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A FILTER ON $[\lambda]^\kappa$

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ABSTRACT. We define a filter on $[\lambda]^\kappa$ with properties similar to those of the closed unbounded filter in $P_\kappa(\lambda)$. This filter's behaviour depends on set theoretical hypotheses.

The study of the combinatorial properties of the collection of subsets of uncountable cardinals has been a main line of research in the theory of large cardinals. For $\kappa < \lambda$ regular uncountable cardinals, the space $P_\kappa(\lambda)$ is the collection of subsets of λ of cardinality smaller than κ. This space was introduced in the investigation of strongly compact cardinals and of supercompact cardinals. In **[Je1]** Jech studied the space $P_\kappa(\lambda)$ on its own and obtained interesting generalizations to the context of this space of classical results pertaining to the theory of the space κ.

The space $[\lambda]^\kappa$, the collection of subsets of λ of cardinality κ, arises in the investigation of the so-called huge cardinals. As shown in Solovay, Reinhardt and Kanamori **[SRK]**, κ is huge with target λ if and only if there exists a κ-complete normal fine ultrafilter on $[\lambda]^\kappa$.

We recall the definition of huge cardinal. We say that κ is huge with target λ if there is an elementary embedding $j\colon V \to M$ of the universe into a transitive model M containing all the ordinals such that κ is the critical point of j, $j(\kappa) = \lambda$ and ${}^\lambda M \subseteq M$. We denote this by $\kappa \to (\lambda)$. (See **[BDPT]**.) In this case the axiom of choice allows us to show that the set $(\lambda)^\kappa = \{P \subseteq \lambda \mid \text{order type of } P = \kappa\}$ belongs to the normal ultrafilter on $[\lambda]^\kappa$, and thus we can characterize the fact that $\kappa \to (\lambda)$ by the fact that there exists a normal, κ-complete, fine ultrafilter on $(\lambda)^\kappa$. Thus, under the axiom of choice the first characterization implies the second. This is not so in the absence of the axiom of choice; for instance, under the axiom of determinateness the implication fails, as shown by Mignone in **[Mig]**.

A natural problem is to find a filter on $[\lambda]^\kappa$ analogous to the closed unbounded filter for $P_\kappa(\lambda)$ constructed by Jech in **[Je1]**. The filter we construct is a κ-complete, normal, fine, nontrivial filter, and, as shown by J. Baumgartner, it is the smallest filter on $[\lambda]^\kappa$ with these properties. Under the assumption that κ is huge with target λ, all elements of our filter have measure 1 with respect to the normal measure

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generated by the witnessing elementary embedding. Thus our filter exhibits a behaviour similar to the closed unbounded filter in $P_\kappa(\lambda)$.

Since the space $[\lambda]^\kappa$ is simply $P_{\kappa^+}(\lambda) - P_\kappa(\lambda)$, one could ask why we consider a new notion of a closed unbounded set at all since we have Jech's notion of a closed unbounded set in $P_{\kappa^+}(\lambda)$ and could restrict it to $[\lambda]^\kappa$. The reason for seeking another filter is that under $\kappa \to (\lambda)$ not all the closed unbounded sets in the sense of Jech have measure 1, and, in fact, the cones over sets of cardinality κ have measure 0. Thus a different notion is necessary.

The behaviour of our filter is not a simple one. As we said, if $\kappa \to (\lambda)$ then our filter is included in every fine normal measure on $[\lambda]^\kappa$, but if the universe is not too fat (for instance if $V = L$) our filter is just the closed unbounded filter on $P_{\kappa^+}(\lambda)$.

We acknowledge the helpful remarks made by J. Baumgartner. He proved Theorem 1(e) using a property of closed unbounded sets of $P_\kappa(\lambda)$ which uses functions from $[\lambda]^{<\omega}$ into λ and which can be obtained from Menas' basis result for closed unbounded sets [**Me1**]. We follow, in general, the notation of Jech in [**Je2**]. If $a \in P_\kappa(\lambda)$, $\hat{a} = \{p \in P_\kappa(\lambda) \mid a \subseteq p\}$ and $\check{a} = \{P \in [\lambda]^\kappa \mid a \subseteq P\}$.

1. The filter $\mathfrak{F}_{\kappa,\lambda}$. Given a set $X \subseteq P_\kappa(\lambda)$, define A_X, the basic set generated by X, as follows:

$$(*)\quad A_X = \left\{P \in [\lambda]^\kappa \mid \text{there exists a directed system } D \subseteq X \text{ such that } P = \bigcup D\right\}.$$

We now define the filter $\mathfrak{F}_{\kappa,\lambda}$ as follows: $A \in \mathfrak{F}_{\kappa,\lambda}$ if and only if there is a *closed and unbounded* subset X of $P_\kappa(\lambda)$ such that $A_X \subseteq A$.

THEOREM 1. *The filter $\mathfrak{F}_{\kappa,\lambda}$ possesses the following properties*:

(a) *The cones $\check{a}$ (for $a \in P_\kappa(\lambda)$) belong to $\mathfrak{F}_{\kappa,\lambda}$.*

(b) *$\mathfrak{F}_{\kappa,\lambda}$ is κ-complete.*

(c) *$\mathfrak{F}_{\kappa,\lambda}$ is normal*; *Fodor's property holds for $\mathfrak{F}_{\kappa,\lambda}$-stationary sets.*

(d) *If $\kappa \to (\lambda)$ and μ is the normal measure induced on $[\lambda]^\kappa$ by a witnessing embedding, then every set in $\mathfrak{F}_{\kappa,\lambda}$ has μ-measure 1. In this case $\mathfrak{F}_{\kappa,\lambda}$ is not κ^+-complete.*

(e) *$\mathfrak{F}_{\kappa,\lambda}$ is the least κ-complete, normal, fine filter on $[\lambda]^\kappa$.*

PROOF. (a) If $a \in P_\kappa(\lambda)$ the the cone $\check{a}$ over a in $[\lambda]^\kappa$ is exactly $A_{\hat{a}}$.

(b) Obvious.

(c) Let $\{A_\nu \mid \nu < \lambda\}$ be a λ-sequence of elements of $\mathfrak{F}_{\kappa,\lambda}$. Choose, for each $\nu < \lambda$, a set X_ν closed and unbounded in $P_\kappa(\lambda)$ such that $A_{X_\nu} \subseteq A_\nu$. Define now $Y = \Delta_{\nu<\lambda} X_\nu$. We show that $A_Y \subseteq \Delta_{\nu<\lambda} A_\nu$. It is enough to verify that $A_Y \subseteq \Delta_{\nu<\lambda} A_{X_\nu}$. The latter set is $\{P \in [\lambda]^\kappa \mid \text{for all } \xi \in P,\ P \in A_{X_\xi}\} = \{P \in [\lambda]^\kappa \mid \text{for all } \xi \in P \text{ there is a directed system } D_\xi \subseteq X_\xi \text{ such that } P = \bigcup D_\xi\}$. Let $P \in A_Y$, and pick $D \subseteq Y$. Given $\xi \in P$, put $D_\xi = \{a \in P_\kappa(\lambda) \mid a \in D \text{ and } \xi \in a\}$. Then D_ξ is directed for each $\xi \in P$, since $D \subseteq \Delta_{\nu<\lambda} X_\nu = Y$, $D_\xi \subseteq X_\xi$, and, finally, $\bigcup D_\xi = \bigcup D$.

(d) Assume now that $\kappa \to (\lambda)$. Let μ be the normal measure in $[\lambda]^\kappa$ generated by the witnessing embedding j; μ is defined by $A \in \mu \Leftrightarrow j''\lambda \in j(A)$.

Thus we must show that $j''\lambda \in j(A_X)$ whenever $X \subseteq P_\kappa(\lambda)$ is a closed unbounded set. By the elementarity of j we have that if $j\colon V \to M$, then $j(A_X) = A^M_{j(X)}$. We need to show that $M \models j''\lambda \in A_{j(X)}$, i.e. we need to exhibit in M a directed system contained in $A_{j(X)}$ whose union is $j''\lambda$. This system is simply $\{j(a) \mid a \in X\}$.

Under the hypothesis that $\kappa \to (\lambda)$, the filter $\mathfrak{F}_{\kappa,\lambda}$ is not κ^+-complete. In fact, put $A_\xi = \{\xi, \kappa + \xi\}\check{}$ (the cone in $[\lambda]^\kappa$ over $\{\xi, \kappa + \xi\}$). If $\mathfrak{F}_{\kappa,\lambda}$ were κ^+-complete then we would have that $\bigcap_{\xi<\kappa} A_\xi = \kappa \dotplus \kappa$ is in the filter and, therefore, that it has measure 1. This is absurd since $(\lambda)^\kappa \cap \kappa \dotplus \kappa = \varnothing$. From the proof it follows that $A_X \in \mu$ for every X unbounded in $P_\kappa(\lambda)$.

(Note that if X is unbounded then A_X is $\mathfrak{F}_{\kappa,\lambda}$ stationary. This follows from the fact that a hand-over-hand construction with alternating choices from a given closed and unbounded set and X is possible. In fact, the family of A_X's for X unbounded generates a κ-complete filter. In opposition to Theorem 1(c), this filter does not seem to be normal.)

To prove (e) we need to establish some facts: Following Menas [**Me1**], given a function $f\colon [\lambda]^n \to P_\kappa(\lambda)$ let $C(f) = \{p \in P_\kappa(\lambda) \mid f(x_1, \ldots, x_n) \subseteq p$ for all $x_1, \ldots, x_n$ contained in $p\}$. For any $f\colon [\lambda]^n \to P_\kappa(\lambda)$, $C(f)$ is a closed and unbounded subset of $P_\kappa(\lambda)$.

Also, let $C_\kappa(f) = \{P \in [\lambda]^\kappa \mid f(x_1, \ldots, x_n) \subseteq P$ for $x_1, \ldots, x_n$ in $P\}$. Given $f\colon [\lambda]^n \to P_\kappa(\lambda)$, $A_{C(f)} = C_\kappa(f)$. To prove this, let $P \in A_{C(f)}$. There is a directed set $D \subseteq C(f)$ such that $P = \bigcup D$. Now, if $\{x_1, \ldots, x_n\} \subseteq P$, there is $p \in D$ such that $\{x_1, \ldots, x_n\} \subseteq p$, but then $f(x_1, \ldots, x_n) \subseteq p \subseteq P$. Conversely, given $P \in C_\kappa(f)$, it is easy to show that $C(f) \cap P(P)$ is a directed subset of $C(f)$ with union P.

Menas showed in [**Me1**] that for any $X \subseteq P_\kappa(\lambda)$ closed and unbounded there is an $f\colon [\lambda]^2 \to P_\kappa(\lambda)$ such that $C(f) \subseteq X$. From this follows

LEMMA 2 (BASIS PROPERTY). *A set A belongs to the filter $\mathfrak{F}_{\kappa,\lambda}$ if and only if there is a function $f\colon [\lambda]^2 \to P_\kappa(\lambda)$ such that $C_\kappa(f) \subseteq A$.*

Now (e) follows. Indeed, if $\mathfrak{F}_{\kappa,\lambda}$ is not the least κ-complete, normal, fine filter on $[\lambda]^\kappa$ then let $\mathfrak{F} \subsetneq \mathfrak{F}_{\kappa,\lambda}$ be the least such filter. Let $A \in \mathfrak{F}_{\kappa,\lambda} - \mathfrak{F}$. Clearly both A and $[\lambda]^\kappa - A$ are $\mathfrak{F}$-stationary (both meet each element of $\mathfrak{F}$). By Lemma 2 there is $f\colon [\lambda]^2 \to P_\kappa(\lambda)$ such that $C_\kappa(f) \subseteq A$. Thus $([\lambda]^\kappa - A) \cap C_\kappa(f) = \varnothing$. Applying normality of $\mathfrak{F}$ twice, we find an $\mathfrak{F}$-stationary set $B \subseteq ([\lambda]^\kappa - A)$ and a fixed pair of ordinals x_1, x_2 in λ such that for every $P \in B$, $\{x_1, x_2\} \subseteq P$ and $f(x_1, x_2) \not\subseteq P$. But then, $B \cap f(x_1, x_2)\check{} = \varnothing$, contradicting the fact that $\mathfrak{F}$ is fine. $\square$

We remark that the filter $\mathfrak{F}_{\kappa,\lambda}$ is never κ^{++} complete (for each $\alpha < \kappa^+$ take $\{\alpha\}\check{} = \{P \in [\lambda]^\kappa \mid \alpha \in P\}$; all these sets are in $\mathfrak{F}_{\kappa,\lambda}$ but $\bigcap_{\alpha<\kappa^+} \{\alpha\}\check{} = \varnothing$).

As in the case of $P_\kappa(\lambda)$ (see Jech [**Je1**]) we tailored our definition of closed unbounded sets of $[\lambda]^\kappa$ to be able to show that all elements of $\mathfrak{F}_{\kappa,\lambda}$ have measure 1 under normal measures on $[\lambda]^\kappa$. It is known that the definition of closed unbounded subset of $P_\kappa(\lambda)$ can be weakened by showing that an unbounded set is closed under directed systems if and only if it is closed under unions of chains (see Magidor [**Ma**]). The same phenomenon occurs for $[\lambda]^\kappa$ since we have the following closure lemma.

LEMMA 3. *Given κ an uncountable regular cardinal, let $\{a_\xi \mid \xi < \kappa\}$ be a sequence of length κ such that $\{a_\xi \mid \xi < \kappa\}$ is a directed subset of $P_\kappa(\lambda)$. Then there exist arbitrarily large initial segments $\{a_\xi \mid \xi < \eta\}$ $(\eta < \kappa)$ which are directed.*

Given a directed $D \subseteq P_\kappa(\lambda)$, if $|\bigcup D| = \kappa$ and $|D| > \kappa$ it is easy to construct $D' \subseteq D$ with $|D'| = \kappa$ and $\bigcup D' = \bigcup D$. Thus we may always assume our directed systems have cardinality κ. We use Lemma 3 to split D into an increasing chain of directed subsystems each of size smaller than κ. The union of each of these subsystems is in X. Thus, given $P \in A_X$, we can present P as the union of an increasing κ-chain of elements of X. We thus have

THEOREM 4. *If statement $(*)$ of the definition of $\mathcal{F}_{\kappa,\lambda}$ is replaced by*

$(*')$ $\quad A'_X = \{P \in [\lambda]^\kappa \mid P$ *is the union of an increasing κ-chain of elements of $X\}$,*

we obtain the same filter.

The argument of the proof does not go through if X is only unbounded and not closed. In fact, if $\kappa \to (\lambda)$ then the following set E is unbounded but A'_E is not in the filter (it has normal measure zero). Let

$$\begin{aligned} E = \{ p \mid \exists\alpha\forall\xi < \lambda \text{ if } p \cap [\kappa\cdot\xi, \kappa\cdot(\xi+1)) \neq \varnothing, \\ \text{then } p \cap [\kappa\cdot\xi, \kappa\cdot(\xi+1)) = (\kappa\cdot\xi) + \alpha - (\kappa\cdot\xi) \\ \text{and } \{\xi \mid p \cap [\kappa\cdot\xi, \kappa\cdot(\xi+1)) \neq \varnothing\} = \kappa \cap p = \alpha\}. \end{aligned}$$

Clearly if $P \in A'_E$ then o.t. $P > \kappa$. So $A'_E \cap (\lambda)^\kappa = \varnothing$ [**DPM**].

Assume κ is huge with a target λ and ν a corresponding normal measure in $[\lambda]^\kappa$ and that, in addition, there exists a normal measure μ in $P_\kappa(\lambda)$ with the partition property (this happens, for instance, if κ is twice huge with λ a first target). The set E has μ-measure 0. However since the measure μ has the partition property there exists a set X, of μ-measure 1 such that $p, q \in X$ and $p \subsetneq q \Rightarrow |p| < |q \cap \kappa|$ (cf. [**Me2**]). The set A'_X is not in our filter (otherwise $A'_X \cap (\lambda)^\kappa$ has ν-measure 1, which is absurd).

We will now prove a lemma which implies that, just as in the case of the closed unbounded filter in $P_\kappa(\lambda)$, it is enough to apply twice the operation Δ (diagonal intersection) to cones to obtain all sets of the form A_X for X closed and unbounded in $P_\kappa(\lambda)$.

LEMMA 5. *Given a collection $\{X_\xi \mid \xi < \lambda\}$ of closed unbounded sets in $P_\kappa(\lambda)$,*

$$\Delta_{\xi<\lambda} A_{X_\xi} = A_{\Delta_{\xi<\lambda} X_\xi}.$$

PROOF. It is enough to prove $\Delta_{\xi<\lambda} A_{X_\xi} \subseteq A_{\Delta_{\xi<\lambda} X_\xi}$. We first prove the following:

Fact 6. If $P \in \Delta_{\xi<\lambda} A_{X_\xi}$ then for every $q \in P_\kappa(P)$, the intersection $\bigcap_{\xi\in q} X_\xi$ is closed and unbounded in $P_\kappa(P)$.

Indeed, given $p \in P_\kappa(P)$, we perform an induction of length $|q|\cdot\omega$ to cover p with elements of each X_ξ $(\xi \in q)$. Since $P \in \Delta_{\xi<\lambda} A_{X_\xi}$, for every $\alpha \in P$ there is an increasing κ-chain of elements of X_α with union P. Therefore if $q = \{\alpha_\nu : \nu < |q|\}$, let $p_0^0 = p$, and for each $\nu < |q|$ let $p_\nu^0 \in X_{\alpha_\nu}$ be such that $\bigcup_{\xi<\nu} p_\xi^0 \subseteq p_\nu^0 \subseteq P$. Similarly, put $p_0^{n+1} = \bigcup_{\nu<|q|} p_\nu^n$ and $p_\nu^{n+1} \in X_{\alpha_\nu}$ such that $\bigcup_{\xi<\nu} p_\xi^{n+1} \subseteq p_\nu^{n+1} \subseteq P$. The sets p_ν^n can always be found as P is the limit of κ-chains of elements of each X_{α_ν}. Finally, $\bigcup_{n\in\omega,\ \nu<|q|} p_\nu^n$ is a subset of P which contains p and belongs to $\bigcap_{\xi\in q} X_\xi$. This completes the proof of the Fact.

To complete the proof of Lemma 5 it is enough to show that if $P \in \Delta_{\xi<\lambda} A_{X_\xi}$ and $p \in P_\kappa(P)$ then there is a $q \in \Delta_{\xi<\lambda} X_\xi$ such that $p \subseteq q \subseteq P$. Using Fact 6 we find such q as follows: Let $q_0 = p$, q_{n+1} is a set in $\bigcap_{\xi\in q_n} X_\xi$ such that $q_n \subseteq q_{n+1} \subseteq P$. Let $q = \bigcup_{n\in\omega} q_n$. $\square$

If $\mathcal{C}$ is a collection of subsets (of $P_\kappa(\lambda)$ or $[\lambda]^\kappa$) we denote by $\Delta\mathcal{C}$ the collection consisting of diagonal intersections of elements of $\mathcal{C}$.

COROLLARY 7. *$\{A_X$: X is closed and unbounded in $P_\kappa(\lambda)\} = \Delta\Delta\{\check{p} \mid p \in P_\kappa(\lambda)\}$.*

PROOF. D. Carr [**Ca**] showed that closed unbounded subsets of $P_\kappa(\lambda)$ are just elements of $\Delta\Delta\{\hat{p} \mid p \in P_\kappa(\lambda)\}$. So given $X \subseteq P_\kappa(\lambda)$ closed and unbounded, $X = \Delta_{\xi<\lambda}\Delta_{\eta<\lambda}\hat{p}_{\xi,\eta}$. Now we apply Lemma 5 twice. $\square$

Consider now the following operator A''_X:

$$A''_X = \Big\{P \in [\lambda]^\kappa \mid \text{there is an end extension chain of length } \kappa \text{ of elements of } X,\ \langle p_\xi \rangle_{\xi<\kappa},\ \text{such that } P = \bigcup_{\xi<\kappa} p_\xi\Big\}.$$

Clearly the elements of A''_X are of order type κ only, and, in fact, a form of the converse property holds as well:

> If $\langle p_\xi \rangle_{\xi<\kappa}$ is a $\subseteq$-chain such that $\overline{\bigcup_{\xi<\kappa} p_\xi} = \kappa$, then there exists a cofinal end extension subchain $\langle p_{\eta_\xi} \rangle_{\xi<\kappa}$.

It follows that if $\kappa \to (\lambda)$ then the filter generated for A''_X (for X closed and unbounded) is exactly $\mathfrak{F}_{\kappa,\lambda}$. However, for the set X considered above, $A''_X = \varnothing$ (even though $\nu(A_X) = 1$).

2. Another identity crisis. As we showed in §1, if $\kappa \to (\lambda)$ then $\mathfrak{F}_{\kappa,\lambda}$ is included in any fine normal measure on $[\lambda]^\kappa$. In particular, $(\lambda)^\kappa$ is $\mathfrak{F}_{\kappa,\lambda}$-stationary since it has measure 1 under any such measure. As we show below this is rather exceptional. First, notice the following:

PROPOSITION 1. *$(\lambda)^\kappa$ does not belong to the filter $\mathfrak{F}_{\kappa,\lambda}$.*

PROOF. Construct, for any closed unbounded subset X of $P_\kappa(\lambda)$, an element of A_X of order type greater than κ. (In fact, for any α, $\kappa \leqslant \alpha < \kappa^+$, there is an element of A_X of order type greater than α.) $\square$

We now prove some auxiliary facts regarding the width of the universe.

PROPOSITION 2. *If $0^\#$ does not exist then for every cardinal κ and $\lambda \geqslant \kappa$, whenever $\langle A, \varepsilon\rangle \prec_1 \langle L_\lambda, \varepsilon\rangle$, $|A| \geqslant \kappa$, then $L_\kappa \subseteq A$.*

PROOF. Otherwise there is $\nu < \kappa$ such that $\nu \notin A$. Consider now η such that π: $A \to L_\eta$ is the transitive Mostowski contraction. Then $\eta \geqslant \kappa$ so ν^{+L}, the successor of

ν in the sense of L, is less than or equal to η. In particular, $P(\nu) \cap L \subseteq L_\eta$ so $\{Y \subseteq \nu \mid \nu \in \pi^{-1}(Y)\}$ is an L-ultrafilter, contradicting the fact that $0^\#$ does not exist. □

Similarly, we prove the following

PROPOSITION 3. *Let κ be a cardinal and $a \subseteq \nu < \kappa$ such that $a^\#$ does not exist. Then, if $\langle A, \varepsilon\rangle \prec_1 \langle L_\lambda[a], \varepsilon\rangle$ and $\{a\} \cup \nu \subseteq A$, it follows that $|A| \geqslant \kappa$ implies $L_\kappa[a] \subseteq A$.*

(This result can be extended even further, for instance, if $0^\dagger$ does not exist, a similar lemma can be obtained for ν greater than the least ordinal measurable in an inner model.)

LEMMA 4. *Let κ and a be as in Proposition 3 and $\lambda > \kappa$. Then the cone $\check{\kappa}$ over κ in $[\lambda]^\kappa$ belongs to $\mathfrak{F}_{\kappa,\lambda}$.*

PROOF. For every $\xi \geqslant \kappa$, $|L_\xi[a]| = |\xi|$. Enumerate $L_\lambda[a]$ in order type λ in such a way that $L_\kappa[a]$ is enumerated by κ. Then every subset of λ codes a subset of $L_\lambda[a]$. Consider now $X = \{p \subseteq \lambda \mid |p| < \kappa$ and p codes an elementary substructure of $L_\lambda[a]$ containing $\{a\} \cup \nu\}$.

The set X is clearly closed and unbounded in $P_\kappa(\lambda)$. If $P \in A_X$ then P also codes an elementary substructure of $L_\lambda[a]$ only now $|P| = \kappa$. By Proposition 3, the model coded by P contains $L_\kappa[a]$, so P must contain all of κ (since κ enumerates $L_\kappa[a]$). We have thus proved that if $P \in A_X$ then $\kappa \subseteq P$, that is to say, $P \in \check{\kappa}$. Therefore $\check{\kappa} \in \mathfrak{F}_{\kappa,\lambda}$. □

THEOREM 5. *Under the same assumptions, $\check{P} \in \mathfrak{F}_{\kappa,\lambda}$ for all $P \in [\lambda]^\kappa$, therefore $(\lambda)^\kappa$ is not $\mathfrak{F}_{\kappa,\lambda}$-stationary.*

PROOF. If π is a permutation of λ, it can be extended to $P_\kappa(\lambda)$ and to $[\lambda]^\kappa$ in the obvious way: if $a \in P_\kappa(\lambda)$, $\pi(a) = \pi''(a)$, and for $P \in [\lambda]^\kappa$, $\pi(P) = \pi''P$. It is clear that $A_{\pi[X]} = \pi[A_X]$ for a closed unbounded $X \subseteq P_\kappa(\lambda)$. The filter generated by the closed unbounded subsets of $P_\kappa(\lambda)$ and the filter $\mathfrak{F}_{\kappa,\lambda}$ are invariant under any permutation of λ and thus, if for any $P \in [\lambda]^\kappa$, $\check{P} \in \mathfrak{F}_{\kappa,\lambda}$, then for every $P \in [\lambda]^\kappa$ we have $\check{P} \in \mathfrak{F}_{\kappa,\lambda}$. The rest follows easily. □

Using this theorem we can proceed toward a connection between $\mathfrak{F}_{\kappa,\lambda}$ and the closed unbounded filter on $P_{\kappa^+}(\lambda)$.

LEMMA 6. *If $X \subseteq P_\kappa(\lambda)$ is closed and unbounded then A_X is closed under unions of increasing chains of length $\leqslant \kappa$.*

PROOF. Let $\{P_\xi\}_{\xi<\eta}$ be an η-chain of elements of A_X with $\eta < \kappa$, and let $P = \bigcup_{\xi<\eta} P_\xi$. For each $\xi < \eta$, $P_\xi = \bigcup_{\zeta<\kappa} p^\xi_\zeta$, where $p^\xi_0 \subseteq p^\xi_1 \subseteq \cdots \subseteq p^\xi_\zeta \subseteq \cdots$, $\zeta < \kappa$, is an increasing κ-chain of elements of X.

We will now construct a κ-chain $\{q_\xi\}_{\xi<\kappa} \subseteq X$ such that $\bigcup_{\zeta<\kappa} q_\zeta = P$. Each q_ζ ($\zeta < \kappa$) will in turn be the union of an η-chain of elements of X,

$$q_\zeta = \bigcup_{\xi<\eta} q_{\zeta,\xi}.$$

Define the $q_{\zeta,\xi}$'s as follows:

$$q_{0,0} = p_0^0, \qquad q_{0,\xi} = \text{the first element of } \left\{ p_\zeta^\xi \right\}_{\zeta<\kappa} \text{ containing } \bigcup_{\delta<\xi} q_{0,\delta}.$$

Note that $q_{0,\xi}$ is defined because $\bigcup_{\delta<\xi} q_{0,\delta}$ has cardinality $< \kappa$ and it is contained in P_ξ. Since P_ξ has cardinality κ and $P_\xi = \bigcup_{\zeta<\kappa} p_\zeta^\xi$ there must be a p_ζ^ξ containing $\bigcup_{\delta<\xi} q_{0,\delta}$.

Now put $q_0 = \bigcup_{\xi<\eta} q_{0,\xi}$. If we have already defined q_δ for all $\delta < \zeta$ then we define $q_{\zeta,0}$ as the first element of $\{p_\zeta^0\}_{\zeta<\kappa}$ covering properly $\bigcup_{\delta<\zeta} q_{\delta,0}$, and $q_{\zeta,\xi}$ as the first element of $\{p_\zeta^\xi\}_{\zeta<\kappa}$ covering $\bigcup_{\tau\leqslant\xi} \bigcup_{\delta<\zeta} q_{\delta,\tau}$. Now put $q_\zeta = \bigcup_{\xi<\eta} q_{\zeta,\xi}$.

Clearly, each $q_\zeta \in X$, and $\bigcup q_\zeta = P$.

For sequences of length κ the construction is somewhat different: Let $\{P_\xi\}_{\xi<\kappa}$ be an increasing κ-chain of elements of A_X. As before for each $\xi < \kappa$, $P_\xi = \bigcup_{\zeta<\kappa} p_\zeta^\xi$, where $\{p_\zeta^\xi\}_{\zeta<\kappa}$ is an increasing κ-chain from X. We now define a κ-chain $\{q_\zeta\}_{\zeta<\kappa} \subseteq X$ such that $\bigcup_{\zeta<\kappa} q_\zeta = \bigcup_{\xi<\kappa} P_\xi$. Let $q_0 = p_0^0$, and if we have defined q_δ for all $\delta < \gamma$, put $q_\gamma =$ the first element of $\{p_\zeta^\gamma\}_{\zeta<\kappa}$ covering $(\bigcup_{\delta<\gamma} q_\delta) \cup (\bigcup \{p_\beta^\alpha \mid \alpha, \beta < \gamma\})$. Clearly $\bigcup_{\zeta<\kappa} q_\zeta = \bigcup_{\xi<\kappa} P_\xi$, and each $q_\zeta \in X$. □

THEOREM 7. *Under the hypothesis of Proposition* 3, $\mathfrak{F}_{\kappa,\lambda} = \mathrm{CLUB}_{\kappa^+,\lambda} \upharpoonright [\lambda]^\kappa$ (*where* $\mathrm{CLUB}_{\kappa^+,\lambda}$ *is the filter generated by the closed unbounded subsets of* $P_{\kappa^+}(\lambda)$).

PROOF. The previous lemma implies that $\mathfrak{F}_{\kappa,\lambda} \subseteq \mathrm{CLUB}_{\kappa^+,\lambda}$ because under our assumptions all the cones $\check{P}$, for $P \in P_{\kappa^+}(\lambda)$, belong to $\mathfrak{F}_{\kappa,\lambda}$ and therefore each $A \in \mathfrak{F}_{\kappa,\lambda}$ is unbounded in $P_{\kappa^+}(\lambda)$.

For the other inclusion we use Carr's result, namely that if $A \in \mathrm{CLUB}_{\kappa^+,\lambda}$ then $A = \Delta_{\xi<\lambda}\Delta_{\eta<\lambda}\hat{r}_{\xi,\eta}$ with $r_{\xi,\eta} \in P_{\kappa^+}(\lambda)$ (and $\hat{r}_{\xi,\eta}$ is its cone in $P_{\kappa^+}(\lambda)$). Now, if $r_{\xi,\eta} \in [\lambda]^\kappa$ then the cone over $r_{\xi,\eta}$ in $P_{\kappa^+}(\lambda)$ is just $\check{r}_{\xi,\eta}$, which under our assumption is in $\mathfrak{F}_{\kappa,\lambda}$. It is easy to verify that $A = \Delta_{\xi<\lambda}\Delta_{\eta<\lambda}\check{r}_{\xi,\eta}$, and therefore $A \in \mathfrak{F}_{\kappa,\lambda}$ since $\mathfrak{F}_{\kappa,\lambda}$ is normal. □

One way of interpreting these results is that under assumptions like $V = L$, besides Menas' basis for the closed unbounded sets in $P_{\kappa^+}(\lambda)$ there are two other bases. One is sets of the the form A_X for X closed and unbounded in $P_\kappa(\lambda)$. Another is sets of the form $C_\kappa(f)$ for $f\colon [\lambda]^2 \to P_\kappa(\lambda)$.

These results show that $\mathfrak{F}_{\kappa,\lambda}$ might or might not be the closed unbounded filter on $P_{\kappa^+}(\lambda)$ restricted to $[\lambda]^\kappa$ depending on the width of the universe.

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