $s = 1$, $p = +\infty$ is not allowed). We shall see on the other hand that the additional condition $s > 3/2$ is required for uniqueness. This merely stems from the fact that we are led to estimate the difference between two solutions in $B^{s-1}_{p,r}$ and that the term $(\partial_x u)^2$ is involved in the right-hand side of (CH).

Due to the use of estimates for the transport equation, we shall get a blow-up criterion (almost) for free: the solution ceases to have the smoothness of the initial data at time $T^\star$ if and only if

$$\int_0^{T^\star} \|u(t)\|_{\mathrm{Lip}}\, dt = +\infty. \tag{0.2}$$

Let us emphasize that this type of explosion criterion is quite universal for nonlinear hyperbolic PDE's (a similar result holds for nonlinear wave equation: see [15]). Actually, getting local existence results and the above blow-up criterion barely utilizes the structure of (CH). An analogous method would yield similar results for more general (possibly d-dimensional) equations

$$\partial_t u + u \cdot \nabla u = f(u, \nabla u).$$

Of course, the exact range of Besov spaces for which local well-posedness holds depends on the dimension and on the nonlinearity $f(u, \nabla u)$.

Next, we shall turn to results which are more specific to (CH). We first aim at getting an explosion criterion more precise than (0.2). For that purpose, we shall suppose that in addition $u_0 \in H^1$. Indeed, this will enable us to use the energy (i.e., the H^1 norm) as a conservation law. In view of Sobolev embeddings, this provides us with a uniform control on $\|u(t)\|_{L^\infty}$. Further considerations based on a result by A. Constantin and J. Escher in [8] will show that only the part $\inf_{x \in A} \partial_x u(t, x)$ of the whole norm $\|u(t)\|_{\mathrm{Lip}}$ is responsible for the blowing-up of solutions. This fact was remarked before in [7] and [6] for smooth solutions, though not explicitly stated. We are also interested in getting a sharp estimate from below for the existence time.

Let us now present our main result. The reader not familiar with Besov spaces can simply take u_0 in H^s $(s>3/2)$ in the statement below.

Theorem 0.1. *Let* $1\leq p\leq +\infty$ *and* $1\leq r<+\infty$. *Suppose that* $u_0\in B^s_{p,r}\cap H^1$ *with* $s>\max(1+1/p,3/2)$ (*or* $s=1+1/p$, $p<2$ *and* $r=1$). *Then* (CH) *has a unique maximal solution in* $C([0,T^\star_{u_0});B^s_{p,r})\cap C^1([0,T^\star_{u_0});B^{s-1}_{p,r})$ *with constant* H^1 *norm.*

Let $c=1$ *if* $A=\mathbb{R}$ *and* $c=\cosh(1/2)/\mathrm{sh}(1/2)$ *if* $A=\mathbb{T}$. *Then the lifespan* $T^\star_{u_0}$ *satisfies*

$$T^\star_{u_0}\geq T_{u_0}\stackrel{\text{def}}{=}-\frac{2}{\sqrt{c}\|u_0\|_{H^1}}\arctan\Big(\frac{\sqrt{c}\|u_0\|_{H^1}}{\inf_{x\in A}\partial_x u_0(x)}\Big). \tag{0.3}$$

Moreover,

$$T^\star_{u_0}<+\infty\Longrightarrow\int_0^{T^\star_{u_0}}\big(\inf_{x\in A}\partial_x u(t,x)\big)\,dt=-\infty. \tag{0.4}$$

If the potential $y_0\stackrel{\text{def}}{=}u_0-\partial^2_{xx}u_0$ *has a sign then* $T^\star_{u_0}=+\infty$ *and* $\mathrm{sgn}(u(t)-\partial^2_{xx}u(t))=\mathrm{sgn}y_0$.

Let us stress the fact that (0.3) is sharp: for any $\epsilon>0$, there exists a smooth u_0 for which the maximal existence time $T^\star_{u_0}$ for solutions to (CH) satisfy $T^\star_{u_0}<(1+\epsilon)T_{u_0}$.

Since for many initial conditions, blow-up occurs in finite time, considering the question of lower semicontinuity for the maximal existence time is of great interest. This is the aim of Theorem 4.1 below. We should mention in passing that our proof easily extends to more general equations.

One of the main interesting features of (CH) is the existence of stable solitary traveling waves which are solitons (see [3] or [7]):

$$u(t,x)=ce^{-|x-x_0-ct|},\ x_0\in\mathbb{R},\ c\in\mathbb{R}.$$

These waves have a discontinuity in the first derivative at their peak (i.e., at $x=x_0+ct$). In particular, the corresponding initial datum u_0 fails to satisfy the regularity assumptions required in Theorem 0.1 (actually u_0 belongs to any Besov space $B^{1+\frac1p}_{p,\infty}$ but not to $B^{1+\frac1p}_{p,1}$). Let us remark on the other hand that in the special case of solitons, the potential $y_0\stackrel{\text{def}}{=}u_0-\partial^2_{xx}u_0$ is a Dirac mass. This motivates the study of well-posedness for initial data such that y_0 is in the space $\mathcal{M}$ of regular Borel measures with bounded total variation.

In [9], A. Constantin and L. Molinet concentrated on the case where, in addition, y_0 is positive. On the basis that, in this particular case, $\mathrm{sgn} y(t)$ and $\|y(t)\|_{L^1}$ are conserved with time, they proved the existence of a "global weak solution" for such initial data. Uniqueness also holds, which means that solitons are indeed relevant solutions. This also means on the other hand that these "weak solutions" are not that weak.

In the present work, we do not make any sign assumption. Our aim is to get a statement similar to Theorem 0.1. We shall indeed prove the result below.

Theorem 0.2. *Suppose that y_0 belongs to $\mathcal{M}$. Then (CH) has a unique maximal solution u belonging to $C([0,T^\star_{u_0});H^s)\cap C^1([0,T^\star_{u_0});H^{s-1})$ for any $s<3/2$, and such that $y \stackrel{\text{def}}{=} u-\partial^2_{xx}u$ stays in $\mathcal{M}$ uniformly on every compact subset of $[0,T^\star_{u_0})$. In addition, the energy is conserved:*

$$T^\star_{u_0} \geq \frac{2}{\|y_0\|_{\mathcal{M}}} \quad \text{and} \quad T^\star_{u_0}<+\infty \Longrightarrow \int_0^{T^\star_{u_0}} \big(\inf_{x\in A} \partial_x u(t,x)\big)\,dt = -\infty.$$

If y_0 has a sign then the solution is global, and $\mathrm{sgn} y(t)$ *and $\|y(t)\|_{\mathcal{M}}$ are conserved.*

Our paper is organized as follows. In the first section, we recall some basic results on Besov spaces. In section 2, we investigate the well-posedness of (CH) in Besov spaces, and state a blow-up criterion. The third section is devoted to the proof of Theorem 0.1. In the fourth part, we prove that the maximal existence time of the solutions has lower semicontinuity with respect to suitably smooth initial data. In section 5, we prove Theorem 0.2. We end the paper with an appendix devoted to the proof of estimates in general Besov spaces for d-dimensional transport equations. We also give there two approximation results used in part 5.

1. A few facts on Besov spaces and some notation. Before reaching into the heart of the matter, we recall some basic results on Besov spaces. They are not specific to the one-dimensional setting, and the positive integer d will stand for the space dimension. A more complete presentation may be found in [19]. We here use [5]'s notation.

The usual definition of Besov spaces relies upon a dyadic partition of unity in Fourier variables: the Littlewood-Paley decomposition. We can use for instance any couple (χ,φ) of C^∞ functions such that χ is supported in

$\{\xi \in \mathbb{R}^d : |\xi| \le 4/3\}$, φ is supported in $\{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$, and

$$\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}^d.$$

Let us first consider the whole-space case. Denote $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$ and $\check{h} = \mathcal{F}^{-1}\chi$. We then define the dyadic blocks as follows:

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if} \quad q \le -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^d} \check{h}(y) u(x-y)\, dy,$$
$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^d} h_q(y) u(x-y)\, dy \quad \text{if} \quad q \ge 0.$$

In the periodical setting, we shall decompose the functions on the torus $\mathbb{T}^d$ in Fourier series: $u(x) = \sum_{\alpha\in\mathbb{Z}^d} u_\alpha e^{i2\pi\alpha\cdot x}$, denote

$$h_q(x) = \sum_{\alpha\in\mathbb{Z}^d} \varphi(2^{-q}\alpha) e^{i2\pi\alpha\cdot x} \quad \text{and} \quad \check{h}(x) = \sum_{\alpha\in\mathbb{Z}^d} \chi(\alpha) e^{i2\pi\alpha\cdot x},$$

and define the dyadic blocks as follows:

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if} \quad q \le -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \sum_{\alpha\in\mathbb{Z}^d} \chi(\alpha) u_\alpha e^{i2\pi\alpha\cdot x} = \int_{\mathbb{T}^d} \check{h}(y) u(x-y)\, dy,$$
$$\Delta_q u \stackrel{\text{def}}{=} \sum_{\alpha\in\mathbb{Z}^d} \varphi(2^{-q}\alpha) u_\alpha e^{i2\pi\alpha\cdot x} = \int_{\mathbb{T}^d} h_q(y) u(x-y)\, dy \quad \text{if} \quad q \ge \quad 0.$$

In both cases, we shall also use the notation $S_q u \stackrel{\text{def}}{=} \sum_{p\le q-1} \Delta_p u$.

From now on, we will not make a distinction between the periodical and the whole-space cases (except where otherwise stated) and the notation A^d will stand for $\mathbb{R}^d$ or $\mathbb{T}^d$.

The formal equality

$$u = \sum_{q\ge -1} \Delta_q u$$

holds in $\mathcal{S}'(A^d)$ and is called the Littlewood-Paley decomposition. It has nice properties of quasi-orthogonality: with our choice of φ, we have

$$\Delta_k \Delta_q u \equiv 0 \ \text{ if } \ |k-q| \ge 2 \ \text{ and } \ \Delta_k(S_{q-1} u \Delta_q u) \equiv 0 \ \text{ if } \ |k-q| \ge 5. \quad (1.1)$$

We can now define the Besov spaces.

Definition 1.1. For $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ and $u \in \mathcal{S}'(A^d)$, we set $\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \left(\sum_{q \geq -1} \left(2^{sq} \|\Delta_q u\|_{L^p}\right)^r\right)^{\frac{1}{r}}$ if $1 \leq r < +\infty$ and $\|u\|_{B^s_{p,\infty}} \stackrel{\text{def}}{=} \sup_{q \geq -1} 2^{sq} \|\Delta_q u\|_{L^p}$. We then define

$$B^s_{p,r} \stackrel{\text{def}}{=} B^s_{p,r}(A^d) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'(A^d) : \|u\|_{B^s_{p,r}} < +\infty\}.$$

The above definition does not depend on the Littlewood-Paley decomposition chosen. Let us indicate how Besov spaces are related to Sobolev and Hölder spaces:

Proposition 1.2. *For $s \in \mathbb{R}$, we have $B^s_{2,2} = H^s$. For $s \in \mathbb{R}^+ \backslash \mathbb{N}$, we have $B^s_{\infty,\infty} = C^s$.*

Remark that we have the following strict embeddings (where $\hookrightarrow$ means continuous embedding): $B^0_{\infty,1} \hookrightarrow L^\infty \hookrightarrow B^0_{\infty,\infty}$. If we denote by Lip the space of bounded functions with bounded first derivatives, we have $B^1_{\infty,1} \hookrightarrow \mathrm{Lip} \hookrightarrow B^1_{\infty,\infty}$. It will be sometimes useful to know the place of L^1 and of the space $\mathcal{M}$ of regular Borel measures with bounded total variation in the range of Besov spaces:

Lemma 1.3. *The following embeddings hold* : $B^0_{1,1} \hookrightarrow L^1 \hookrightarrow \mathcal{M} \hookrightarrow B^0_{1,\infty}$.

Proof. Let us prove $\mathcal{M} \hookrightarrow B^0_{1,\infty}$. The other embeddings are straightforward. If $u \in \mathcal{M}$, then

$$\Delta_q u(x) = \int_{A^d} h_q(y)\, du(x - y)$$

so that, denoting by $\|u\|_{\mathcal{M}}$ the total variation of u, we have according to Fubini's theorem,

$$\|\Delta_q u\|_{L^1} \leq \int_{A^d} |h_q(y)|\, dy \int_{A^d} d|u|(z) \leq C\|u\|_{\mathcal{M}}.$$

We of course also have $\|\Delta_{-1} u\|_{L^1} \leq \left\|\check{h}\right\|_{L^1} \|u\|_{\mathcal{M}}$, which completes the proof. $\square$

Straightforward computations show that the Dirac measure δ (which is in some sense the less regular element of $\mathcal{M}$) belongs to $B^0_{1,\infty}$ but not to $B^0_{1,r}$ for $r < +\infty$.

Let us give some classical properties for the Besov spaces (see the proofs in [19] for example).

Proposition 1.4. *The following properties hold :*

i) *Density: if* $p, r < +\infty$ *then* $\mathcal{S}(A^d)$ *is dense in* $B^s_{p,r}$.

ii) *Generalized derivatives: Let* $f \in C^\infty(\mathbb{R}^d)$ *be a homogeneous function of degree* $m \in \mathbb{R}$ *away from a neighborhood of the origin. There exists a constant* C *depending only on* f *and such that* $\|f(D)u\|_{B^{s-m}_{p,r}} \le C\|u\|_{B^s_{p,r}}$.

iii) *Sobolev embeddings: if* $p_1 \le p_2$ *and* $r_1 \le r_2$, *then* $B^s_{p_1,r_1} \hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$. *If* $s_1 < s_2$, $1 \le p \le +\infty$ *and* $1 \le r_1, r_2 \le +\infty$, *then the embedding* $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ *is locally compact.*

iv) *Algebraic properties: for* $s > 0$, $B^s_{p,r} \cap L^\infty$ *is an algebra. Moreover,* ($B^s_{p,r}$ *is an algebra*) $\iff$ ($B^s_{p,r} \hookrightarrow L^\infty$) $\iff$ ($s > p/d$ *or* ($s \ge p/d$ *and* $r = 1$)).

v) *Interpolation:* $(B^{s_1}_{p,q_1}, B^{s_2}_{p,q_2})_{\theta,q} = B^{\theta s_1+(1-\theta)s_2}_{p,q}$ *for* $0 < \theta < 1$.

vi) *Fatou property: If the sequence* $(f_k)_{k\in\mathbb{N}}$ *is uniformly bounded in* $B^s_{p,r}$ *and converges weakly in* $\mathcal{S}'$ *to* f, *then* $f \in B^s_{p,r}$ *and* $\|f\|_{B^s_{p,r}} \le \liminf_{k\to\infty} \|f_k\|_{B^s_{p,r}}$.

We have the following continuity properties for the product of two functions:

Proposition 1.5. *For any* $s > 0$ *and* $1 \le p, r \le +\infty$, *there exists* $C = C(d, s)$ *such that*

$$\|uv\|_{B^s_{p,r}} \le C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}\big). \tag{1.2}$$

If $1 \le p, r \le \infty$, $s_1, s_2 < d/p$ *if* $r > 1$ ($s_1, s_2 \le d/p$ *if* $r = 1$) *and* $s_1 + s_2 > 0$, *there exists* $C = C(s_1, s_2, p, r, d)$ *such that*

$$\|uv\|_{B^{s_1+s_2-\frac{d}{p}}_{p,r}} \le C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}. \tag{1.3}$$

If $1 \le p, r \le +\infty$, $s_1 \le d/p$, $s_2 > d/p$ ($s_2 \ge d/p$ *if* $r = 1$) *and* $s_1 + s_2 > 0$, *there exists* $C = C(s_1, s_2, p, r, d)$ *such that*

$$\|uv\|_{B^{s_1}_{p,r}} \le C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}. \tag{1.4}$$

Notation. Throughout, C will stand for a "harmless" constant. For any Banach space X, $0 < T \le +\infty$ and $1 \le r \le +\infty$, we shall denote by $L^r(0, T; X)$ the set of measurable functions on $(0, T)$ valued in X and such that the map $t \mapsto \|u(t)\|_X$ belongs to the Lebesgue space $L^r(0,T)$. The space $C([0,T];X)$ will stand for the set of continuous functions on $[0,T]$ with values in X, and we shall use the notation $C^1([0,T];X)$ for functions f in $C([0,T];X)$ differentiable with respect to t and such that $\partial_t f$ also belongs to $C([0,T];X)$.

2. Local well-posedness. In this section, we address the question of local well-posedness for (CH) with initial data u_0 in $\mathrm{Lip} \cap B^s_{p,r}$. We shall see that existence holds true as soon as $s > 1$, and uniqueness as soon as $s > 3/2$. We also prove a first blow-up criterion. More precise criteria will be derived in section 3.

Uniqueness and continuity with respect to the initial data are a corollary of the following:

Proposition 2.1. *Let $1 \le p, r \le +\infty$ and $s > 3/2$. Suppose that we are given $(u,v) \in \big(L^\infty(0,T; B^s_{p,r} \cap \mathrm{Lip}) \cap C([0,T];\mathcal{S}')\big)^2$ two solutions of* (CH) *with initial data $u_0, v_0 \in B^s_{p,r} \cap \mathrm{Lip}$. Then we have for every $t \in [0,T]$:*

$$\|u(t)-v(t)\|_{B^{s-1}_{p,r}\cap L^\infty} \le \|u_0 - v_0\|_{B^{s-1}_{p,r}\cap L^\infty} e^{C\int_0^t \left(\|u(\tau)\|_{B^s_{p,r}\cap \mathrm{Lip}} + \|v(\tau)\|_{B^s_{p,r}\cap\mathrm{Lip}}\right)d\tau}. \tag{2.1}$$

Proof. Let us assume for the sake of simplicity that $B^s_{p,r} \subset \mathrm{Lip}$. Denote $w \overset{\text{def}}{=} v - u$. Obviously, w is in $C([0,T]; B^{s-1}_{p,r})$ and solves the following transport equation:

$$\partial_t w + u\partial_x w = -w\partial_x v + P(D)\big(w(u+v) + \frac{1}{2}\partial_x w\,\partial_x(u+v)\big).$$

According to estimate (A.1) and Proposition 1.4 *ii)*, the following inequality holds true:

$$\|w(t)\|_{B^{s-1}_{p,r}} \le \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|\partial_x u(\tau')\|_{B^{s-1}_{p,r}}d\tau'} + C\int_0^t e^{C\int_\tau^t \|\partial_x u(\tau')\|_{B^{s-1}_{p,r}}d\tau'} \tag{2.2}$$
$$\times \Big(\|(w\partial_x v)(\tau)\|_{B^{s-1}_{p,r}} + \big\|\big(w(u+v)\big)(\tau)\big\|_{B^{s-2}_{p,r}} + \big\|\big(\partial_x w\,\partial_x(u+v)\big)(\tau)\big\|_{B^{s-2}_{p,r}}\Big)d\tau.$$

Since $B^{s-1}_{p,r} \subset L^\infty$, the space $B^{s-1}_{p,r}$ is an algebra according to Proposition 1.4 *ii)*. Therefore,

$$\|w\partial_x v\|_{B^{s-1}_{p,r}} \le C\|\partial_x v\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-1}_{p,r}}, \tag{2.3}$$

$$\|w(u+v)\|_{B^{s-2}_{p,r}} \le C\|w(u+v)\|_{B^{s-1}_{p,r}} \le C\big(\|u\|_{B^{s-1}_{p,r}} + \|v\|_{B^{s-1}_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}. \tag{2.4}$$

If $s > 2+1/p$ or $s \geq 2+1/p$ and $r=1$, then $B^{s-2}_{p,r}$ is also an algebra so that

$$\begin{aligned}\|\partial_x w\,\partial_x(u+v)\|_{B^{s-2}_{p,r}} &\leq C\|\partial_x w\|_{B^{s-2}_{p,r}}\big(\|\partial_x u\|_{B^{s-2}_{p,r}}+\|\partial_x v\|_{B^{s-2}_{p,r}}\big),\\ &\leq C\|w\|_{B^{s-1}_{p,r}}\big(\|u\|_{B^{s-1}_{p,r}}+\|v\|_{B^{s-1}_{p,r}}\big).\end{aligned} \tag{2.5}$$

Otherwise, we still have $s > 3/2$ and $s > 1+1/p$ (or $s \geq 1+1/p$ if $r=1$) because $B^s_{p,r} \subset \mathrm{Lip}$. Therefore, we can apply (1.4) and get

$$\begin{aligned}\|\partial_x w\,\partial_x(u+v)\|_{B^{s-2}_{p,r}} &\leq C\|\partial_x w\|_{B^{s-2}_{p,r}}\big(\|\partial_x u\|_{B^{s-1}_{p,r}}+\|\partial_x v\|_{B^{s-1}_{p,r}}\big),\\ &\leq C\|w\|_{B^{s-1}_{p,r}}\big(\|u\|_{B^{s}_{p,r}}+\|v\|_{B^{s}_{p,r}}\big).\end{aligned} \tag{2.6}$$

Plugging (2.3), (2.4), and (2.5) or (2.6) in (2.2), and applying Gronwall's lemma, we end up with

$$\|u(t)-v(t)\|_{B^{s-1}_{p,r}} \leq \|u_0-v_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t\left(\|u(\tau)\|_{B^s_{p,r}}+\|v(\tau)\|_{B^s_{p,r}}\right)d\tau}. \tag{2.7}$$

If $B^s_{p,r} \not\subset \mathrm{Lip}$, the proof of (2.1) relies on inequality (A.2). Next, this is just a matter of replacing everywhere $\|\cdot\|_{B^{s-1}_{p,r}}$ (respectively $\|\cdot\|_{B^s_{p,r}}$) by $\|\cdot\|_{B^{s-1}_{p,r}\cap L^\infty}$ (respectively $\|\cdot\|_{B^s_{p,r}\cap\mathrm{Lip}}$) in the proof above. The important fact is that

$$\|P(D)z\|_{\mathrm{Lip}} \leq C\,\|z\|_{L^\infty}\,. \tag{2.8}$$

This may be seen by considering the explicit expression of the kernel of $P(D)$ or by noticing that $\partial_x P(D)z = z-(1-\partial^2_{xx})^{-1}z$. $\square$

Before stating our local existence result, we define some functional spaces.

Definition 2.2. For $T>0$, $s\in\mathbb{R}$ and $1\leq p\leq+\infty$, we set

$E^s_{p,r}(T) \stackrel{\mathrm{def}}{=} C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})\cap \mathrm{Lip}([0,T]\times A)$ if $r<+\infty$,

$E^s_{p,\infty}(T) \stackrel{\mathrm{def}}{=} L^\infty(0,T;B^s_{p,\infty})\cap\mathrm{Lip}([0,T];B^{s-1}_{p,\infty})\cap\mathrm{Lip}([0,T]\times A)$.

The notation $E^s_{p,r}$ stands for $\cap_{T>0}E^s_{p,r}(T)$.

Let us state our well-posedness result:

Theorem 2.3. *Suppose that $s>1$, $1\leq p,r\leq+\infty$ and $u_0\in B^s_{p,r}\cap\mathrm{Lip}$. Then there exists a time $T>0$ such that* (CH) *has a solution $u\in E^s_{p,r}(T)$. If in addition $u_0\in H^1$, one can arrange that the solution satisfies*

$$\|u(t)\|_{H^1} \leq \|u_0\|_{H^1}. \tag{2.9}$$

If moreover $s > 3/2$, then uniqueness holds in $E^s_{p,r}(T)$, equality holds in (2.9) if $u_0 \in H^1$, and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $B^s_{p,r}$ into $C([0,T]; B^{s'}_{p,r}) \cap C^1([0,T]; B^{s'-1}_{p,r})$ for every $s' < s$ if $r = +\infty$, and $s' = s$ if $r < +\infty$.

Proof. Uniqueness when $s > 3/2$ is an immediate consequence of Proposition 2.1.

Now, let us assume for the sake of simplicity that $B^s_{p,r} \hookrightarrow \mathrm{Lip}$. We use a standard iterative process to build a solution. Introduce a nonnegative mollifier $\rho \in C^\infty_0(\mathbb{R}^d)$ such that $\int \rho = 1$, and denote $\rho^n(x) = n^d \rho(nx)$. We then choose $u^0 = 0$ and define by induction a sequence of smooth functions $(u^n)_{n\in\mathbb{N}}$ solving the following linear transport equation:

$$\begin{aligned}(\partial_t + u^n \partial_x) u^{n+1} &= P(D)\big((u^n)^2 + \frac{(\partial_x u^n)^2}{2}\big),\\ u^{n+1}{}_{|t=0} &= u_0^{n+1} \stackrel{\text{def}}{=} \rho^{n+1} \star u_0.\end{aligned} \tag{2.10}$$

Denote $U^n(t) = \int_0^t \|u^n(\tau)\|_{B^s_{p,r}}\, d\tau$. According to Propositions A.1 and 1.4 *ii)*, we have

$$\begin{aligned}\|u^{n+1}(t)\|_{B^s_{p,r}} &\le e^{CU^n(t)}\Big(\|u_0^{n+1}\|_{B^s_{p,r}} \\ &+ C\int_0^t e^{-CU^n(\tau)}\big(\|(u^n(\tau))^2\|_{B^{s-1}_{p,r}} + \|(\partial_x u^n(\tau))^2\|_{B^{s-1}_{p,r}}\big)\, d\tau\Big).\end{aligned} \tag{2.11}$$

As $\rho^{n+1} \star \Delta_q u_0 = \Delta_q(\rho^{n+1} \star u_0)$ and $\left\|\rho^{n+1}\right\|_{L^1} = 1$, we have $\|u_0^{n+1}\|_{B^s_{p,r}} \le \|u_0\|_{B^s_{p,r}}$. On the other hand, $B^{s-1}_{p,r}$ is an algebra and $B^s_{p,r} \hookrightarrow B^{s-1}_{p,r}$ so that the last term of (2.11) may be easily bounded. We eventually conclude that

$$\|u^{n+1}(t)\|_{B^s_{p,r}} \le e^{CU^n(t)}\Big(\|u_0\|_{B^s_{p,r}} + C\int_0^t e^{-CU^n(\tau)} \|u^n(\tau)\|^2_{B^{s-1}_{p,r}}\, d\tau\Big). \tag{2.12}$$

Let us fix a $T > 0$ such that $2C\|u_0\|_{B^s_{p,r}} T < 1$ and suppose that

$$\forall t \in [0,T],\ \|u^n(t)\|_{B^s_{p,r}} \le \frac{\|u_0\|_{B^s_{p,r}}}{1 - 2C\|u_0\|_{B^s_{p,r}} t}. \tag{2.13}$$

Plugging (2.13) in (2.12) eventually yields

$$\|u^{n+1}(t)\|_{B^s_{p,r}} \leq \frac{1}{\sqrt{1-2C\|u_0\|_{B^s_{p,r}}t}}$$
$$\times\left(\|u_0\|_{B^s_{p,r}} + C\|u_0\|^2_{B^s_{p,r}}\int_0^t \frac{d\tau}{(1-2C\|u_0\|_{B^s_{p,r}}\tau)^{\frac{3}{2}}}\right) \leq \frac{\|u_0\|_{B^s_{p,r}}}{1-2C\|u_0\|_{B^s_{p,r}}t}.$$

Therefore, $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. This clearly entails that $u^n\partial_x u^n$ is uniformly bounded in $C([0,T];B^{s-1}_{p,r})$, and the right-hand side of (2.10) has been shown to be uniformly bounded in $C([0,T];B^s_{p,r})$. We conclude that the sequence $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$.

As the embedding $B^s_{p,r}\hookrightarrow B^{s-1}_{p,r}$ is locally compact, the Arzela-Ascoli theorem and Cantor's diagonal process enable us to conclude that, up to an extraction, the sequence $(u^n)_{n\in\mathbb{N}}$ tends to a limit $u\in \mathrm{Lip}([0,T];B^{s-1}_{p,r})$ for the topology of $C([0,T];(B^{s-1}_{p,r})_{loc})$; i.e., for any $\phi\in C_0^\infty(A)$,

$$\|\phi u_n(t)-\phi u(t)\|_{B^{s-1}_{p,r}} \underset{n\to+\infty}{\longrightarrow} 0 \text{ uniformly on } [0,T].$$

Since $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})$, we have, according to Proposition 1.4 *vi)*, $u\in L^\infty(0,T;B^s_{p,r})$. By interpolation, convergence of ϕu^n to ϕu holds in $C([0,T];B^{s'}_{p,r})$ for any $s'<s$ and $\phi\in C_0^\infty(A)$. As $s>1$, this enables us to prove that u indeed solves (CH) in the sense of distributions.

Now, $(\partial_t+u\partial_x)u$ is in $L^\infty(0,T;B^s_{p,r})$. Hence, if in addition $r<+\infty$, we have $u\in C([0,T];B^s_{p,r})$ according to Proposition A.1. From this, we readily gather that $\partial_t u\in C([0,T];B^{s-1}_{p,r})$.

If v_0 is in a small neighborhood of u_0 in $B^s_{p,r}$, the arguments above give the existence of a solution $v\in E^s_{p,r}(T)$ to (CH) with initial datum v_0. If $s>3/2$, Proposition 2.1 combined with an obvious interpolation ensures continuity with respect to the initial data in $C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})$ for any $s'<s$.

The fact that continuity also holds in $C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$ when $r<+\infty$ is not obvious but belongs to the mathematical folklore. It may be proved through the use of a sequence of approximate solutions $(u_\epsilon)_{\epsilon>0}$ for (CH) which converges uniformly in $C([0,T];B^s_{p,r})\cap C^1([0,T];$

$B^{s-1}_{p,r}$). A viscosity approximation gives the desired property of convergence. Since this question of continuity up to the index s is of secondary importance, we shall not give more details in the present paper.

The case where $B^s_{p,r} \not\hookrightarrow \mathrm{Lip}$ may be treated in a similar way. We just have to use the norms $\|\cdot\|_{B^s_{p,r}\cap\mathrm{Lip}}$ instead of $\|\cdot\|_{B^s_{p,r}}$. Indeed, inequalities (2.8) and (A.2) provide a control on the Lip norms. The details are left to the reader.

Let us assume now that, in addition, the initial datum u_0 is in H^1. Using the approximation scheme above to prove that $\|u(t)\|_{H^1} \leq \|u_0\|_{H^1}$ is not judicious since, in contrast to (CH), it does not conserve the H^1 norm.

This is actually more convenient to mollify the initial datum u_0 as above and to define u^n as the maximal solution of (CH) corresponding to u^n_0. Since, for instance, $u^n_0 \in H^4$ and $\|u^n_0\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}}$, the proof above combined with uniqueness ensures that for any T such that $2C\|u_0\|_{B^s_{p,r}} T < 1$, $u^n \in C([0,T]; H^4 \cap B^s_{p,r}) \cap C^1([0,T]; H^3 \cap B^{s-1}_{p,r})$. In particular, the smoothness of u^n allows us to derive directly from (0.1) that

$$\forall t \in [0,T],\ \|u^n(t)\|_{H^1} = \|u^n_0\|_{H^1} \leq \|u_0\|_{H^1}. \tag{2.14}$$

On the other hand, applying Proposition A.1 to the equation (CH) satisfied by u^n and following the steps of the proof above, we conclude again (2.13). We can therefore prove the convergence of $(u^n)_{n\in\mathbb{N}}$ to a solution u of (CH) as before. Since (2.14) holds, we besides get $\|u(t)\|_{H^1} \leq \|u_0\|_{H^1}$.

Let us remark now that the construction above enables us to solve (CH) backwards on the whole interval $[0,T']$ from any time $T' < T$. Starting with $u(T')$, this yields a solution $v \in E^s_{p,r}(T')$ defined on $[0,T']$. Since $u(T') \in H^1$, we will have in particular $\|v(0)\|_{H^1} \leq \|u(T')\|_{H^1}$. If moreover $s > 3/2$, uniqueness ensures that $v \equiv u$ on $[0,T']$; hence, $\|u(0)\|_{H^1} \leq \|u(T')\|_{H^1}$. Therefore, the H^1 norm is preserved with time. $\square$

Let us now tackle the problem of breakdown. The first explosion criterion we shall obtain is a corollary of the following estimate, which is similar to that deduced by S. Klainerman in [15] for Sobolev norms in the framework of nonlinear wave equations:

Lemma 2.4. *Let* $1 \leq p,r \leq +\infty$ *and* $s > 0$. *Let* $u \in L^\infty(0,T; B^s_{p,r} \cap \mathrm{Lip})$ *solving* (CH) *with* $u_0 \in B^s_{p,r} \cap \mathrm{Lip}$ *as an initial datum. The following inequalities hold on* $[0,T)$ (*with a constant* C *depending on* s *and* p) :

$$\|u(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} e^{C\int_0^t \|u(\tau)\|_{\mathrm{Lip}}\,d\tau}, \tag{2.15}$$

$$\|u(t)\|_{\mathrm{Lip}} \leq \|u_0\|_{\mathrm{Lip}} e^{3\int_0^t \|u(\tau)\|_{\mathrm{Lip}}\,d\tau}. \tag{2.16}$$

Proof. Let us apply Δ_q to (CH). This yields

$$(\partial_t + u\partial_x)\Delta_q u = [u, \Delta_q]\partial_x u + P(D)\Delta_q\big(u^2 + \frac{1}{2}(\partial_x u)^2\big).$$

As $s > 0$, we have according to the estimate (A.9) of the appendix,

$$\|[u, \Delta_q]\partial_x u\|_{L^p} \leq Cc_q 2^{-qs}\|u\|_{B^s_{p,r}}\,\|\partial_x u\|_{L^\infty}\,.$$

On the other hand, according to Propositions 1.5 and 1.4 *ii)*,

$$\|P(D)\big(u^2 + \frac{(\partial_x u)^2}{2}\big)\|_{B^s_{p,r}} \leq \|u\|_{\mathrm{Lip}}\|u\|_{B^s_{p,r}}.$$

Going along the lines of the proof of Proposition A.1, this leads to

$$\|u(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} + C\int_0^t \|u(\tau)\|_{\mathrm{Lip}}\|u(\tau)\|_{B^s_{p,r}}\,d\tau.$$

Gronwall's lemma yields (2.15). Making use of inequalities (3.4) and (3.5) below, the proof of (2.16) is straightforward. □

Definition 2.5. Let $u_0 \in B^s_{p,r} \cap \mathrm{Lip}$. We define the lifespan $T^\star_{u_0}$ of the solutions of (CH) with initial data as

$$T^\star_{u_0} \stackrel{\mathrm{def}}{=} \sup\{T > 0 : \text{(CH) has a solution } \ u \in E^s_{p,r}(T)\}.$$

We then have the following result:

Proposition 2.6. *Suppose that $u_0 \in B^s_{p,r} \cap \mathrm{Lip}$ with $1 \leq p, r \leq +\infty$ and $s > 3/2$. If $T^\star_{u_0}$ is finite, then we have*

$$\int_0^{T^\star_{u_0}} \|u(\tau)\|_{\mathrm{Lip}}\,d\tau = +\infty. \tag{2.17}$$

If in addition $u_0 \in H^1$, then $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ on $[0, T^\star_{u_0})$ and

$$T^\star_{u_0} < +\infty \Longrightarrow \int_0^{T^\star_{u_0}} \|\partial_x u(\tau)\|_{L^\infty}\;d\tau = \infty. \tag{2.18}$$

Proof. Let $u \in \cap_{T<T^\star} E^s_{p,r}(T)$ be such that $\int_0^{T^\star} \|u(\tau)\|_{\mathrm{Lip}}\, d\tau$ is finite. According to Lemma 2.4,

$$\forall t \in [0, T^\star),\ \|u(t)\|_{B^s_{p,r}} \leq M_{T^\star} \stackrel{\text{def}}{=} \|u_0\|_{B^s_{p,r}\cap \mathrm{Lip}} e^{C\int_0^{T^\star} \|u(\tau)\|_{\mathrm{Lip}}\, d\tau}. \tag{2.19}$$

Let $\epsilon > 0$ be such that $2C\epsilon M_{T^\star} < 1$ where C is the constant used in the proof of Theorem 2.3. We then have a solution $\widetilde{u} \in E^s_{p,r}(\epsilon)$ to (CH) with initial datum $u(T^\star - \epsilon/2)$. For the sake of uniqueness, $\widetilde{u}(t) = u(t+T^\star-\epsilon/2)$ on $[0, \epsilon/2[$ so that $\widetilde{u}$ extends the solution u beyond $T^\star$. We conclude that $T^\star < T^\star_{u_0}$ and (2.17) is proved.

If in addition $u_0 \in H^1$, we saw in Theorem 2.3 that $\|u(t)\|_{H^1}$ is constant on a small time interval. Making use of Proposition 2.1 and Lemma 2.4, we can easily check that $\|u(t)\|_{H^1}$ is a constant on the whole interval $[0, T^\star_{u_0})$. Indeed, if it were not the case, there would exist a maximal $\widetilde{T} < T^\star_{u_0}$ such that $\|u(t)\|_{H^1}$ is a constant on $[0, \widetilde{T})$. Defining $M_{\widetilde{T}}$ as in (2.19) and using Theorem 2.3, we would get an $\epsilon > 0$ such that (CH) has a solution in $E^s_{p,r}(\epsilon)$ with constant H^1 norm and initial data $u(\widetilde{T} - \epsilon/2)$. Therefore, uniqueness implies that $\|u(t)\|_{H^1}$ is constant up to time $\widetilde{T} + \epsilon/2$, which contradicts the definition of $\widetilde{T}$. Now, thanks to the following Sobolev inequality,

$$\|v\|_{L^\infty} \leq \frac{\|v\|_{H^1}}{\sqrt{2}}, \tag{2.20}$$

we have a uniform control on $\|u(t)\|_{L^\infty}$ for $t < T^\star_{u_0}$ so that (2.17) reduces to (2.18). □

Remark 2.7. In the supercritical case $s > \max(1+1/p, 3/2)$ and if $u_0 \in H^1$, we have a more general explosion criterion:

$$T^\star_{u_0} < +\infty \Longrightarrow \int_0^{T^\star_{u_0}} \|\partial_x u(\tau)\|_{B^0_{\infty,\infty}}\, d\tau = +\infty. \tag{2.21}$$

The fact that $\|\partial_x u(t)\|_{L^\infty}$ may be replaced with the weaker norm $\|\partial_x u(t)\|_{B^0_{\infty,\infty}}$ is not surprising. A similar property has been stated for Euler equations (see for example [1] or [5]). This merely stems from the following standard interpolation inequality for $s > 1 + 1/p$:

$$\|\partial_x u\|_{L^\infty} \leq \|\partial_x u\|_{B^0_{\infty,\infty}} \log\big(e + \frac{\|\partial_x u\|_{B^s_{p,r}}}{\|\partial_x u\|_{B^0_{\infty,\infty}}}\big).$$

3. Lower bounds for the maximal existence time. This part is mainly devoted to the proof of Theorem 0.1. Our first result however states that even for more general initial data with possibly infinite energy, one can get a lower bound depending only on $\|u_0\|_{\mathrm{Lip}}$ for the lifespan of the solutions.

Proposition 3.1. *If $u_0 \in B^s_{p,r} \cap \mathrm{Lip}$ with $s > 3/2$, then we have*

$$T^\star_{u_0} \geq \widetilde{T}_{u_0} \stackrel{\text{def}}{=} \frac{1}{(1+\sqrt{2}/2)\,\|\partial_x u_0\|_{L^\infty} + (\sqrt{2}+1)\,\|u_0\|_{L^\infty}}.$$

Proof. From (CH), we derive the following equality:

$$\partial_t\partial_x u + u\partial_x\partial_x u = u^2 - \frac{1}{2}(\partial_x u)^2 - (1-\partial^2_{xx})^{-1}\big(u^2 + \frac{1}{2}(\partial_x u)^2\big).$$

This clearly implies that

$$\begin{aligned}\|\partial_x u(t)\|_{L^\infty} \leq \|\partial_x u_0\|_{L^\infty} + \int_0^t \Big(&\|u(\tau)\|^2_{L^\infty} + \frac{1}{2}\|\partial_x u(\tau)\|^2_{L^\infty} \\ &+ \Big\|(1-\partial^2_{xx})^{-1}\Big(u(\tau)^2 + \frac{1}{2}(\partial_x u(\tau))^2\Big)\Big\|_{L^\infty}\Big)\,d\tau.\end{aligned} \tag{3.1}$$

On the other hand, the operator $(1-\partial^2_{xx})^{-1}$ is the convolution operator with kernel

$$p(x) = \frac{1}{2}e^{-|x|} \quad \text{if} \quad A = \mathbb{R} \quad \text{and} \quad p(x) = \frac{\mathrm{ch}(x-[x]-1/2)}{2\mathrm{sh}1/2} \quad \text{if} \quad A = \mathbb{T}, \tag{3.2}$$

so that the following estimates hold true:

$$\big\|(1-\partial^2_{xx})^{-1}v\big\|_{L^\infty} \leq \|v\|_{L^\infty} \quad \text{and} \quad \big\|\partial_x(1-\partial^2_{xx})^{-1}v\big\|_{L^\infty} \leq \|v\|_{L^\infty}. \tag{3.3}$$

Coming back to (3.1), we thus gather that

$$\|\partial_x u(t)\|_{L^\infty} \leq \|\partial_x u_0\|_{L^\infty} + \int_0^t (\|\partial_x u(\tau)\|^2_{L^\infty} + 2\,\|u(\tau)\|^2_{L^\infty})\,d\tau. \tag{3.4}$$

Similarly, we directly get from (CH) and (3.3) that

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \frac{1}{2}\int_0^t (\|\partial_x u(\tau)\|^2_{L^\infty} + 2\,\|u(\tau)\|^2_{L^\infty})\,d\tau. \tag{3.5}$$

From (3.4) and (3.5), we conclude that

$$z(t) \le z(0) + (1+\sqrt{2}/2)\int_0^t (z(\tau))^2\,d\tau$$

with $z(t) \stackrel{\text{def}}{=} \|\partial_x u(t)\|_{L^\infty} + \sqrt{2}\,\|u(t)\|_{L^\infty}$ so that

$$z(t) \le \frac{z_0}{1-(1+\sqrt{2}/2)z_0 t} \quad \text{for} \quad t < \min\big(\widetilde{T}_{u_0}, T^\star_{u_0}\big). \tag{3.6}$$

In view of Proposition 2.6, this implies that $T^\star_{u_0} \ge \widetilde{T}_{u_0}$. $\square$

Remark. We shall see below that the coefficient $(1+\sqrt{2}/2)$ is not optimal in the case of finite-energy solutions. Let us notice that plugging (3.6) in (2.15) yields the following bound:

$$\|u(t)\|_{B^s_{p,r}} \le \|u_0\|_{B^s_{p,r}} \big(1 - \frac{t}{\widetilde{T}_{u_0}}\big)^{-\beta}$$

for a positive β depending only on s, p and r.

In the case where the initial data u_0 are in H^1, one can improve the explosion criterion (2.18) and get a sharp estimate from below for the existence time $T^\star_{u_0}$:

Theorem 3.2. *Suppose that $u_0 \in B^s_{p,r} \cap H^1$ with $s > 3/2$ and $B^s_{p,r} \subset$ Lip. Then* (CH) *has a unique maximal solution $u \in C([0,T^\star_{u_0}); B^s_{p,r}) \cap C^1([0,T^\star_{u_0}); B^{s-1}_{p,r})$ with constant H^1 norm. If we denote*

$$M(t) \stackrel{\text{def}}{=} \sup_{x\in A} \partial_x u(t,x) \quad \textit{and} \quad m(t) \stackrel{\text{def}}{=} \inf_{x\in A} \partial_x u(t,x),$$

we have for every $t < T^\star_{u_0}$

$$\begin{aligned} M(t) &\le \frac{\|u_0\|_{H^1}}{\sqrt{2}} \max\Big(1, \frac{\sqrt{2}M(0) + \|u_0\|_{H^1}\tanh(\frac{\|u_0\|_{H^1} t}{2\sqrt{2}})}{\|u_0\|_{H^1} + \sqrt{2}M(0)\tanh(\frac{\|u_0\|_{H^1} t}{2\sqrt{2}})}\Big) \\ &\le \max\Big(M(0), \frac{\|u_0\|_{H^1}}{\sqrt{2}}\Big), \end{aligned} \tag{3.7}$$

and the lifespan $T^\star_{u_0}$ satisfies

$$T^\star_{u_0} \ge T_{u_0} \stackrel{\text{def}}{=} -\frac{2}{\sqrt{c}\|u_0\|_{H^1}} \arctan\Big(\frac{\sqrt{c}\|u_0\|_{H^1}}{m(0)}\Big) \tag{3.8}$$

with $c = 1$ if $A = \mathbb{R}$ and $c = \cosh(1/2)/\mathrm{sh}(1/2)$ if $A = \mathbb{T}$. The lower bound above is sharp: for any $\epsilon > 0$, there exists a $u_0 \in H^3$ such that $T^\star_{u_0} < (1+\epsilon)T_{u_0}$. In addition, we have the following blow-up criterion:

$$T^\star_{u_0} < +\infty \Longrightarrow \int_0^{T^\star_{u_0}} m(t)\,dt = -\infty. \tag{3.9}$$

Proof. Throughout the proof, we shall assume that $u_0 \not\equiv 0$.
Step 1: The case of smooth data: $u_0 \in H^3$. The proof of Proposition 3.2 relies on an "abstract key result" by A. Constantin and J. Escher (see [8]) which holds true for functions belonging to $C^1([0,T];H^2)$. Since we shall apply this result to u, we shall suppose as a first step that $u_0 \in H^3$. The "abstract key result" is the following:

Lemma 3.3. *Let $T > 0$ and $v \in C^1([0,T];H^2(A))$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t)$ (respectively $\zeta(t)$) in A such that $m(t) \stackrel{\text{def}}{=} \inf_{x\in A} \partial_x v(t,x) = \partial_x v(t,\xi(t))$ $\big($resp. $M(t) \stackrel{\text{def}}{=} \sup_{x\in A} \partial_x v(t,x) = \partial_x v(t,\zeta(t))\big)$ and the function m (respectively M) is almost everywhere differentiable on $(0,T)$ with*

$$m'(t) = \partial_x\partial_t v(t,\xi(t)) \quad \big(\text{respectively} \;\; M'(t) = \partial_x\partial_t v(t,\zeta(t))\big).$$

This lemma was first proved in [8] for m only and in the case $A = \mathbb{R}$. However, since the proof relies on local arguments and on the existence of $\xi(t)$, it works for $A = \mathbb{T}$ as well. Besides, changing v into $-v$, we obtain the desired result for M.

Let us tackle the proof of Theorem 3.2 in the case $u_0 \in H^3$. According to Theorem 2.3 and Proposition 2.6, (CH) has a unique maximal solution $u \in C([0,T^\star_{u_0});H^3) \cap C^1([0,T^\star_{u_0});H^2)$ with constant H^1 norm, and $T^\star_{u_0}$ satisfies (2.18). In view of Lemma 3.3 applied to $v = u$, we have for almost every $t \in (0,T^\star_{u_0})$,

$$\frac{dm}{dt} + \frac{m^2}{2} = u^2(t,\xi(t)) - \Big(p \star \Big(u^2 + \frac{(\partial_x u)^2}{2}\Big)\Big)(t,\xi(t)), \tag{3.10}$$

$$\frac{dM}{dt} + \frac{M^2}{2} = u^2(t,\zeta(t)) - \Big(p \star \Big(u^2 + \frac{(\partial_x u)^2}{2}\Big)\Big)(t,\zeta(t)). \tag{3.11}$$

In the case $A = \mathbb{R}$, it was proved in [8] that

$$p \star \Big(u^2 + \frac{(\partial_x u)^2}{2}\Big) \ge \frac{u^2}{2}. \tag{3.12}$$

Since in the case $A = \mathbb{T}$ the kernel of $(1 - \partial_{xx}^2)^{-1}$ satisfies

$$p(x) = \frac{\mathrm{ch}(x - [x] - 1/2)}{2\mathrm{sh}(1/2)} = \sum_{k \in \mathbb{Z}} \frac{e^{-|x-k|}}{2},$$

it is easy to prove that (3.12) still holds.

From (3.11) and (3.12), we now gather that for almost every $t \in (0, T_{u_0}^\star)$,

$$\frac{dM}{dt} + \frac{M^2}{2} \le \frac{u^2}{2}(t, \zeta(t)).$$

In view of inequality (2.20) and as $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$, we get

$$\frac{dM}{dt} + \frac{M^2}{2} \le \frac{\|u_0\|_{H^1}^2}{4}. \tag{3.13}$$

The continuity of $\partial_x u$ on $A \times [0, T_{u_0}^\star)$ obviously entails the continuity of M. Therefore, according to (3.13), we easily gather (by contradiction) that $M(t) \le \|u_0\|_{H^1}/\sqrt{2}$ if $M(0) \le \|u_0\|_{H^1}/\sqrt{2}$.

If $M(0) > \|u_0\|_{H^1}/\sqrt{2}$, there exists a maximal $T^\star \le T_{u_0}^\star$ such that $M > \|u_0\|_{H^1}/\sqrt{2}$ on $[0, T^\star)$. According to (3.13), M is nonincreasing on this interval. More precisely, denoting $N(t) = \sqrt{2}M(t)/\|u_0\|_{H^1}$, we have for almost every $t < T^\star$,

$$\frac{2N'(t)}{1 - N^2(t)} \ge \frac{\|u_0\|_{H^1}}{\sqrt{2}},$$

so that routine computations yield

$$M(t) \le \frac{\|u_0\|_{H^1}}{\sqrt{2}} \Big(\frac{\sqrt{2}M(0) + \|u_0\|_{H^1} \tanh(\frac{\|u_0\|_{H^1} t}{2\sqrt{2}})}{\|u_0\|_{H^1} + \sqrt{2}M(0) \tanh(\frac{\|u_0\|_{H^1} t}{2\sqrt{2}})} \Big).$$

Now, if $T^\star < T_{u_0}^\star$, we then have $M(t) \le \|u_0\|_{H^1}/\sqrt{2}$ on $[T^\star, T_{u_0}^\star)$. This achieves the proof of (3.7) in the case $u_0 \in H^3$. Combining (2.18) and (3.7), we readily gather the explosion criterion (3.9).

Let us now turn to the proof of the bound from below for $T_{u_0}^\star$. Since $\|p\|_{L^\infty} \le c/2$ with c given in the statement of Theorem 3.2, we obviously have

$$\frac{dm}{dt} + \frac{m^2}{2} \ge -\frac{c}{2}\|u(t)\|_{H^1}^2 = -\frac{c}{2}\|u_0\|_{H^1}^2.$$

After a time integration, we deduce that, for all $t<\min(T^\star_{u_0},T_{u_0})$,

$$\arctan\frac{m(t)}{\sqrt{c}\|u_0\|_{H^1}}\geq\arctan\frac{m(0)}{\sqrt{c}\|u_0\|_{H^1}}-\frac{\sqrt{c}\|u_0\|_{H^1}}{2}t,$$

or in other words,

$$-m(t)\leq\frac{\sqrt{c}\|u_0\|_{H^1}\tan\Bigl(\frac{\sqrt{c}\|u_0\|_{H^1}}{2}t\Bigr)-m(0)}{1+\frac{m(0)}{\sqrt{c}\|u_0\|_{H^1}}\tan\Bigl(\frac{\sqrt{c}\|u_0\|_{H^1}}{2}t\Bigr)}. \tag{3.14}$$

Therefore, $\int_0^t(\min_{x\in A}\partial_x u(\tau))\,d\tau$ remains bounded as long as $t<\min(T^\star_{u_0}, T_{u_0})$. In view of (3.9), this implies that $T^\star_{u_0}\geq T_{u_0}$.

Step 2: The case of rough data. Let us consider a $u_0\in B^s_{p,r}\cap H^1$ with $s>3/2$ and $B^s_{p,r}\subset\mathrm{Lip}$. Existence and uniqueness in $E^s_{p,r}(T)$ for a small positive T is given by Theorem 2.3. We also know according to Proposition 2.6 that the H^1 norm is constant on the whole interval $[0,T^\star_{u_0})$. We now want to prove that $T^\star_{u_0}$ is bounded from below according to (3.8).

Let u^n be the solution corresponding to the approximate initial data $u_0^n\stackrel{\text{def}}{=}S_nu_0$. Since in particular, $u_0\in B^{1+\frac1p}_{p,1}\cap H^1$, we gather from the definition of Besov spaces that $(u_0^n)_{n\in\mathbb{N}}$ tends to u_0 in $\mathrm{Lip}\cap H^1$ so that $T^\star_{u_0^n}$ tends to $T^\star_{u_0}$. Note also that $(u_0^n)_{n\in\mathbb{N}}$ is uniformly bounded in $B^s_{p,r}$.

Let us fix an $\epsilon\in(0,1)$. For a N large enough, we have

$$n\geq N\Longrightarrow T_{u_0^n}>T_\epsilon\stackrel{\text{def}}{=}-\frac{2}{\|u_0\|_{H^1}}\arctan\Bigl(\frac{(1-\epsilon)\|u_0\|_{H^1}}{m(0)}\Bigr).$$

According to step 1, we thus have $T^\star_{u_0^n}>T_\epsilon$ and for $t\in[0,T_\epsilon]$,

$$\begin{aligned}\sup_{x\in A}\partial_x u^n(t)&\leq\frac{\|u_0^n\|_{H^1}}{\sqrt2}\max\Bigl(1,\frac{\sqrt2\sup_{x\in A}\partial_x u_0^n(x)+\|u_0^n\|_{H^1}\tanh(\frac{\|u_0^n\|_{H^1}t}{2\sqrt2})}{\|u_0^n\|_{H^1}+\sqrt2\sup_{x\in A}\partial_x u_0^n(x)\tanh(\frac{\|u_0^n\|_{H^1}t}{2\sqrt2})}\Bigr),\\&\leq C\max\Bigl(\frac{\|u_0\|_{H^1}}{\sqrt2},M(0)\Bigr).\end{aligned}\tag{3.15}$$

On the other hand, taking N larger if needed, we have according to (3.14) and for $n\geq N$

$$\begin{aligned}-\inf_{x\in A}\partial_x u^n(t)&\leq\frac{\|u_0^n\|_{H^1}\tan\bigl(\frac{\|u_0^n\|_{H^1}}{2}t\bigr)-(\inf_{x\in A}\partial_x u_0^n(x))}{1+\frac{(\inf_{x\in A}\partial_x u_0^n(x))}{\|u_0^n\|_{H^1}}\tan\bigl(\frac{\|u_0^n\|_{H^1}}{2}t\bigr)},\\&\leq\frac{2}{\epsilon}\Bigl(\frac{\|u_0\|^2_{H^1}+m(0)^2}{-m(0)}\Bigr).\end{aligned}\tag{3.16}$$

From (3.16), (3.15) and Lemma 2.4, we get uniform bounds for $(u^n)_{n\in\mathbb{N}}$ in $C([0,T_\epsilon];B^s_{p,r})\cap C^1([0,T_\epsilon];B^{s-1}_{p,r})$. We can therefore conclude exactly as in the proof of Theorem 2.3 that, up to an extraction, $(u^n)_{n\in\mathbb{N}}$ converges weakly to a solution $u\in E^s_{p,r}(T_\epsilon)$ of (CH) which satisfies (3.7) on $[0,T_\epsilon]$. Since $s>3/2$, uniqueness holds true so that we conclude, having ϵ tend to zero, that $T^\star_{u_0}\geq T_{u_0}$ and that (3.7) is satisfied on $[0,T_{u_0})$.

Let us prove that (3.7) holds on $[0,T^\star_{u_0})$. Let $[0,T^\star]$ be the largest interval on which (3.7) holds,

$$\widetilde{m}=\inf_{(t,x)\in[0,T^\star]\times A}\partial_x u(t,x)\quad\text{and}\quad \widetilde{T}=-\frac{2}{\sqrt{c}\|u_0\|_{H^1}}\arctan\Big(\frac{\sqrt{c}\|u_0\|_{H^1}}{\widetilde{m}}\Big).$$

Assuming that $T^\star<T^\star_{u_0}$, we know that $\|\partial_x u(t)\|_{L^\infty}$ is uniformly bounded on $[0,T^\star]$ so that $\widetilde{m}>-\infty$. On the other hand, if we define $v_0\stackrel{\text{def}}{=}u(T^\star-\widetilde{T}/2)$, we have just proved that the corresponding solution v of (CH) satisfies (3.7) on $[0,T^\star_{v_0})$, thus, on $[0,\widetilde{T})$. In view of uniqueness, this means that u satisfies (3.7) on $[0,T^\star+\widetilde{T}/2)$, which contradicts the definition of $\widetilde{T}$. Therefore, $T^\star=T^\star_{u_0}$. Now, we can conclude thanks to (2.18) that the explosion criterion reduces to (3.9).

Last step: The estimate (3.8) is sharp. Let us notice that, as u_0 is in $B^{1+\frac{1}{p}}_{p,1}$, u_0 and $\partial_x u_0$ are continuous and tend to zero at infinity. Since we assumed that $u_0\not\equiv 0$, the term $m(0)$ is negative so that $T_{u_0}\in(0,+\infty]$.

The fact that the lower bound above is sharp may be deduced from Theorem 4.1 in [7]. Indeed, it is proved there in the case $A=\mathbb{R}$ (but it is easy to extend the results to $A=\mathbb{T}$) that for any odd $u_0\in H^3$ such that $\partial_x u_0(0)<0$, we have $T^\star_{u_0}\leq -2(\partial_x u_0(0))^{-1}$ (actually there is a missprint in the statement of the theorem; the right inequality is at the last line of page 319). Let us choose a $u_0\in H^3$ such that $\partial_x u_0(0)=\inf_{x\in A}\partial_x u_0(x)<0$ (for example $u_0(x)=-xe^{-x^2}$ in the case $A=\mathbb{R}$) and define $u_0^n(x)\stackrel{\text{def}}{=}n^{-1/2}u_0(nx)$ for $n\geq 1$. We clearly have

$$\partial_x u_0^n(0)=-n^{1/2}\partial_x u_0(0)\quad\text{and}\quad \|u_0^n\|_{H^1}\longrightarrow_{n\to+\infty}\|\partial_x u_0\|_{L^2}$$

so that

$$\liminf_{n\to+\infty}\frac{T_{u_0^n}}{T^\star_{u_0^n}}\geq\lim_{n\to+\infty}\Big(\frac{-\partial_x u_0^n(0)}{2}\Big)T_{u_0^n}=1.\quad\square$$

Remark. If $u_0 \in B^s_{p,r} \cap H^1$ with $B^s_{p,r} \subset \mathrm{Lip}$ and $s > 1$, the proof above shows that for any $T < T_{u_0}$, (CH) has a solution in $E^s_{p,r}(T) \cap L^\infty(0,T;H^1)$ satisfying (3.7). If $s \leq 3/2$, the question of uniqueness remains open, though.

We end this section by showing that under a sign assumption for the potential $y_0 \stackrel{\text{def}}{=} u_0 - \partial^2_{xx} u_0$, (CH) has a unique global solution. This result finds its origin in [6] where the regularity assumption $u_0 \in H^3$ was needed.

Theorem 3.4. *Suppose that $B^s_{p,r} \subset \mathrm{Lip}$ and that $u_0 \in B^s_{p,r} \cap H^1$. If $y_0 \stackrel{\text{def}}{=} u_0 - \partial^2_{xx} u_0$ has a sign, then (CH) has a global solution $u \in E^s_{p,r} \cap L^\infty(\mathbb{R}^+;H^1)$. If $s > 3/2$, then uniqueness holds in $E^s_{p,r}$. Moreover, $\|u(t)\|_{H^1}$ is a constant and $y(t)$ has the sign of y_0.*

Proof. Approximate the initial datum u_0 with a nonnegative, smooth, compactly supported mollifier ρ such that $\int \rho = 1$, and set $u^n_0 \stackrel{\text{def}}{=} \rho^n \star u_0$ with $\rho^n(x) = n^d \rho(nx)$. Since u_0 is in $H^1 \cap B^s_{p,r}$, we have (for instance) $u^n_0 \in H^4 \cap B^s_{p,r}$. According to Theorem 2.3, (CH) has a unique classical solution $u^n \in \cap_{T<T^n} \big(E^4_{2,2}(T) \cap E^s_{p,r}(T)\big)$ with constant H^1 norm.

On the other hand, ρ is nonnegative so that $y^n_0 \stackrel{\text{def}}{=} u^n_0 - \partial^2_{xx} u^n_0 = \rho^n \star y_0$ has the sign of y_0. Moreover, if we denote by ψ^n the flow of u^n (i.e. the solution to $\psi^n(t,x) = x + \int_0^t u^n(\tau, \psi^n(\tau,x))\, d\tau$), we have according to [6],

$$\forall t \in [0,T^n), \forall x \in A,\ y^n_0(x) = y^n(t, \psi^n(t,x))(\partial_x \psi^n(t,x))^2 \tag{3.17}$$

so that $\mathrm{sgn} y^n(t) = \mathrm{sgn} y^n_0 = \mathrm{sgn} y_0$ for all $t \in [0,T^n)$. Therefore, using the explicit expression of $u^n = p \star y^n$ and $\partial_x u^n = \partial_x p \star y^n$ with p given in (3.2), we readily get

$$\|\partial_x u^n\|_{L^\infty} \leq \|u^n\|_{L^\infty},$$

whence

$$\|\partial_x u^n(t)\|_{L^\infty} \leq \|u^n(t)\|_{L^\infty} \leq \|u^n(t)\|_{H^1} = \|u^n_0\|_{H^1} \leq \|u_0\|_{H^1}. \tag{3.18}$$

Therefore $T^n = +\infty$. Moreover, according to Lemma 2.4 and (3.18), we have

$$\forall t \in \mathbb{R}^+,\ \|u^n(t)\|_{B^s_{p,r}} \leq \|u^n_0\|_{B^s_{p,r}} e^{C\int_0^t \|u(\tau)\|_{\mathrm{Lip}}\, d\tau} \leq \|u_0\|_{B^s_{p,r}} e^{Ct\|u_0\|_{H^1}},$$

and we can conclude that $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}$.

Following the end of the proof of Theorem 2.3, the above result suffices to get that, up to an extraction, $(u^n)_{n\in\mathbb{N}}$ tends weakly to a global solution $u \in E^s_{p,r} \cap L^\infty(\mathbb{R}^+; H^1)$ of (CH) such that $\mathrm{sgn} y(t) = \mathrm{sgn} y_0$. If $s > 3/2$, uniqueness stems from Proposition 2.1, and it was proved in Proposition 2.6 that $\|u(t)\|_{H^1}$ is a constant. $\square$

4. Lower semicontinuity of existence time. In this section, we address the question of lower semicontinuity of existence time with respect to sufficiently smooth initial data. We shall prove the following:

Theorem 4.1. *Assume that $v_0 \in B^s_{p,r}$ where (s, p, r) are such that $B^s_{p,r} \subset$ Lip and $s > 3/2$. Let $u_0 \in B^{2+\frac{1}{p}}_{p,1}$ and $T < T^\star_{u_0}$. Then there exists a constant $C = C(p)$ such that if*

$$\|v_0 - u_0\|_{B^{\frac{1}{p}+1}_{p,1}} < \frac{1}{C\int_0^T \exp\Bigl(C\int_0^\tau \|u(\tau')\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau'\Bigr)d\tau} \tag{4.1}$$

then (CH) *has a* (*unique*) *solution $v \in E^s_{p,r}(T)$ with initial datum v_0.*

Proof. Let us remark first that the right-hand side of (4.1) is positive. Indeed, it may be bounded from below by $(CT\exp\{C\int_0^T \|u(t)\|_{B^{\frac{1}{p}+2}_{p,1}}\,dt\})^{-1}$.

Let v be the maximal solution of (CH) with initial datum v_0 (whose existence is ensured by Theorem 2.3). Denoting $w \stackrel{\text{def}}{=} v - u$, we gather from (CH) that

$$\partial_t w + (u + w)\partial_x w = -w\partial_x u + F(u, w) \tag{4.2}$$

with $F(u, w) = P(D)(w^2 + 2uw + (\partial_x w)^2/2 + \partial_x u\partial_x w)$.

Denote $T^\star \stackrel{\text{def}}{=} \min(T^\star_{u_0}, T^\star_{v_0})$ (where $T^\star_{u_0}$ and $T^\star_{v_0}$ are the lifespans of u and v), and let us turn to the proof of estimates for w in $C([0, T^\star); B^{\frac{1}{p}+1}_{p,1})$. Apply inequality (A.10) in the appendix. We get for $0 \le t < T^\star$,

$$\begin{aligned}\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}} \le \|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} &+ \int_0^t \|w\partial_x u(\tau)\|_{B^{\frac{1}{p}+1}_{p,1}}\,d\tau + \int_0^t \|F(u(\tau), w(\tau))\|_{B^{\frac{1}{p}+1}_{p,1}}\,d\tau\\ &+ C\int_0^t (\|\partial_x u(\tau)\|_{B^{\frac{1}{p}}_{p,1}} + \|\partial_x w(\tau)\|_{B^{\frac{1}{p}}_{p,1}})\|w(\tau)\|_{B^{\frac{1}{p}+1}_{p,1}}\,d\tau;\end{aligned}$$

hence, thanks to Proposition 1.5,

$$\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}} \le \|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} + C\int_0^t \Bigl(\|w(\tau)\|_{B^{\frac{1}{p}+1}_{p,1}} + \|u(\tau)\|_{B^{\frac{1}{p}+2}_{p,1}}\Bigr)\|w(\tau)\|_{B^{\frac{1}{p}+1}_{p,1}}\,d\tau.$$

According to Gronwall's lemma, we infer that

$$\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}} \leq e^{C\int_0^t (\|u(\tau')\|_{B^{\frac{1}{p}+2}_{p,1}}+\|w(\tau')\|_{B^{\frac{1}{p}+1}_{p,1}})\,d\tau'} \|w_0\|_{B^{\frac{1}{p}+1}_{p,1}}. \tag{4.3}$$

Denoting $W(t) = \exp\bigl(-C\int_0^t \|w(\tau)\|_{B^{\frac{1}{p}+1}_{p,1}}\,d\tau\bigr)$ and

$$G(t) = C\|w_0\|_{B^{1+\frac{1}{p}}_{p,1}} \exp\bigl(C\int_0^t \|u(\tau)\|_{B^{\frac{1}{p}+2}_{p,1}}\bigr)\,d\tau,$$

inequality (4.3) reads

$$W' \geq -G. \tag{4.4}$$

Let

$$T^{\star\star} \stackrel{\mathrm{def}}{=} \min\Bigl\{0 \leq t \leq T^\star_{u_0} : C\|w_0\|_{B^s_{p,r}}\int_0^t \exp\bigl(C\int_0^\tau \|u(\tau')\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau'\bigr)d\tau \geq 1\Bigr\}.$$

Integrating (4.4) between 0 and t and plugging the resulting inequality in (4.3), we infer that $\forall t < \min(T^\star, T^{\star\star})$,

$$\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}} \leq \frac{\|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} \exp\Bigl(C\int_0^t \|u(\tau)\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau\Bigr)}{1 - C\|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} \int_0^t \exp\Bigl(C\int_0^\tau \|u(\tau')\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau'\Bigr)d\tau}. \tag{4.5}$$

Assumption (4.1) clearly entails that $T < T^{\star\star}$. Let us argue by contradiction and assume that $T^\star_{v_0} \leq T$. Then, according to (4.5), we have, for $t < T^\star_{v_0}$,

$$\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}} \leq \frac{\|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} \exp\Bigl(C\int_0^T \|u(\tau)\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau\Bigr)}{1 - C\|w_0\|_{B^{\frac{1}{p}+1}_{p,1}} \int_0^T \exp\Bigl(C\int_0^\tau \|u(\tau')\|_{B^{\frac{1}{p}+2}_{p,1}}\,d\tau'\Bigr)d\tau} < +\infty$$

so that $\|w(t)\|_{B^{\frac{1}{p}+1}_{p,1}}$ is uniformly bounded on $[0, T^\star_{v_0})$. Since $B^{1+\frac{1}{p}}_{p,1} \hookrightarrow \mathrm{Lip}$, Proposition 2.6 shows that v may be extended beyond $T^\star_{v_0}$. $\square$

Remark. Though the solution associated with $u_0 = 0$ is global, one can find arbitrary small initial data for which the corresponding maximal solution is not global (see [6], [7], [8] and [16]).

5. Weak solutions. The results of the previous parts are not completely satisfactory since they do not yield global existence of solitons for instance. Indeed, as explained in the introduction, for these latter, y_0 is a Dirac measure so that $u_0 = (1-\partial_{xx}^2)^{-1}y_0$ is in every $B^{1+\frac{1}{p}}_{p,\infty}$ $(1 \le p \le +\infty)$ but not in $B^{1+\frac{1}{p}}_{p,1}$. This motivates the study of the initial value problem with data u_0 such that y_0 is in $\mathcal{M}$. Clearly, Lemma 1.3 ensures us that u_0 is in the space $B^2_{1,\infty}$ and, in addition, $u_0 \in \mathrm{Lip} \cap H^1$. Therefore, Theorem 2.3 yields local existence and uniqueness of a solution u in $E^2_{1,\infty}(T)$ with constant energy. It would be interesting to prove a more precise result, namely that $y(t)$ stays in $\mathcal{M}$, and to study whether the same explosion criterion as in Theorem 3.2 holds. Under the sign assumption for y_0, we shall obtain as in Theorem 3.4 that the solution is global. This latter result has been proved before by A. Constantin and L. Molinet in [9]. Our main result is the following:

Theorem 5.1. *Suppose that y_0 belongs to $\mathcal{M}$. Then (CH) has a unique maximal solution $u \in L^\infty_{loc}(0, T^\star_{u_0}; B^2_{1,\infty} \cap \mathrm{Lip}) \cap \mathrm{Lip}_{loc}([0, T^\star_{u_0}); B^1_{1,\infty} \cap L^\infty)$. In addition, the energy is conserved, $y(t) \stackrel{\mathrm{def}}{=} (u - \partial^2_{xx}u)(t)$ stays in $\mathcal{M}$ uniformly on every compact interval of $[0, T^\star_{u_0})$ and*

$$\sup_{(t,x)\in[0,T^\star_{u_0})\times A} \partial_x u(t,x) \le \max\Big(\frac{\|u_0\|_{H^1}}{\sqrt{2}}, \sup_{x\in A} \partial_x u_0(x)\Big). \tag{5.1}$$

We also have

$$T^\star_{u_0} \ge \frac{2}{\|y_0\|_{\mathcal{M}}} \quad \textit{and} \quad T^\star_{u_0} < +\infty \Longrightarrow \int_0^{T^\star_{u_0}} \big(\inf_{x\in A} \partial_x u(t,x)\big)\, dt = -\infty.$$

If y_0 has a definite sign, then the solution is global and $\|y(t)\|_{\mathcal{M}} = \|y_0\|_{\mathcal{M}}$ for all $t \ge 0$.

Proof. Uniqueness is given by Proposition 2.1.

Step 1: Existence on a small time interval. Let us fix a $T < 2\|y_0\|^{-1}_{\mathcal{M}}$. We aim at building a solution on $[0,T]$ with constant energy, which satisfies (5.1) and $y(t) \in \mathcal{M}$ uniformly on $[0,T]$. To achieve it, approximate the initial data as in the proof of Theorem 3.4. Obviously u_0^n belongs to $H^\infty \stackrel{\mathrm{def}}{=} \cap_{s\in\mathbb{R}} H^s$, tends to u_0 in the sense of the distributions and $y_0^n \stackrel{\text{def}}{=} u_0^n - \partial_{xx}^2 u_0^n = \rho^n \star y_0$ so that

$$\|y_0^n\|_{L^1} \leq \|y_0\|_{\mathcal{M}}. \tag{5.2}$$

Note also that $\|u_0^n\|_{H^1} \leq \|y_0^n\|_{L^1} \leq \|y_0\|_{\mathcal{M}}$. According to Proposition 2.3, (CH) has a unique maximal solution $u^n \in C([0,T^n); H^\infty)$ with initial datum u_0^n and constant energy. On the other hand, $y^n \stackrel{\text{def}}{=} u^n - \partial_{xx}^2 u^n$ solves

$$\partial_t y^n + \partial_x(u^n y^n) = -y^n \partial_x u^n$$

so that

$$\partial_t |y^n| + \partial_x(u^n |y^n|) = -|y^n| \partial_x u^n. \tag{5.3}$$

Integrating in space and time, we get

$$\|y^n(t)\|_{L^1} \leq \|y_0^n\|_{L^1} - \int_0^t \|y^n(\tau)\|_{L^1} \left(\inf_{x\in A} \partial_x u^n(\tau)\right) d\tau, \tag{5.4}$$

whence

$$\|y^n(t)\|_{L^1} \leq \|y_0^n\|_{L^1} \, e^{-\int_0^t (\inf_{x\in A} \partial_x u^n(\tau))\, d\tau}. \tag{5.5}$$

Now, let us notice that, as $\partial_x u^n = \partial_x p * y^n$ with p given in (3.2), we have

$$\|\partial_x u^n\|_{L^\infty} \leq \frac{1}{2} \|y^n\|_{L^1}. \tag{5.6}$$

Plugging (5.2) and (5.6) in (5.5), and performing an explicit integration, we get

$$\forall t < \min(T^n, 2\|y_0\|_{\mathcal{M}}^{-1}),\ \|y^n(t)\|_{L^1} \leq \frac{2\|y_0\|_{\mathcal{M}}}{2 - t\|y_0\|_{\mathcal{M}}}. \tag{5.7}$$

Coming back to (5.6), this gives us a uniform control on $\|\partial_x u^n\|_{L^\infty}$ for $t < \min(T^n, 2\|y_0\|_{\mathcal{M}}^{-1})$. This means, according to Proposition 2.6, that $T^n \geq 2\|y_0\|_{\mathcal{M}}^{-1} > T$. We therefore have uniform estimates for $\partial_x u^n$ in $L^\infty([0,T] \times A)$. Moreover, according to Theorem 3.2,

$$\begin{aligned}
\sup_{(t,x)\in[0,T^n)\times A} \partial_x u^n(t,x) &\leq \max\Big(\frac{\|u_0^n\|_{H^1}}{\sqrt{2}}, \sup_{x\in A} \partial_x u_0^n(x)\Big) \\
&\leq \max\Big(\frac{\|u_0\|_{H^1}}{\sqrt{2}}, \sup_{x\in A} \partial_x u_0(x)\Big).
\end{aligned}$$

We conclude that the sequence $(y^n)_{n\in\mathbb{N}}$ is uniformly bounded on $L^\infty(0,T;L^1)$. According to Lemma 1.3, we thus get that $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^2_{1,\infty})$. Coming back to equation (CH), we infer that $(\partial_t u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^1_{1,\infty})$ so that, finally, $(u^n)_{n\in\mathbb{N}}$ is also uniformly bounded in $\mathrm{Lip}([0,T];B^1_{1,\infty})$.

Since the injection $B^2_{1,\infty}\hookrightarrow B^1_{1,\infty}$ is locally compact, Ascoli's theorem followed by a diagonalization process shows us that, up to an extraction, $(u_n)_{n\in\mathbb{N}}$ tends weakly to a limit $u\in\mathrm{Lip}([0,T];B^1_{1,\infty})$. Coming back to the uniform bounds for the sequence, we gather that $u\in L^\infty(0,T;B^2_{1,\infty})$ has constant energy, that $y(t)$ is in $\mathcal{M}$ uniformly on $[0,T]$ and that (5.1) is fulfilled on $[0,T]$.

Interpolation arguments enable us to prove that convergence holds locally in any space $L^\infty(0,T;B^s_{1,\infty})$ such that $s<2$, which suffices to check that u is indeed a solution to (CH). Having T tend to $2\|y_0\|^{-1}_{\mathcal{M}}$ and using uniqueness, we can conclude that $T^\star_{u_0}\geq 2\|y_0\|^{-1}_{\mathcal{M}}$.

Step 2: The potential $y(t)$ stays in $\mathcal{M}$ locally uniformly on $[0,T^\star_{u_0})$. Step 2 stems from Proposition 2.6 and from the following lemma:

Lemma 5.2. *Assume that $y_0\in\mathcal{M}$, $u\in E^2_{1,\infty}(T)$ solves* (CH) *on $[0,T]$ and that $y\in L^\infty(0,T;\mathcal{M})$. Then there exists a universal constant C such that on $[0,T]$, we have*

$$\|y(t)\|_{\mathcal{M}}\leq\|y_0\|_{\mathcal{M}}e^{\int_0^t\|\partial_x u(\tau)\|_{L^\infty}\,d\tau}. \tag{5.8}$$

Proof. Let ρ be the compactly supported mollifier used in Theorem 3.4. Let us apply the operator $(1-\partial^2_{xx})\rho^n\star$ to (CH). Denoting $y^n=\rho^n\star y$ and $u^n=\rho^n\star u$, we gather

$$\partial_t y^n+(1-\partial^2_{xx})\big(\rho^n\star u\partial_x u\big)+(1-\partial^2_{xx})\big(\rho^n\star\partial_x\big(u^2+\frac{(\partial_x u)^2}{2}\big)\big)=0.$$

Using the fact that $u(t)\in\mathrm{Lip}$ and that $y(t)\in\mathcal{M}$, we can write $\partial_x(u\partial_x u)\equiv u\partial^2_{xx}u+(\partial_x u)^2$ in the sense of the distributions. After some computations, this implies that

$$\partial_t y^n+u^n\partial_x u^n+\partial_x(u^n y^n)-\partial_x\big(\frac{(\partial_x u^n)^2}{2}\big)=S^n_1+S^n_2+S^n_3 \tag{5.9}$$

with

$$S^n_1=u^n\partial_x u^n-\rho^n\star(u\partial_x u),\quad S^n_2=\partial_x(u^n y^n)-\partial_x\big(\rho^n\star(uy)\big),$$
$$S^n_3=\partial_x\big(\rho^n\star\frac{(\partial_x u)^2}{2}\big)-\partial_x\big(\frac{(\partial_x u^n)^2}{2}\big).$$

According to Lemma A.4 and to Lemma A.5 in the appendix, we have

$$\|S_1^n\|_{L^1} \le C\,\|\partial_x u\|_{L^\infty}\,\|u\|_{L^1}\,, \quad \|S_2^n\|_{L^1} \le C\,\|\partial_x u\|_{L^\infty}\,\|y\|_{\mathcal{M}},$$
$$\|S_3^n\|_{L^1} \le C\,\|\partial_x u\|_{L^\infty}\,\|\partial_{xx}^2 u\|_{\mathcal{M}}.$$

Since obviously $\|u\|_{L^1} \le \|y\|_{\mathcal{M}}$ and $\|\partial_{xx}^2 u\|_{\mathcal{M}} \le 2\|y\|_{\mathcal{M}}$, we have

$$\|S_1^n + S_2^n + S_3^n\|_{L^1} \le C\,\|\partial_x u\|_{L^\infty}\,\|y\|_{\mathcal{M}}.$$

On the other hand, from (5.9), we get

$$\partial_t|y^n| + \partial_x(u^n|y^n|) = -|y^n|\partial_x u^n + (S_1^n + S_2^n + S_3^n)\mathrm{sgn}(y^n)$$

so that standard computations based on Gronwall's lemma lead to the desired estimate. □

Given a solution $u \in L^\infty_{loc}(0, T^\star_{u_0}; B^2_{1,\infty} \cap \mathrm{Lip}) \cap \mathrm{Lip}_{loc}(0, T^\star_{u_0}; B^1_{1,\infty} \cap L^\infty)$ with initial data u_0 such that $y_0 \in \mathcal{M}$, we are now going to show that (5.8) is satisfied on $[0, T^\star_{u_0})$.

Combining step 1, uniqueness and Lemma 5.2, we gather that

$$\text{(5.8) holds on} \quad [0, 2\|y_0\|_{\mathcal{M}}^{-1}). \tag{5.10}$$

Assume that it holds on $[0, T^\star)$ for a $T^\star < T^\star_{u_0}$. Lemma 5.2 tells us that $\|y(t)\|_{\mathcal{M}}$ is uniformly bounded by a constant $M_{T^\star}$ on $[0, T^\star)$. Let $\epsilon \stackrel{\text{def}}{=} M_{T^\star}^{-1}$. According to (5.10), the solution $\widetilde{u}$ corresponding to the initial datum $u(T^\star - \epsilon/2)$ satisfies (5.8) on $[0, \epsilon]$. Uniqueness ensures that $u(t) = \widetilde{u}(t - T^\star + \epsilon/2)$ so that (5.8) holds on $[0, T^\star + \epsilon/2]$. Therefore we can conclude that (5.8) is satisfied on $[0, T^\star_{u_0})$.

Step 3: End of the proof of the local result. The conservation of the energy up to time $T^\star_{u_0}$ has been proved in Proposition 2.6. On the other hand, it is known that (5.1) holds on a small nontrivial interval $[0, T]$. Arguing as in step 2 for the proof of (5.8) on $[0, T^\star_{u_0}[$, we conclude that (5.1) holds on $[0, T^\star_{u_0})$.

Now, Proposition 2.6 gives us the wanted explosion criterion.

Last step: Global solutions. Let us assume that y_0 has a definite sign. Building a sequence $(u^n)_{n\in\mathbb{N}}$ of approximate solutions according to step 1, we get from Theorem 3.4 that $\mathrm{sgn}y^n(t) = \mathrm{sgn}y_0^n = \mathrm{sgn}y_0$. Therefore, integrating (5.3), we infer that

$$\|y^n(t)\|_{L^1} = \|y^n(t)\|_{L^1} \le \|y_0\|_{\mathcal{M}}. \tag{5.11}$$

According to (5.6), and to Proposition 2.6, we conclude that the solution u^n is global. Moreover, thanks to (5.11), we get uniform bounds in $L^\infty_{loc}(\mathbb{R}^+; B^2_{2,\infty})$ for $(u^n)_{n\in\mathbb{N}}$, and we can conclude exactly as in step one, to the existence of a global solution $u \in E^2_{1,\infty}$. Thanks to (5.11), we have in addition $\|y(t)\|_{\mathcal{M}} \leq \|y_0\|_{\mathcal{M}}$. Obviously, we would have obtained similar global results by solving (CH) backwards. Indeed, this amounts to changing u_0 into $-u_0$. Therefore, in view of uniqueness, $\|y(t)\|_{\mathcal{M}}$ is conserved. $\square$

Remark 5.3. By Sobolev embeddings and interpolation, we have $u \in C([0, T^\star_{u_0}); H^s) \cap C^1([0, T^\star_{u_0}); H^{s-1})$ for any $s < 3/2$, which completes the proof of Theorem 0.2.

Remark 5.4. The estimate from below $T^\star_{u_0} \geq 2\|u_0\|^{-1}_{\mathcal{M}}$ is probably not sharp: indeed, it stems from inequality (5.4), which is quite rough. On the other hand, the "best" estimate from below that we can hope is $T^\star_{u_0} \geq 4\|u_0\|^{-1}_{\mathcal{M}}$. Indeed, if $A = \mathbb{R}$, the sequence of initial data

$$u_0^n(x) = -\frac{n\mathrm{sgn}(x)}{n^2-1}\left(e^{-|x|} - e^{-n|x|}\right)$$

gives a sequence of solutions which satisfy $\|y_0^n\|_{L^1} = 2$ and, according to Theorem 4.1 in [7], blows up before the time $2 + 2n^{-1}$.

Appendix.

A.1. Linear transport equations. This section is devoted to the statement of estimates in general Besov spaces for d-dimensional linear transport equations:

$$\partial_t f + v \cdot \nabla f = F, \qquad f_{t=0} = f_0, \tag{$\mathcal{T}$}$$

where the vector field v has bounded space derivatives. Such estimates are standard in the framework of Sobolev or Hölder spaces with positive regularity indices. The case of negative regularity indices for Hölder spaces has been tackled in [5], chapter 4.

Here is the statement in general Besov spaces:

Proposition A.1. *Suppose that $(p, r) \in [1, +\infty]^2$ and $s > -d/p$. Let v be a vector field such that ∇v belongs to $L^1(0, T; B^{s-1}_{p,r})$ if $s > d/p+1$ or to $L^1(0, T; B^{d/p}_{p,r} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B^s_{p,r}$, $F \in L^1(0, T; B^s_{p,r})$ and that $f \in L^\infty(0, T; B^s_{p,r}) \cap C([0, T]; \mathcal{S}')$ solves ($\mathcal{T}$). Then there exists a constant C depending only on s, p and d, and such that the following inequalities hold:*

1) *If $r = 1$ or $s \neq d/p + 1$,*

$$\|f(t)\|_{B^s_{p,r}} \leq e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}}\,d\tau\right) \tag{A.1}$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{d/p}_{p,r}\cap L^\infty}\,d\tau$ if $s < d/p + 1$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}}\,d\tau$ else.

2) *If $s \le d/p+1$ and, in addition, $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty(0,T\times A^d)$ and $\nabla F \in L^1(0,T;L^\infty)$ then*

$$\|f(t)\|_{B^s_{p,r}} + \|\nabla f(t)\|_{L^\infty} \le e^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \|\nabla f_0\|_{L^\infty} \\ + \int_0^t e^{-CV(\tau)}\big(\|F(\tau)\|_{B^s_{p,r}} + \|\nabla F(\tau)\|_{L^\infty}\,\big)\,d\tau\Big) \tag{A.2}$$

with $V(t)=\int_0^t \|\nabla v(\tau)\|_{B^{d/p}_{p,r}\cap L^\infty}\,d\tau$.

If $r<+\infty$, then $f\in C([0,T];B^s_{p,r})$. If $r=+\infty$ then $f\in C([0,T];B^{s'}_{p,1})$ for all $s'<s$.

If $\operatorname{div} v=0$ and $v\cdot\nabla f$ stands for $\operatorname{div}(vf)$, then all the above results hold true for $s>-1-d/p$.

Proof. Slight modifications of Chemin's proof (devoted to the case of Hölder spaces) give estimates (A.1) and (A.2) so that we will not write out all the details.

The basic idea is to split $(\mathcal{T})$ in dyadic blocks according to Littlewood-Paley decomposition:

$$(\partial_t + v\cdot\nabla)\Delta_q f = \Delta_q F + R_q, \qquad \Delta_q f_{|t=0} = \Delta_q f_0, \tag{A.3}$$

where we denoted by R_q the remainder $[v,\Delta_q]\cdot\nabla f$. Using standard energy arguments and a convenient integration by parts, we end up with

$$\|\Delta_q f(t)\|_{L^p} \le \|\Delta_q f_0\|_{L^p} + \int_0^t \Big(\|\Delta_q F(\tau)\|_{L^p} + \|R_q(\tau)\|_{L^p} + \frac{\|\operatorname{div} v(\tau)\|_{L^\infty}\,\|\Delta_q f(\tau)\|_{L^p}}{p}\Big)d\tau. \tag{A.4}$$

Multiply both sides of (A.4) by 2^{qs}, take the ℓ^r norm and apply Minkowski's inequality. It follows that

$$\|f(t)\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}}\,d\tau \\ + \int_0^t \bigg(\Big(\sum_{q\ge -1}\big(2^{qs}\,\|R_q(\tau)\|_{L^p}\big)^r\Big)^{\frac1r} + \frac1p\,\|\operatorname{div} v(\tau)\|_{L^\infty}\,\|f(\tau)\|_{B^s_{p,r}}\bigg)\,d\tau. \tag{A.5}$$

Now, the main problem lies in estimating the remainder. We have the following:

Lemma A.2. *If $1\le p,r\le+\infty$ and $s>-d/p$ (or $s>-d/p-1$ if $\operatorname{div} v=0$), the following estimates hold for a constant $C=C(s,d,p,r)$:*

$$\|R_q\|_{L^p} \le Cc_q 2^{-qs}\|\nabla v\|_{B^{d/p}_{p,r}\cap L^\infty}\|f\|_{B^s_{p,r}} \quad \text{if} \quad s<d/p+1, \tag{A.6}$$

$$\|R_q\|_{L^p} \le Cc_q 2^{-qs}\|\nabla v\|_{B^{d/p}_{p,r}\cap L^\infty}\|f\|_{B^s_{p,r}\cap \mathrm{Lip}} \quad \text{if} \quad s=d/p+1, \tag{A.7}$$

$$\|R_q\|_{L^p} \le Cc_q 2^{-qs}\|\nabla v\|_{B^{s-1}_{p,r}}\|f\|_{B^s_{p,r}} \quad \text{if} \quad s>d/p+1. \tag{A.8}$$

Moreover, if $s>0$, we also have

$$\big\|[v^j,\Delta_q]\cdot\partial_j v^i\big\|_{L^p} \le Cc_q 2^{-qs}\,\|\nabla v\|_{L^\infty}\,\|v\|_{B^s_{p,r}}, \tag{A.9}$$

where the Einstein convention on the summation over repeated indices has been used.

Assume that Lemma A.2 holds true and consider first the case $s \neq d/p+1$, or $s = d/p+1$ and $r = 1$. Plugging inequalities (A.6), (A.7) or (A.8) in (A.5) yields

$$\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}}\, d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}}\, d\tau \tag{A.10}$$

and Gronwall's lemma enables us to conclude to inequality (A.1).

Let us now consider the case $s = d/p+1$ and $r > 1$. Since $B^{d/p}_{p,r} \hookrightarrow L^\infty$ does not hold true, we further need an estimate for $\|\nabla f\|_{L^\infty}$. Let us notice that

$$(\partial_t + v\cdot\nabla)\partial_j f = -\partial_j v \cdot \nabla f + \partial_j F$$

so that

$$\|\nabla f(t)\|_{L^\infty} \leq \|\nabla f_0\|_{L^\infty} + \int_0^t \|\nabla F(\tau)\|_{L^\infty}\ d\tau + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\nabla f(\tau)\|_{L^\infty}\ d\tau. \tag{A.11}$$

Now, summing (A.5) and (A.11), plugging inequality (A.7) in the right-hand side and using Gronwall's lemma completes the proof of (A.2).

Let us tackle the question of continuity in time. Since $f \in C([0,T];\mathcal{S}')$, this is not hard to check that $\Delta_q f \in C([0,T];L^p)$ for every $q \geq -1$ and $p \in [1,+\infty]$. This implies that $S_q f \in C([0,T];B^s_{p,r})$ for all $q \in \mathbb{N}$.

Suppose first that $r < +\infty$. Then the sequence of functions $(S_q f)_{q\in\mathbb{N}}$ defined on the interval $[0,T]$ and valued in $B^s_{p,r}$ converges uniformly on $[0,T]$. Indeed, according to (1.1),

$$\Delta_{q'}(f - S_q f) = \sum_{\substack{|q''-q'|\geq 1\\ q''\geq q}} \Delta_{q'}\Delta_{q''} f$$

whence

$$\|f - S_q f\|_{B^s_{p,r}} \leq C\Big(\sum_{q'\geq q-1} \big(2^{q's}\,\|\Delta_{q'} f\|_{L^p}\big)^r\Big)^{\frac{1}{r}}. \tag{A.12}$$

Using inequality (A.4) to bound the right-hand side of (A.12), we gather

$$\|f - S_q f\|_{L^\infty_T(B^s_{p,r})} \leq C\bigg(\Big(\sum_{q'\geq q-1}\Big(2^{q's}\,\|\Delta_{q'} f_0\|_{L^p}\Big)^r\Big)^{\frac{1}{r}}$$
$$+\int_0^T\Big(\sum_{q'\geq q-1}\big(2^{qs}\,\|\Delta_{q'} F(\tau)\|_{L^p}\big)^r\Big)^{\frac{1}{r}} d\tau + \|f\|_{L^\infty_T(B^s_{p,r})}\int_0^T\Big(\sum_{q'\geq q-1} c_{q'}(\tau)^r\Big)^{\frac{1}{r}} V'(\tau)\, d\tau\bigg).$$

The first term clearly tends to zero when q tends to infinity. The terms in the integral also tend to zero for almost every τ. Lebesgue's dominated convergence theorem enables us to conclude that $\|f - S_q f\|_{L^\infty_T(B^s_{p,r})}$ tends to zero when q tends to infinity. This proves that $u \in C([0,T];B^s_{p,r})$ in the case $r < +\infty$.

When $r=+\infty$, we just utilize that for any $s'<s$ we have the embedding $B^s_{p,\infty}\hookrightarrow B^{s'}_{p,1}$ so that the above argument may be repeated in the space $B^{s'}_{p,1}$. This yields $f\in C([0,T];B^{s'}_{p,1})$. $\square$

Proof of Lemma A.2. We shall use some basic results in paradifferential calculus. Let us recall that paradifferential calculus is a convenient way to define a generalized product between distributions which is continuous in functional spaces where the usual product does not make sense (see the pioneering work by J.-M. Bony in [2]). The paraproduct between u and v is defined by $T_u v \stackrel{\text{def}}{=} \sum_{q\in\mathbb{N}} S_{q-1}u\Delta_q v$. We have the following formal decomposition:

$$uv=T_uv+T_vu+R(u,v) \ \text{ with } \ R(u,v)\stackrel{\text{def}}{=}\sum_{q\geq -1}\Delta_q u\widetilde{\Delta}_q v \ \text{ and } \ \widetilde{\Delta}_q=\Delta_{q-1}+\Delta_q+\Delta_{q+1}.$$

The estimates below are standard (see [19] for example).

Proposition A.3. *For all $s\in\mathbb{R}$, $p\in[1,+\infty]$ and $r\in[1,+\infty]$, we have*

$$\|T_uv\|_{B^s_{p,r}}\leq C\,\|u\|_{L^\infty}\,\|v\|_{B^s_{p,r}}.$$

Suppose that $t<0$; then we also have

$$\|T_uv\|_{B^{s+t}_{p,r}}\leq C\|u\|_{B^t_{\infty,\infty}}\|v\|_{B^s_{p,r}}.$$

Let $(s_1,s_2)\in\mathbb{R}^2$, $(p,p_1,p_2,r,r_1,r_2)\in[1,+\infty]^6$ be such that $s_1+s_2>0$, $\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{r}\leq\frac{1}{r_1}+\frac{1}{r_2}$. Then the following estimate holds:

$$\|R(u,v)\|_{B^{s_1+s_2-d\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)}_{p,r}}\leq C\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}}.$$

In the above estimates, C depends only on d and on the parameters defining the Besov norms.

Following [5, pp. 67–70], we decompose R_q into $R_q=\sum_{i=1}^5 R^i_q$ with $R^1_q=[T_{v^j},\Delta_q]\partial_j f$, $R^2_q=T_{\partial_j\Delta_q f}v^j$, $R^3_q=-\Delta_q T_{\partial_j f}v^j$, $R^4_q=\partial_j R(v^j,\Delta_q f)-\Delta_q\partial_j R(v^j,f)$, and $R^5_q=\Delta_q R(\mathrm{div}v,f)-R(\mathrm{div}v,\Delta_q f)$. The original proof by Chemin was devoted to the case of Hölder spaces. It easily extends to the general Besov spaces: this is just a matter of replacing L^∞ with L^p norms, and ℓ^∞ bounds over $\mathbb{N}\cup -1$ by ℓ^r summations. Under the assumptions of Lemma A.2 on s,p,r, we get

$$\left\|R^1_q\right\|_{L^p}\leq Cc_q2^{-qs}\,\|\nabla v\|_{L^\infty}\,\|f\|_{B^s_{p,r}},\tag{A.13}$$

$$\left\|R^2_q\right\|_{L^p}\leq Cc_q2^{-qs}\|\nabla v\|_{B^0_{\infty,\infty}}\|f\|_{B^s_{p,r}},\tag{A.14}$$

$$\left\|R^3_q\right\|_{L^p}\leq Cc_q2^{-qs}\|\nabla v\|_{B^{d/p}_{p,r}}\|\nabla f\|_{B^{s-1}_{p,r}}\tag{A.15}$$

$$\text{if}\quad s<d/p+1\quad\text{or}\quad(s=d/p+1\quad\text{and}\quad r=1)\,,$$

$$\left\|R^3_q\right\|_{L^p}\leq Cc_q2^{-qs}\|\nabla v\|_{B^{s-1}_{p,r}}\,\|\nabla f\|_{L^\infty}\,,\tag{A.16}$$

$$\left\|R^4_q\right\|_{L^p}\leq Cc_q2^{-qs}\|\nabla v\|_{B^{d/p}_{p,r}}\|f\|_{B^s_{p,r}},\tag{A.17}$$

$$\left\|R^5_q\right\|_{L^p}\leq Cc_q2^{-qs}\|\mathrm{div}v\|_{B^{d/p}_{p,r}}\|f\|_{B^s_{p,r}},\tag{A.18}$$

with $\sum_{q\geq -1} c_q^r = 1$. This yields (A.6), (A.7) and (A.8). Let us observe that the first five estimates hold true for $s > -d/p - 1$. The stronger condition $s > -d/p$ is required only for the last term R_q^5, which does not appear in the case of a solenoidal vector field v.

To prove (A.9), we write $R_q = \sum_{i=1}^3 R_q^i + \widetilde{R}_q^4$ with R_q^1, R_q^2, R_q^3 as above (with $f = v$) and

$$\widetilde{R}_q^4 = R(v^j, \Delta_q \partial_j v^i) - \Delta_q R(v^j, \partial_j v^i).$$

We bound R_q^1, R_q^2 and R_q^3 according to (A.13), (A.14) and (A.16). Using Proposition A.3, we easily get, for $s > 0$,

$$\left\| \widetilde{R}_q^4 \right\|_{L^p} \leq C c_q 2^{-qs} \left\| \nabla v \right\|_{L^\infty} \left\| v \right\|_{B^s_{p,r}},$$

and the proof of (A.9) is achieved. $\square$

Remark. The estimate (A.1) also holds for homogeneous Besov spaces when $s < d/p+1$, or $s \leq d/p + 1$ and $r = 1$. This is just a matter of replacing $\mathbb{N} \cup \{-1\}$ with $\mathbb{Z}$ in the summations.

A.2. Approximation lemma. This last section is devoted to the proof of the two approximation lemmas used in section 5. Results in the same spirit may be found in [9] or in [17], page 43.

Throughout this part, we are given a smooth function ρ with support in the ball $B(0, R)$. For any tempered distribution h, the notation h^n will stand for $h \star \rho^n$ where $\rho^n(x) \stackrel{\text{def}}{=} n^d \rho(nx)$.

Lemma A.4. *There exists a constant C depending only on ρ and such that*

$$\left\| \rho^n \star \partial_x(vg) - \partial_x(v^n g^n) \right\|_{L^1} \leq C \left\| \partial_x v \right\|_{L^\infty} \left\| g \right\|_{\mathcal{M}}$$

for any function v with first derivative in L^∞, and $g \in \mathcal{M}$.

Proof. Arguing by density, one can assume with no loss of generality that g is a smooth integrable function. Next, we use the decomposition

$$\rho^n \star \partial_x(vg) - \partial_x(v^n g^n) = \underbrace{\rho^n \star \partial_x(vg) - \partial_x(vg^n)}_{R_1^n} + \underbrace{(v - v^n)\partial_x g^n}_{R_2^n} + \underbrace{g^n \partial_x(v - v^n)}_{R_3^n}. \quad \text{(A.19)}$$

For almost every x, the term $R_1^n(x)$ is written

$$R_1^n(x) = n^2 \int \partial_x \rho(n(x-y))(v(y) - v(x))g(y)\, dy - n\partial_x v(x) \int g(y)\rho(n(x-y))\, dy. \quad \text{(A.20)}$$

Whence, for almost every x,

$$|R_1^n(x)| \leq \left\| \partial_x v \right\|_{L^\infty} \Big(\int n^2 |\partial_x \rho(n(x-y))| |x-y| |g(y)|\, dy + \int n |\rho(n(x-y))| |g(y)|\, dy \Big).$$

Therefore, a convolution inequality yields

$$\left\| R_1^n \right\|_{L^1} \leq C \left\| \partial_x v \right\|_{L^\infty} \left\| g \right\|_{L^1}. \quad \text{(A.21)}$$

Let us notice that

$$(v-v^n)(x)=\int(v(x)-v(y))n\rho(n(x-y))\,dy$$

so that

$$\|v-v^n\|_{L^\infty}\leq n\,\|\partial_x v\|_{L^\infty}\int|z||\rho(nz)|\,dz\leq Cn^{-1}\,\|\partial_x v\|_{L^\infty}\,.$$

Therefore, since obviously $\|\partial_x g^n\|_{L^1}\leq Cn\,\|g\|_{L^1}$, we conclude that

$$\|R_2^n\|_{L^1}\leq C\,\|\partial_x v\|_{L^\infty}\,\|g\|_{L^1}\,. \tag{A.22}$$

To bound $\left\|R_3^n\right\|_{L^1}$, we use that $\|\partial_x v^n\|_{L^\infty}\leq C\,\|\partial_x v\|_{L^\infty}$ and $\|g^n\|_{L^1}\leq C\,\|g\|_{L^1}$. Coming back to (A.21) and (A.22), this completes the proof of Lemma A.4. □

Lemma A.5. *There exists a constant C depending only on ρ and such that*

$$\|\rho^n\star\partial_x(vg)-\partial_x(v^ng^n)\|_{L^1}\leq C\|\partial_x v\|_{\mathcal{M}}\,\|g\|_{L^\infty}$$

for any functions $v\in BV$ and $g\in L^\infty$.

Proof. Arguing by density, one can assume that v is smooth and that v and $\partial_x v$ belong to L^1. Again, making use of the decomposition (A.19), it suffices to prove that, for $i=1,2,3$, we have

$$\|R_i^n\|_{L^1}\leq C\,\|\partial_x v\|_{L^1}\,\|g\|_{L^\infty}\,.$$

Let us notice first that for any function $\phi\in C_0^\infty(\mathbb{R})$ supported in $B(0,R)$ and $x\in\mathbb{R}$, we have

$$n\Big|\int(v(x)-v(y))\phi(n(x-y))\,dy\Big|\leq\|\phi\|_{L^1}\left(|\partial_x v|\star 1_{[-R/n,R/n]}\right)(x) \tag{A.23}$$

where $1_{[a,b]}$ denotes the characteristic function of $[a,b]$. Indeed, we have

$$\begin{aligned}n\int(v(x)-v(y))\phi(n(x-y))\,dy&=\int\phi(z)(v(x)-v(x-n^{-1}z))\,dz,\\&=\int\phi(z)\Big(\int_0^{z/n}\partial_x v(x-\lambda)\,d\lambda\Big)\,dz\end{aligned}$$

so that

$$\begin{aligned}|n\int(v(x)-v(y))\phi(n(x-y))\,dy|&\leq\int_{-R}^{R}|\phi(z)|\,dz\int_{-R/n}^{R/n}|\partial_x v(x-y)|\,dy,\\&\leq\|\phi\|_{L^1}\int|\partial_x v(x-y)|1_{[-R/n,R/n]}(y)\,dy.\end{aligned}$$

Now, let us decompose R_1^n according to (A.20). The last term of the right-hand side obviously satisfies the required estimate. For the first term, we use (A.23) with $\phi=\partial_x\rho$ and apply a convolution inequality. As $\left\|1_{[-R/n,R/n]}\right\|_{L^1}=2R/n$, this yields the desired estimate.

Using again (A.23), we see that $\|v - v^n\|_{L^1} \leq Cn^{-1} \|\partial_x v\|_{L^1}$. Since $\|\partial_x g^n\|_{L^\infty} \leq Cn \|g\|_{L^\infty}$, we gather that R_2^n satisfies the wanted estimate. The case of R_3^n is straightforward. $\square$

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