Open access • Journal Article • DOI:10.1002/NUM.1018

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Published on: 01 Jul 2001 - Numerical Methods for Partial Differential Equations (John Wiley & Sons, Inc.)

Topics: Pressure-correction method, Finite difference method, Lattice Boltzmann methods, Finite difference and Discretization

A Finite Difference Interpretation of the Lattice Boltzmann Method

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Abstract

Compared to conventional techniques in computational fluid dynamics, the lattice Boltzmann method (LBM) seems to be a completely different approach to solve the incompressible Navier-Stokes equations. The aim of this article is to correct this impression by showing the close relation of LBM to two standard methods: relaxation schemes and explicit finite difference discretizations. As a side effect, new starting points for a discretization of the incompressible Navier-Stokes equations are obtained.

Keywords. discrete velocity models, lattice Boltzmann method, low Mach number limit, incompressible Navier-Stokes equations, finite difference method, relaxation systems, pseudo-compressibility methods

AMS subject classifications. 76P05, 76D05, 65M06, 35B25

1 Introduction

In recent years, the lattice Boltzmann method has been proposed as a potential alternative to conventional methods in computational fluid dynamics. The basic idea of LBM is to use a very simple microscopic model of a gas which is nevertheless capable of correctly describing the macroscopic flow behavior. The microscopic approach of LBM has its origins in the theory of lattice gas automata (see [1]) and is closely related to discrete velocity models of the Boltzmann equation [2]. Essentially, the velocities of the gas particles are restricted in such a way, that microscopic movement can only take place between the nodes of a regular space lattice. However, since the density of particles per direction can vary continuously, the average velocity of the gas is not discrete. If the collision process among the gas particles is modeled appropriately and if the system is in a particular asymptotic situation (diffusion

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limit), the average velocity approximates a solution of the incompressible Navier-Stokes equations.

In its standard form, LBM is an explicit finite difference approximation of a velocity-discrete Boltzmann equation with a collision operator of relaxation type. The variables in the kinetic description are the particle densities per discrete velocity (so called occupation numbers). They always outnumber the macroscopic flow variables which are obtained as averages based on the occupation numbers (see [3, 4] for detailed reviews and references on LBM).

Since LBM reduces complicated macroscopic phenomena to a simple microscopic dynamics, it is an interesting object for many researchers. However, if LBM is considered mainly as a numerical method for macroscopic equations, like the incompressible Navier-Stokes system, it is natural to ask for its relation to already existing schemes. The answer to this question is complicated by the fact that LBM is formulated in kinetic terms and not directly in terms of the target equations. Nevertheless, the closeness of LBM to finite difference methods has already been mentioned. For example, in the interesting article [5], lattice Boltzmann methods are presented as subclass of so called *fully Lagrangian schemes* which are shown to be directly related to standard finite difference methods for some particular cases.

In the present article, we investigate the frequently used lattice Boltzmann model for two-dimensional Navier-Stokes flow based on nine discrete velocities and a regular square lattice with the aim to show the close relation to standard methods. It turns out that the lattice Boltzmann algorithm can be viewed as a non-standard way of writing an explicit finite difference approximation of either the Navier-Stokes equations directly or of some relaxation system for the Navier-Stokes equations. The close relation between the kinetic approach of LBM and finite difference methods rests on the basic observation presented in Section 5: discrete microscopic transport plus velocity averaging is equivalent to a finite difference approximation. Since transport and averaging are always ingredients of LBM, the observations in Section 5 can be used to translate other lattice Boltzmann methods into finite difference schemes so that the presented results are not restricted to the example under consideration.

In contrast to the standard treatment, which is based on a two-scale Chapman-Enskog expansion to relate LBM and Navier-Stokes equations, we use the diffusion scaling of the Boltzmann equation (see Section 3) which seems to be a simpler approach.

In Section 6, it is shown that for a special choice of the collision parameter in LBM, the method can be rewritten as explicit finite difference approximation of a compressible Navier-Stokes system (which reduces to the incompressible case in low Mach number flows). In this reformulation, all aspects of the kinetic approach have disappeared or, more precisely, are condensed in the structure of the finite difference stencils for the differential operators in the compressible Navier-Stokes equations. As a common feature, these stencils involve diagonal neighbors and can be viewed

as convex combinations of usual central differences. It is interesting to note that the derivatives in the pressure gradient, for example, are discretized different from those in the nonlinear convective terms. Despite the unusual size of the stencils, their evaluation is very efficient in the original lattice-Boltzmann formulation. We also find, that the scheme contains no special treatment of the stiff velocity-pressure coupling which arises in nearly incompressible situations. In fact, LBM in its standard form works in the stability constellation $\Delta t/\Delta x^2 = \mathcal{O}(1)$ known from the explicit scheme for advection diffusion equations. The approximate divergence-free condition is automatically assured by a pseudo-compressibility approach which contains a pressure stabilization, Chorin's idea of artificial compressibility, as well as convective terms which are usually not considered.

For choices of the collision parameter different from the one in Section 6, the situation is slightly more complicated and leads to an interpretation of LBM as superposition of finite difference schemes (Section 7) or as finite difference approximation of a relaxation system for the incompressible Navier-Stokes equations (Section 8).

It is a general observation that an interpretation of a new method in terms of other approaches can have two advantages: First, the new method is better understood and further research can take advantage from the knowledge already available. Secondly, new developments can be triggered in already established fields. In this respect, it is important to note that LBM leads to stable and reliable results so that it is worthwhile to investigate the mechanisms leading to this behavior.

2 The Lattice Boltzmann Method

The basic kinetic model is given by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = J(f) \tag{1}$$

which describes the evolution of a particle density $f(\mathbf{x}, \mathbf{v}, t)$. The left hand side of (1) represents free transport of the particles while the right hand side describes interactions through collisions. Continuous and discrete velocity models only differ in the structure of the phase space $\mathcal{X} \times \mathcal{V}$. In the classical Boltzmann equation, the velocity domain $\mathcal{V}$ is the full space $\mathbb{R}^3$, while discrete models are based on

$$\mathcal{V} = \{\mathbf{c}_0, \ldots, \mathbf{c}_{N-1}\} \qquad \mathbf{c}_i \in \mathbb{R}^d.$$

The space part $\mathcal{X}$ is continuous in both cases. In the following, we will consider a two-dimensional model ($d = 2$) with nine velocities ($N = 9$)

$$\begin{array}{llll} \mathbf{c}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) & \mathbf{c}_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) & \mathbf{c}_3 = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) & \mathbf{c}_4 = \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) \\ \mathbf{c}_5 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & \mathbf{c}_6 = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) & \mathbf{c}_7 = \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right) & \mathbf{c}_8 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right) \end{array}$$

and $\mathbf{c}_0 = \mathbf{0}$ which point to the corners and edges of the unit square. However, our observations are not limited to this special situation and we will formulate most steps in a general notation.

We start by introducing the discrete velocity integral for functions $\psi : \mathcal{V} \mapsto \mathbb{R}$

$$\langle \psi \rangle = \sum_{i=0}^{N-1} \psi(\mathbf{c}_i).$$

Note that ψ can be identified with the N-vector of its values $\psi(\mathbf{c}_0), \ldots, \psi(\mathbf{c}_{N-1})$. For particle distributions $f(\mathbf{x}, \mathbf{v}, t)$, the N functions

$$f_i(\mathbf{x}, t) = f(\mathbf{x}, \mathbf{c}_i, t) \qquad i = 0, \ldots, N-1$$

are usually called *occupation numbers*. Macroscopic quantities like mass density ρ and momentum density $\rho\mathbf{u}$ are obtained by taking velocity moments of f

$$\rho(\mathbf{x}, t) = \langle f(\mathbf{x}, \mathbf{v}, t) \rangle, \qquad \rho\mathbf{u}(\mathbf{x}, t) = \langle \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) \rangle \tag{2}$$

In many lattice Boltzmann applications, the collision operator $J(f)$ in (1) is of BGK-type

$$J(f) = -\frac{1}{\tau}(f - f^{eq}(\rho, \mathbf{u})). \tag{3}$$

where $\tau > 0$ is called *relaxation time* and f^{eq} is the *equilibrium distribution*, which depends on f through the parameters ρ and $\mathbf{u}$ in (2). Other models are based on

$$J(f)(\mathbf{w}) = \langle \mathcal{A}(\mathbf{w}, \mathbf{v})(f - f^{eq})(\mathbf{v}) \rangle$$

which can be viewed as linearizations of general nonlinear collision operators. Conservation of mass and momentum in the collision process translates into

$$\langle J(f) \rangle = 0 \qquad \text{and} \qquad \langle \mathbf{v} J(f) \rangle = \mathbf{0}$$

which puts additional conditions on the kernel $\mathcal{A}$. In (3), it is a direct consequence of the construction, since f and f^{eq} have identical mass and momentum densities. In the following, we assume the simple structure (3) but the considerations can also be applied to collision matrices.

In the standard D2Q9-model [6], the equilibrium distribution has the form

$$f^{eq}(\rho, \mathbf{u}; \mathbf{v}) = \rho \left(1 + 3u_k v_k + \frac{9}{2} u_k u_l (v_k v_l - \delta_{kl}/3) \right) f^*(\mathbf{v}) \tag{4}$$

where Einstein's summation convention is used for the indices k, l (taking values $1, 2$) and f^* is defined by

$$f^*(\mathbf{c}_i) = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{1}{9} & i = 1, \cdots, 4 \\ \frac{1}{36} & i = 5, \cdots, 8 \end{cases} \tag{5}$$

The actual lattice Boltzmann evolution is obtained from (1) by a discretization in space and time. First, (1) is split into a collision step $\partial_t f = J(f)$ and a free flow step $\partial_t f + \mathbf{v}\nabla f = 0$. Then, the time derivative in the collision part is approximated by an explicit Euler step

$$\tilde{f}(\mathbf{x}, \mathbf{v}, t + \delta t) = f(\mathbf{x}, \mathbf{v}, t) + \delta t J(f)(\mathbf{x}, \mathbf{v}, t).$$

Finally, the free flow equation is solved explicitly with $\tilde{f}(\mathbf{x}, \mathbf{v}, t + \delta t)$ as initial value, giving rise to the lattice Boltzmann evolution

$$f(\mathbf{x} + \mathbf{v}\delta t, \mathbf{v}, t + \delta t) = f(\mathbf{x}, \mathbf{v}, t) - \frac{\delta t}{\tau}(f - f^{eq})(\mathbf{x}, \mathbf{v}, t). \tag{6}$$

The space discretization is obtained by simply restricting $\mathbf{x}$ to the nodes of a regular square lattice with side length $\delta x = \delta t$. Due to the structure of the discrete velocities $\mathbf{c}_0, \ldots, \mathbf{c}_8$, the shifted positions $\mathbf{x} + \mathbf{v}\delta t$ are then automatically nodes of the same grid and (6) is completely discretized. Apart from this regular discretization, which makes (6) a finite difference approximation of (1), other approaches including curvilinear and unstructured meshes, have been discussed (see [7, 8]).

If the simple updating rule (6) is used iteratively, while obeying certain conditions which we detail in the next section, it turns out that $\rho = \langle f \rangle$ and $\mathbf{u} = \langle \mathbf{v} f \rangle / \rho$ corresponding to f give rise to an approximate solution of the incompressible Navier-Stokes equations with Reynolds number related to the relaxation parameter τ

$$\frac{1}{Re} = \frac{1}{3}\left(\frac{\tau}{\delta t} - \frac{1}{2}\right). \tag{7}$$

Usually, this surprising fact is explained by an asymptotic closeness of Boltzmann and Navier-Stokes description. However, as we will show in the following, the discrete evolution (6) can also be interpreted directly as finite difference approximation of the Navier-Stokes system.

3 The diffusion scaling

In order to use the lattice Boltzmann evolution as approximation method for the incompressible Navier-Stokes equations, several side conditions have to be observed. First, δx has to be small compared to the typical length scale L of the problem to obtain a reasonable space resolution. In other words, $\Delta x = \epsilon = \delta x / L$ should be small. Secondly, the typical flow speed U should be small to ensure incompressibility. In [9], it is shown that $U = \mathcal{O}(\epsilon)$ is required to get a consistent approximation. Since U is only a scale for the speed, we set $U = \epsilon$, absorbing possible factors in the scaled velocity field $\hat{\mathbf{u}} = \mathbf{u}/\epsilon$. Together with the length scale L, the speed U determines the

relevant time scale $T = L/U$ for the Navier-Stokes problem. Taking into account that $\delta x = |\mathbf{c}_1 \delta t| = \delta t$, we find

$$\Delta t = \frac{\delta t}{T} = \frac{\delta x}{L} U = \epsilon^2.$$

Finally, the relaxation time τ is typically chosen of the order of δt so that $\hat{\tau} = \tau/\delta t = \mathcal{O}(1)$. Introducing scaled quantities $\hat{\mathbf{x}} = \mathbf{x}/L$, $\hat{t} = t/T$, $\hat{f}(\hat{\mathbf{x}}, \mathbf{v}, \hat{t}) = f(L\hat{\mathbf{x}}, \mathbf{v}, T\hat{t})$ and similar definitions for $\hat{\rho}$ and $\hat{\mathbf{u}} = \mathbf{u}/\epsilon$, we obtain the scaled version of (6)

$$\hat{f}(\hat{\mathbf{x}} + \epsilon\mathbf{v}, \mathbf{v}, \hat{t} + \epsilon^2) = \hat{f}(\hat{\mathbf{x}}, \mathbf{v}, \hat{t}) - \frac{1}{\hat{\tau}}\Big(\hat{f}(\hat{\mathbf{x}}, \mathbf{v}, \hat{t}) - f^{eq}(\hat{\rho}(\hat{\mathbf{x}}, \hat{t}), \epsilon\hat{\mathbf{u}}(\hat{\mathbf{x}}, \hat{t}); \mathbf{v})\Big).$$

In the following, we will always work in the described scaling and therefore skip the hat-superscripts again. Also, the relations $\Delta x = \epsilon$ and $\Delta t = \epsilon^2$ will frequently be used. In the new notation, LBM has the form

$$f(\mathbf{x} + \mathbf{v}\Delta x, \mathbf{v}, t + \Delta t) = f(\mathbf{x}, \mathbf{v}, t) - \frac{1}{\tau}\Big(f(\mathbf{x}, \mathbf{v}, t) - f^{eq}(\rho(\mathbf{x}, t), \epsilon\mathbf{u}(\mathbf{x}, t); \mathbf{v})\Big) \quad (8)$$

where $\mathbf{x}$ is a point on a regular square lattice with spacing Δx. Since (8) has been obtained by a discretization of the Boltzmann equation (1), we also expect consistency to (1). However, this is only true if Δt and Δx are of the same order. Due to the relation $\Delta t = \Delta x^2$, we find that (8) is consistent to a modification of the Boltzmann equation which can be calculated by applying a Taylor expansion around $\Delta x = \Delta t = 0$. To get first order consistency in $\Delta t = \Delta x^2$, space derivatives have to be considered up to third order. It turns out that (8) is first order consistent (in Δt) to

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{1}{\epsilon} v_k \frac{\partial f}{\partial x_k} = &-\frac{1}{\epsilon^2\tau}\Big(f - f^{eq}(\rho, \epsilon\mathbf{u})\Big) + \frac{1}{2} v_k v_l \frac{\partial^2 f}{\partial x_k \partial x_l} \\ &+ \frac{1}{\epsilon\tau} v_k \frac{\partial}{\partial x_k}\Big(f - f^{eq}(\rho, \epsilon\mathbf{u})\Big) + \frac{1}{3}\epsilon v_k v_l v_m \frac{\partial^3 f}{\partial x_k \partial x_l \partial x_m}. \quad (9)\end{aligned}$$

The space derivative of the collision operator is due to a term $\epsilon v_k \partial^2 f/\partial t \partial x_k$ in the expansion of $f(\mathbf{x} + \mathbf{v}\Delta x, \mathbf{v}, t + \Delta t)$. Replacing the time part ofthis derivative with the help of the modified Boltzmann equation then leads to derivatives of the collision term. Moreover, equation (9) contains additional second and third order derivatives compared to the Boltzmann equation (1) in the *diffusion scaling* $\mathbf{x} \to \mathbf{x}/\epsilon$, $t \to t/\epsilon^2$, $\mathbf{u} \to \epsilon\mathbf{u}$

$$\frac{\partial f}{\partial t} + \frac{1}{\epsilon}\mathbf{v}\nabla f = -\frac{1}{\epsilon^2\tau}\Big(f - f^{eq}(\rho, \epsilon\mathbf{u})\Big). \quad (10)$$

A formal expansion (similar to the much more detailed approach in [10] and [11]) shows that, in lowest ϵ-order, the velocity $\mathbf{u}$ is a solution of the incompressible Navier-Stokes equations with the pressure related to ϵ^2-fluctuations of ρ.

In the following section, we discuss the limit $\epsilon \to 0$ for a system of velocity moments which is equivalent to the velocity-discrete Boltzmann equation. While (10) leads to the Navier-Stokes equations with $Re = 3/\tau$, the additional term in (9) involving second order derivatives modifies the viscosity. It turns out, that (9) is related to the Navier-Stokes system with Reynolds number

$$\frac{1}{Re} = \frac{1}{3}\left(\tau - \frac{1}{2}\right) \tag{11}$$

which is the scaled version of (7).

4 Equivalent moment systems

Generally speaking, equation (10) consists of a linear hyperbolic differential operator and a nonlinear, stiff relaxation term on the right hand side. Obviously, this behavior does not change under a linear transformation of variables. In particular, if we choose an invertible linear mapping which includes the components

$$f \mapsto \begin{pmatrix} \langle f \rangle \\ \langle \mathbf{v}/\epsilon f \rangle \end{pmatrix} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \end{pmatrix} \tag{12}$$

the resulting system contains mass and momentum equations as a subsystem (see also [12] and [13] for a similar reformulation of LBM). The new system will again be hyperbolic with stiff relaxation terms which suggests the interpretation as *relaxation system*. To make these ideas more precise, we extend (12) to an invertible mapping by considering additional velocity moments. Apart from

$$Q_0(\mathbf{v}) = 1, \qquad Q_1(\mathbf{v}) = v_1/\epsilon, \qquad Q_2(\mathbf{v}) = v_2/\epsilon$$

we take

$$Q_3(\mathbf{v}) = \frac{1}{\epsilon^2}\left(v_1^2 - \frac{1}{3}\right), \qquad Q_4(\mathbf{v}) = \frac{1}{\epsilon^2} v_1 v_2, \qquad Q_5(\mathbf{v}) = \frac{1}{\epsilon^2}\left(v_2^2 - \frac{1}{3}\right)$$

and

$$Q_6(\mathbf{v}) = \frac{(3|\mathbf{v}|^2 - 4)v_1}{\epsilon^3} \qquad Q_7(\mathbf{v}) = \frac{(3|\mathbf{v}|^2 - 4)v_2}{\epsilon^3} \qquad Q_8(\mathbf{v}) = \frac{9|\mathbf{v}|^4 - 15|\mathbf{v}|^2 + 2}{\epsilon^4}$$

The polynomials Q_i are chosen in such a way that they are mutually f^* orthogonal, i.e. $\langle P_i P_j f^* \rangle = 0$ for $i \neq j$. Since also $\langle Q_i^2 f^* \rangle > 0$ for all i, one can show that

$$\mathbf{Q}f = (\langle Q_0 f \rangle, \ldots, \langle Q_8 f \rangle)^T$$

is a linear, invertible mapping. Indeed, given $\mathbf{M} \in \mathbb{R}^9$, we have

$$\mathbf{Q}^{-1}\mathbf{M} = \sum_{i=0}^{8} \frac{M_i Q_i}{\langle Q_i^2 f^* \rangle} f^*, \qquad \text{or} \qquad f = \sum_{i=0}^{8} \frac{\langle Q_i f \rangle}{\langle Q_i^2 f^* \rangle} Q_i f^*. \tag{13}$$

Taking into account that

$$\begin{aligned} &\langle Q_0^2 f^* \rangle^{-1} = 1, && \langle Q_1^2 f^* \rangle^{-1} = 3\epsilon^2, && \langle Q_2^2 f^* \rangle^{-1} = 3\epsilon^2, \\ &\langle Q_3^2 f^* \rangle^{-1} = 9\epsilon^4/2, && \langle Q_4^2 f^* \rangle^{-1} = 9\epsilon^4, && \langle Q_5^2 f^* \rangle^{-1} = 9\epsilon^4/2 \end{aligned}$$

we see that $f^{eq}(\rho, \epsilon\mathbf{u}; \mathbf{v})$ in (4) is precisely of the form (13) with

$$\mathbf{M}^{eq} = \mathbf{Q} f^{eq}(\rho, \epsilon\mathbf{u}) = (\rho, \rho u_1, \rho u_2, \rho u_1^2, \rho u_1 u_2, \rho u_2^2, 0, 0, 0)^T. \tag{14}$$

By applying $\mathbf{Q}$ to (10), a system of equations for the moments $\mathbf{M} = \mathbf{Q}f$ is obtained

$$\frac{\partial \mathbf{M}}{\partial t} + \frac{1}{\epsilon} \mathbf{Q} v_k \mathbf{Q}^{-1} \frac{\partial \mathbf{M}}{\partial x_k} = -\frac{1}{\epsilon^2 \tau} (\mathbf{M} - \mathbf{M}^{eq}) \tag{15}$$

(where v_k is used as abbreviation for the product operator $f(\mathbf{v}) \mapsto v_k f(\mathbf{v})$). To get (15) in a more convenient form, we introduce auxiliary names for the variables. Apart from $\langle Q_0 f \rangle = \rho$ and $\langle Q_i f \rangle = \rho u_i$ for $i = 1, 2$, the second order moments form a symmetric tensor Θ, the third order polynomials lead to a vector $\mathbf{q}$ and Q_8 gives rise to a scalar quantity s

$$\Theta = \begin{pmatrix} \langle Q_3 f \rangle & \langle Q_4 f \rangle \\ \langle Q_4 f \rangle & \langle Q_5 f \rangle \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} \langle Q_6 f \rangle \\ \langle Q_7 f \rangle \end{pmatrix}, \qquad s = \langle Q_8 f \rangle .$$

In these variables, the first two equations of (15) are related to mass and momentum conservation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{u} &= 0 \\ \frac{\partial \rho\mathbf{u}}{\partial t} + \operatorname{div} \Theta + \frac{1}{3\epsilon^2} \nabla \rho &= 0 \end{aligned} \tag{16}$$

Here, the divergence is applied to the rows of Θ. The equation for Θ is

$$\frac{\partial \Theta}{\partial t} + \frac{1}{3} \begin{pmatrix} \partial_{x_2} q_2 & \partial_{x_2} q_1 + \partial_{x_1} q_2 \\ \partial_{x_2} q_1 + \partial_{x_1} q_2 & \partial_{x_1} q_1 \end{pmatrix} = -\frac{1}{\epsilon^2 \tau} (\Theta - \rho\mathbf{u} \otimes \mathbf{u} + \frac{2\tau}{3} S[\rho\mathbf{u}]) \tag{17}$$

where

$$S_{ij}[\rho\mathbf{u}] = \frac{1}{2} \left(\frac{\partial \rho u_i}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_i} \right).$$

Finally, the third and fourth order moments satisfy

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{6} \nabla s &= -\frac{1}{\epsilon^2 \tau} \left(\mathbf{q} + \tau \operatorname{div} \begin{pmatrix} \Theta_{22} & 2\Theta_{12} \\ 2\Theta_{12} & \Theta_{11} \end{pmatrix} \right) \\ \frac{\partial s}{\partial t} &= -\frac{1}{\epsilon^2 \tau} (s + 4\tau \operatorname{div} \mathbf{q}) \end{aligned} \tag{18}$$

The diffusion limit $\epsilon \to 0$ of the system (16), (17) and (18) is easily determined. First we assume that ρ is of the form

$$\rho = \bar{\rho}(1 + 3\epsilon^2 p) \tag{19}$$

with a constant $\bar{\rho} > 0$ and an order one function p. Relation (19) is motivated by (16). Indeed, since all terms are scaled to order one, $\nabla\rho/\epsilon^2$ can only be balanced if $\nabla\rho = \mathcal{O}(\epsilon^2)$. Using (19), equation (16) transforms into

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{1}{3\epsilon^2} \operatorname{div} \mathbf{u} &= -\operatorname{div} p\mathbf{u} \\ \frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \frac{1}{\bar{\rho}}\Theta + \nabla p &= -3\epsilon^2 \frac{\partial p\mathbf{u}}{\partial t} \end{aligned} \tag{20}$$

and for $\epsilon \to 0$, equation (17) yields in lowest order

$$\frac{1}{\bar{\rho}}\Theta = \mathbf{u} \otimes \mathbf{u} - \frac{2\tau}{3} S[\mathbf{u}]. \tag{21}$$

Since (18) decouples completely from the other equations (in lowest order) and since $\operatorname{div} S[\mathbf{u}] = (\Delta + \nabla \operatorname{div})\mathbf{u}/2$, we obtain from (20) and (21) as limiting system the incompressible Navier-Stokes equations

$$\begin{aligned} \operatorname{div} \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \mathbf{u} \otimes \mathbf{u} + \nabla p &= \frac{\tau}{3} \Delta \mathbf{u} \end{aligned} \tag{22}$$

The Reynolds number is related to the relaxation time by $1/Re = \tau/3$. We remark that (16), (17), and (18) is a *relaxation system* for (22) as described for nonlinear hyperbolic systems in [14]. Moreover, the equations for $\mathbf{q}$ and s are not relevant in lowest ϵ-order. This indicates that the original discrete velocity equation carries too much information if the only aim is to approximate the limiting Navier-Stokes system. In fact, one can set up a lattice Boltzmann method based on only six discrete velocities which is compatible with a regular hexagonal space grid [15]. In this case, the equivalent moment system just consists of equations for mass, momentum and Θ which obviously is the minimal requirement in view of the underlying relaxation system. Thus, the overhead of three variables in the nine-velocity model is only due to the interplay between the required symmetries of the velocity space $\mathcal{V}$ and the condition that the grid should be a square lattice. (Note that storing three extra variables per node on a fine grid is numerically quite expensive.)

The basic idea of relaxation schemes, to replace nonlinear conservation systems by linear equations with nonlinear relaxation terms, comes with the price of introducing new variables. If the extended system is solved in a bounded domain, this leads to the problem of prescribing boundary conditions for the extra variables. In the system (16), (17), and (18), some of the variables still have a clear physical meaning (for example, Θ is the momentum flux up to the pressure term). This kind of

interpretation is lost for higher moments (like s) so that it is difficult to find suitable boundary conditions for particular physical situations. A similar problem appears in the classical theory of moment systems derived from the Boltzmann equation and is yet unsolved [16, 17]. Translated back into the original kinetic formulation, the problem to determine boundary values for higher moments is recovered in the sense that now occupation numbers have to be prescribed for all directions which enter the domain at a boundary.

As we have seen in the previous section, the discrete Lattice Boltzmann evolution with $\Delta t = \Delta x^2$ is consistent to (9) which is a modification of the original Boltzmann equation. Repeating the above argument for (9) instead of (10), we just have to consider additional contributions due to

$$\mathbf{Q}\frac{1}{\epsilon\tau}v_k\frac{\partial}{\partial x_k}\Big(f - f^{eq}(\rho,\epsilon\mathbf{u})\Big) = \frac{1}{\epsilon}\mathbf{Q}v_k\mathbf{Q}^{-1}\frac{\partial}{\partial x_k}\frac{1}{\tau}(\mathbf{M}-\mathbf{M}^{eq}),$$

adding to the first order derivatives in (15), as well as

$$\frac{1}{2}\frac{\partial^2}{\partial x_k\partial x_l}\mathbf{Q}v_kv_l\mathbf{Q}^{-1}\mathbf{M} + \frac{\epsilon}{3}\frac{\partial^3}{\partial x_k\partial x_l\partial x_m}\mathbf{Q}v_kv_lv_m\mathbf{Q}^{-1}\mathbf{M}$$

which adds second and third order derivative terms to the moment system. Introducing $\omega = 1/\tau$ and $\Theta^\omega = (1-\omega)\Theta + \omega\rho\mathbf{u}\otimes\mathbf{u}$, mass and momentum equation (16) now have the form

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{u} &= \frac{1}{6}\Delta\rho\\ \frac{\partial\rho\mathbf{u}}{\partial t} + \operatorname{div}\Theta^\omega + \frac{1}{3\epsilon^2}\nabla\rho &= \frac{1}{6}(\Delta + 2\nabla\operatorname{div})\rho\mathbf{u} - \frac{1}{18}\nabla\Delta\rho.\end{aligned} \tag{23}$$

Equation (17) is modified to

$$\begin{aligned}\frac{\partial\Theta}{\partial t} + \frac{1-\omega}{3}\begin{pmatrix}\partial_{x_2}q_2 & \partial_{x_2}q_1+\partial_{x_1}q_2\\ \partial_{x_2}q_1+\partial_{x_1}q_2 & \partial_{x_1}q_1\end{pmatrix}\\ = -\frac{\omega}{\epsilon^2}(\Theta-\rho\mathbf{u}\otimes\mathbf{u}) - \frac{1}{3\epsilon^2}\left(2S[\rho\mathbf{u}] - \frac{1}{9}\nabla\otimes\nabla\rho\right) + B^\Theta\end{aligned} \tag{24}$$

where $(\nabla\otimes\nabla\rho)_{ij} = \partial_{x_i}\partial_{x_j}\rho$ and B^Θ contains third derivatives of momentum and second derivatives of Θ and $\rho\mathbf{u}\otimes\mathbf{u}$. Since B^Θ is irrelevant in the lowest ϵ-order, we omit details. Similarly, the additional terms $A^{\mathbf{q}}$, $B^{\mathbf{q}}$ and A^s, B^s in

$$\begin{aligned}\frac{\partial\mathbf{q}}{\partial t} + \frac{1-\omega}{6}\nabla s &= -\frac{\omega}{\epsilon^2}\mathbf{q} - \frac{1}{\epsilon^2}\left(\operatorname{div}\begin{pmatrix}\Theta^\omega_{22} & 2\Theta^\omega_{12}\\ 2\Theta^\omega_{12} & \Theta^\omega_{11}\end{pmatrix} - A^{\mathbf{q}}\right) + B^{\mathbf{q}}\\ \frac{\partial s}{\partial t} &= -\frac{\omega}{\epsilon^2}s - \frac{1}{\epsilon^2}(4(1-\omega)\operatorname{div}\mathbf{q} - A^s) + B^s\end{aligned} \tag{25}$$

are not important in the limit $\epsilon\to 0$ and need not be specified. As before, we can show that (23), (24) and (25) lead to the incompressible Navier-Stokes equations in

lowest ϵ-order. Indeed, inserting the assumption (19) into the equation for ρ leads to $\operatorname{div}\mathbf{u} = \mathcal{O}(\epsilon^2)$ and (23) turns into

$$\operatorname{div}\mathbf{u} = \mathcal{O}(\epsilon^2)$$
$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}\left(\frac{(1-\omega)}{\bar{\rho}}\Theta + \omega \mathbf{u}\otimes\mathbf{u}\right) + \nabla p = \frac{1}{6}\Delta\mathbf{u} + \mathcal{O}(\epsilon^2).$$

Moreover, from (24) we see that

$$\frac{1}{\bar{\rho}}\Theta = \mathbf{u}\otimes\mathbf{u} - \frac{2\tau}{3}S[\mathbf{u}] + \mathcal{O}(\epsilon^2) \tag{26}$$

so that with

$$\frac{2\tau(1-\omega)}{3}\operatorname{div} S[\mathbf{u}] = \frac{\tau-1}{3}(\Delta\mathbf{u} + \nabla\operatorname{div}\mathbf{u}) = \frac{\tau-1}{3}\Delta\mathbf{u} + \mathcal{O}(\epsilon^2),$$

the velocity $\mathbf{u}$ in (23), (24), and (25) satisfies

$$\operatorname{div}\mathbf{u} = \mathcal{O}(\Delta t)$$
$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}\mathbf{u}\otimes\mathbf{u} + \nabla p = \frac{1}{3}\left(\tau - \frac{1}{2}\right)\Delta\mathbf{u} + \mathcal{O}(\Delta t).$$

From this result we can draw three conclusions. First, since the discrete lattice Boltzmann evolution with $\Delta t = \Delta x^2$ is consistent to (23), (24), and (25), it is also consistent to the incompressible Navier-Stokes equation with Reynolds number given in (11) (consistency is of order Δt). Secondly, the equations (23), (24), and (25) form a relaxation-type system for the incompressible Navier-Stokes equations. Note, however, that the nonlinearity in the convective part of the momentum equation is not completely replaced by a new variable. Instead, we find the linear combination $\operatorname{div}((1-\omega)\Theta + \omega\rho\mathbf{u}\otimes\mathbf{u})$ so that the equations are a combination of a direct and a relaxation system. Another difference to usual relaxation systems is the coupling between the relaxation parameter $\tau\epsilon^2$ and the space and time steps. Finally, if LBM is considered only as numerical method to approximate the Navier-Stokes equations, the information contained in the highest order moments is not needed. If we rewrite f in terms of its moments using the unique representation (13)

$$f = \left(\rho + 3\epsilon\rho u_k v_k + \frac{9\epsilon^2}{2}\Theta_{kl}(v_k v_l - \delta_{kl}/3) + \frac{3\epsilon^3}{2}(3|\mathbf{v}|^2 - 4)v_k q_k + \frac{\epsilon^4}{16}P_8\, s\right) f^* \tag{27}$$

(with $P_8 = \epsilon^4 Q_8$) we can use the fact that $\mathbf{q}$ and s are not required in the limit and that the only relevant information in Θ is relation (26). Setting $\mathbf{q} = \mathbf{0}$ and $s = 0$ in (27) and replacing Θ with $\rho(\mathbf{u}\otimes\mathbf{u} - \frac{2\tau}{3}S[\mathbf{u}])$, as suggested by (26), we obtain the distribution function

$$F(\rho, \mathbf{u}; \mathbf{v}) = \rho\left(1 + 3u_k v_k + \frac{9}{2}\left(u_k u_l - \frac{2\tau}{3}S_{kl}[\mathbf{u}]\right)(v_k v_l - \delta_{kl}/3)\right) f^*(\mathbf{v}).$$

where $S[\mathbf{u}]$ has to be calculated by taking derivatives of $\mathbf{u}$. In [18], a distribution function similar to F has been used to set up suitable boundary and initial values for the lattice Boltzmann method. Since the higher moments $\mathbf{q}$ and s are negligible and Θ is given essentially through ρ and $\mathbf{u}$, F can be viewed as an approximate inverse of the moment map $f \mapsto (\langle f \rangle, \langle \mathbf{v} f \rangle / \epsilon)$. This observation can be used to translate initial and boundary conditions which are given in terms of ρ and $\mathbf{u}$ into corresponding conditions for the occupation numbers (see [18] for details). In [19], a scheme has been introduced which is similar to the lattice Boltzmann method but which is based on F and avoids the high memory consumption in LBM which results from the necessity to store all occupation numbers.

While the system (23), (24), and (25) is obtained by applying $\mathbf{Q}$ to the continuous version (9) of the lattice Boltzmann evolution (8), a discretization of the moment system follows by applying $\mathbf{Q}$ directly to (8). In fact, since $\mathbf{Q}$ is an invertible mapping, this discretization of the moment system is equivalent to LBM. To simplify notation, we first rewrite (8) as in [6] by using $\omega = 1/\tau$, changing $\mathbf{x}$ to $\mathbf{x} - \mathbf{v}\Delta x$ and denoting the number of the time step by an upper index

$$f^{n+1}(\mathbf{x}, \mathbf{v}) = (1-\omega) f^n(\mathbf{x} - \mathbf{v}\Delta x, \mathbf{v}) + \omega f^{eq}(\rho^n(\mathbf{x} - \mathbf{v}\Delta x), \epsilon \mathbf{u}^n(\mathbf{x} - \mathbf{v}\Delta x); \mathbf{v}). \quad (28)$$

Applying $\mathbf{Q}$ to (28) amounts to multiplying the equation by Q_i and integrating over $\mathbf{v}$. On the left hand side, we obtain immediately the i-th component of the moment vector at the new time step

$$M_i^{n+1}(\mathbf{x}) = \left\langle Q_i f^{n+1}(\mathbf{x}, \mathbf{v}) \right\rangle .$$

To treat the terms on the right hand side, we use the unique representation (13) of distribution functions in terms of their moments

$$\begin{aligned} (1-\omega) f^n(\mathbf{x} - \mathbf{v}\Delta x, \mathbf{v}) + \omega f^{eq}(\rho^n(\mathbf{x} - \mathbf{v}\Delta x), \epsilon \mathbf{u}^n(\mathbf{x} - \mathbf{v}\Delta x); \mathbf{v}) \\ = \sum_{j=0}^{8} \frac{Q_j(\mathbf{v})}{\left\langle Q_j^2 f^* \right\rangle} \left((1-\omega) M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^*(\mathbf{v}) \end{aligned}$$

where we have set $\mathbf{M}^n = \mathbf{Q} f^n$ and $\mathbf{M}^{n,eq} = \mathbf{Q} f^{eq}(\rho^n, \epsilon \mathbf{u}^n)$. Altogether, we obtain the equivalent form of (28)

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^{8} \frac{1}{\left\langle Q_j^2 f^* \right\rangle} \left\langle Q_i Q_j \left((1-\omega) M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle \quad (29)$$

with $i = 0, \ldots, 8$. Again, $\mathbf{x}$ is restricted to a regular square lattice with spacing Δx. In the next section, we show that (29) is, in fact, a compact way of writing a finite difference discretization.

5 Microscopic transport and finite difference stencils

In order to interpret the discretization (29), we analyze expressions of the form

$$\langle \psi(\mathbf{x} - \mathbf{v}\Delta x) P f^* \rangle \tag{30}$$

where ψ is a smooth function, f^* is defined in (5), and P is a function in $\mathbf{v}$ (typically a polynomial). Note that $g(\mathbf{x}, \mathbf{v}, t) = \psi(\mathbf{x} - t\mathbf{v}/\epsilon) f^*(\mathbf{v})$ is the exact solution of the transport problem

$$\frac{\partial g}{\partial t} + \frac{1}{\epsilon}\mathbf{v} \cdot \nabla g = 0, \qquad g(\mathbf{x}, \mathbf{v}, 0) = \psi(\mathbf{x}) f^*(\mathbf{v}).$$

Hence, (30) is the P-moment of the solution of the transport problem at time $\Delta t = \epsilon^2$ (taking into account that $\Delta x = \epsilon$).

Introducing the notation $\mathbf{x}_{ij} = (i\Delta x, j\Delta x)$ and using the definition of $\langle \cdot \rangle$, we find

$$\langle \psi(\mathbf{x}_{ij} - \mathbf{v}\Delta x)(P f^*)(\mathbf{v}) \rangle = \sum_{m=0}^{8} \psi(\mathbf{x}_{ij} + \mathbf{c}_m \Delta x)(P f^*)(-\mathbf{c}_m) = \sum_{k,l=-1}^{1} \alpha_{kl} \psi(\mathbf{x}_{i+k,j+l})$$

so that (30) is, in fact, a finite difference expression. Usually, the coefficients α_{kl} are arranged in a table (the so called stencil). Here, we have

$$\begin{bmatrix} \alpha_{-1\,1} & \alpha_{01} & \alpha_{11} \\ \alpha_{-10} & \alpha_{00} & \alpha_{10} \\ \alpha_{-1-1} & \alpha_{0-1} & \alpha_{1-1} \end{bmatrix} = \begin{bmatrix} Pf^*(\mathbf{c}_8) & Pf^*(\mathbf{c}_4) & Pf^*(\mathbf{c}_7) \\ Pf^*(\mathbf{c}_1) & Pf^*(\mathbf{c}_0) & Pf^*(\mathbf{c}_3) \\ Pf^*(\mathbf{c}_5) & Pf^*(\mathbf{c}_2) & Pf^*(\mathbf{c}_6) \end{bmatrix} \tag{31}$$

If f^* is kept fixed, we can thus identify polynomials with finite difference approximations.

To find out the approximation properties of the stencil belonging to a particular polynomial, we replace $\psi(\mathbf{x} - \mathbf{v}\Delta x)$ by a Taylor expansion

$$\begin{aligned} \langle \psi(\mathbf{x} - \mathbf{v}\Delta x) P f^* \rangle &= \langle P f^* \rangle \, \psi(\mathbf{x}) - \Delta x \, \langle v_i P f^* \rangle \, \partial_{x_i} \psi(\mathbf{x}) \\ &+ \frac{\Delta x^2}{2} \langle v_i v_j P f^* \rangle \, \partial_{x_i} \partial_{x_j} \psi(\mathbf{x}) - \frac{\Delta x^3}{6} \langle v_i v_j v_k P f^* \rangle \, \partial_{x_i} \partial_{x_j} \partial_{x_k} \psi(\mathbf{x}) + \mathcal{O}(\Delta x^4). \end{aligned}$$

Note that the coefficients in front of the derivatives are just velocity moments of f^*. An immediate consequence of the symmetry

$$f^*(\mathbf{v}) = f^*(-\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{V}$$

is that all odd derivatives in the Taylor expansion vanish if $P(-\mathbf{v}) = P(\mathbf{v})$ and, vice versa, all even derivatives disappear provided $P(-\mathbf{v}) = -P(\mathbf{v})$. Hence, polynomials P involving only even (odd) orders are related to second order approximations of

even (odd) derivatives. These observations apply in particular to the discretization (29) because all polynomials Q_i are either even or odd.

The following examples are given in the form

$$\text{polynomial} \leftrightarrow \text{stencil} \leftrightarrow \text{approximated operator}.$$

We begin with $P = Q_0 Q_0 / \langle Q_0^2 f^* \rangle = 1$ which essentially describes the nine-point formula for the Laplacian

$$1 \leftrightarrow \frac{1}{36} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix} \leftrightarrow 1 + \frac{\Delta x^2}{6} \Delta + \mathcal{O}(\Delta x^4). \tag{32}$$

Our next example is based on $P = Q_0 Q_i / \langle Q_i^2 f^* \rangle = 3\epsilon v_i$ for $i = 1, 2$ which gives rise to first derivatives

$$-\frac{3v_1}{\Delta x} \leftrightarrow \frac{1}{12\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2) \tag{33}$$

and

$$-\frac{3v_2}{\Delta x} \leftrightarrow \frac{1}{12\Delta x} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & -4 & -1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_2} + \mathcal{O}(\Delta x^2). \tag{34}$$

We remark that both stencils can be viewed as convex combinations of standard central difference approximations. For example, the ∂_{x_1}-stencil in (33) can be written as

$$\frac{1}{12} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{6} \left(\frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4}{2} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right)$$

This observation applies to many stencils found in (29). However, the convex combinations are not always those of (33) and (34). Related to the odd polynomial $P = Q_2 Q_4 / \langle Q_4^2 f^* \rangle = 9\epsilon v_1 v_2^2$, we get

$$-\frac{9v_1 v_2^2}{\Delta x} \leftrightarrow \frac{1}{4\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2). \tag{35}$$

and similarly for $P = Q_1 Q_4 / \langle Q_4^2 f^* \rangle = 9\epsilon v_1^2 v_2$

$$-\frac{9v_1^2 v_2}{\Delta x} \leftrightarrow \frac{1}{4\Delta x} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_2} + \mathcal{O}(\Delta x^2). \tag{36}$$

The third approximation of first derivatives is obtained, for example, from the combination $P = Q_5 Q_6 / \langle Q_6^2 f^* \rangle = 3/2\epsilon v_1 (v_2^2 - 1/3)(3v_1^2 + 3v_2^2 - 4)$

$$-\frac{3}{2\Delta x}(3v_1^2 - 1)(3v_1^2 + 3v_2^2 - 4) \leftrightarrow \frac{1}{6\Delta x}\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2). \tag{37}$$

The corresponding approximation of ∂_{x_2} follows by exchanging the roles of the coordinates. For the stencil, this exchange amounts to a reflection of the weights at the diagonal and, in view of (31), leads to the transformation $P(v_1, v_2) \to P(v_2, v_1)$ of the polynomial. Note that all polynomials Q_i respect this symmetry, i.e. $(v_1, v_2) \mapsto Q(v_2, v_1)$ is again some Q_j.

Continuing with the even polynomials

$$\begin{pmatrix} Q_0 Q_3 / \langle Q_3^2 f^* \rangle & Q_0 Q_4 / \langle Q_4^2 f^* \rangle \\ Q_0 Q_4 / \langle Q_4^2 f^* \rangle & Q_0 Q_5 / \langle Q_5^2 f^* \rangle \end{pmatrix} = \frac{9}{2}\epsilon^2 \begin{pmatrix} v_1^2 - 1/3 & 2v_1 v_2 \\ 2v_1 v_2 & v_2^2 - 1/3 \end{pmatrix}$$

we find the second order derivative

$$\frac{9}{\Delta x^2}\left(v_1^2 - \frac{1}{3}\right) \leftrightarrow \frac{1}{6\Delta x^2}\begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix} \leftrightarrow \frac{\partial^2}{\partial x_1^2} + \mathcal{O}(\Delta x^2) \tag{38}$$

and the reflected version for $9(v_2^2 - 1/3)$. A mixed derivative is given by

$$\frac{9 v_1 v_2}{\Delta x^2} \leftrightarrow \frac{1}{4\Delta x^2}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \leftrightarrow \frac{\partial^2}{\partial x_1 \partial x_2} + \mathcal{O}(\Delta x^2). \tag{39}$$

We remark that the size of the stencils is directly related to the number of discrete velocities in the lattice Boltzmann model. In particular, usual central differences and the five point stencil for the Laplacian are obtained with a model consisting only of $\mathbf{c}_0, \ldots, \mathbf{c}_4$. Standard lattice Boltzmann methods with this particular choice of discrete velocities, however, are not consistent to the Navier-Stokes equations (see [20]).

6 LBM as finite difference scheme

For the special case $\tau = 1$, the lattice Boltzmann evolution (8) simplifies to

$$f(\mathbf{x} + \mathbf{v}\Delta x, t + \Delta t) = f^{eq}(\rho(\mathbf{x}, t), \epsilon \mathbf{u}(\mathbf{x}, t); \mathbf{v}). \tag{40}$$

Since $\omega = 1/\tau = 1$, the equivalent moment system (29) has the form

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^{8} \frac{1}{\langle Q_j^2 f^* \rangle} \left\langle Q_i Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle. \tag{41}$$

Note that f^{eq}, and thus also $\mathbf{M}^{eq}$, depend only on ρ and $\mathbf{u}$ so that the lattice Boltzmann evolution (with $\tau = 1$) is formulated completely in terms of the flow variables. In particular, (41) only has to be considered for $i = 0, 1, 2$ and since $M_i^{eq} = 0$ for $i \geq 6$ (see (14)) we can rewrite (41) as

$$\begin{aligned} \rho^{n+1}(\mathbf{x}) &= \sum_{j=0}^{5} \frac{1}{\left\langle Q_j^2 f^* \right\rangle} \left\langle Q_0 Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle \\ (\rho\mathbf{u})^{n+1}(\mathbf{x}) &= \sum_{j=0}^{5} \frac{1}{\left\langle Q_j^2 f^* \right\rangle} \left\langle \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle \end{aligned} \tag{42}$$

In [21] it has been shown that (42) falls into the class of *kinetic schemes* which are routinely used in CFD (see [22]). The particular version (42) is a finite difference realization of the general concept of kinetic schemes which easily follows from the considerations of the previous section. Indeed, using the identification of the polynomials $Q_0 Q_j / \left\langle Q_j^2 f^* \right\rangle$ with finite difference stencils as presented in Section 5, we can rewrite the equation for ρ^{n+1} and get

$$\rho^{n+1} = \rho^n - \epsilon \Delta x D_k \rho^n u_k^n + \frac{\Delta x^2}{6} L\rho^n + \frac{\epsilon^2 \Delta x^2}{2} D_{kl} \rho u_k^n u_l^n \tag{43}$$

where D_1, D_2 are the stencils (33) and (34), L is the discretization of the Laplacian based on (32), and D_{kl} are the second order derivatives given by (38) and (39). Similarly, we can write the momentum equation in (42) as

$$\begin{aligned} (\rho u_1)^{n+1} = \rho^n u_1^n - \epsilon \Delta x (D_1 \rho^n u_1^n u_1^n + \tilde{D}_2 \rho^n u_1^n u_2^n) - \frac{\Delta x}{3\epsilon} D_1 \rho^n \\ + \frac{\Delta x^2}{6} (L\rho^n u_1^n + 2 D_{1k} \rho^n u_k^n) \end{aligned} \tag{44}$$

where $\tilde{D}_2$ is the stencil (36) (for the second component of momentum, $\tilde{D}_1$ given by (35) and D_2 are used correspondingly). Since $\epsilon \Delta x = \epsilon^2 = \Delta t$, we see that (43) and (44) are indeed approximations of (23). (The discretization of $\nabla \Delta \rho / 18$ in the momentum equation (23) is contained in $\frac{\Delta x}{3\epsilon} D_k \rho$ which has a leading factor $1/\epsilon^2$ after division by Δt. Hence, the third order parts in the discretizations D_k appear in relevant order.) While (44) essentially approximates the Navier-Stokes equations, (43) assures the approximate divergence-free condition. Basically, it is a pseudo-compressibility method [23, 24] which can be seen by setting $\rho^n = \bar{\rho}(1 + 3\epsilon^2 p^n)$ in (43) which leads to

$$\frac{p^{n+1} - p^n}{\Delta t} + \frac{1}{3\epsilon^2} D_i u_i^n + D_i (p^n u_i^n) = \frac{1}{6} L p^n + \frac{\epsilon^2}{2\bar{\rho}} D_{kl} \rho u_k^n u_l^n .$$

This equation for the pressure contains elements of Chorin's artificial compressibility method [25] to replace $\operatorname{div} \mathbf{u} = 0$ by the equation

$$\frac{\partial p}{\partial t} + \frac{1}{\epsilon^2} \operatorname{div} \mathbf{u} = 0$$

and of the pressure stabilization method

$$\frac{1}{\epsilon^2}\operatorname{div}\mathbf{u} - \Delta p = 0$$

which was originally used in [26]. However, the convection term $\operatorname{div} p\mathbf{u}$ as well as the higher order nonlinear terms $\epsilon^2 \partial_{x_i}\partial_{x_j}\rho u_i u_j$, which appear in the discretization, are usually not considered.

Thus, in the case $\omega = 1$, the lattice Boltzmann method is nothing but a direct finite difference discretization of the Navier-Stokes equations (based on second order accurate nine-point stencils) together with a pseudo-compressibility approach.

As far as the structure of the stencils is concerned, it is interesting to note that first order derivatives are discretized differently (by D_k or $\tilde{D}_k$), depending on the term they appear in (similarly, the discretization of the Laplacian L differs from the combination $D_{11} + D_{22}$ of the other second order derivatives). We also remark that the original lattice Boltzmann evolution (40) together with the averaging $\langle f^{n+1}\rangle$ and $\langle \mathbf{v}/\epsilon f^{n+1}\rangle$ is an efficient way to evaluate the relatively large stencils. The evaluation mechanism can be described as follows: first, information about the old data ρ^n and $\mathbf{u}^n$ is preprocessed at each node by multiplication with weights and summation. Then, each node receives preprocessed data from its neighbors (eight values are communicated per node) and calculates new quantities for ρ and $\mathbf{u}$ by averaging. Compared to that, an equivalent, direct implementation of the stencils needs more communication. Indeed, just to calculate $\nabla\rho$ with the stencils (33) and (34), ρ has to be obtained from all neighbors which already amounts to a communication of eight values. On top of that, u_1 and u_2 are needed from the neighboring sites which increases communication by a factor three. Even if standard central differences are used to discretize the Navier-Stokes equations, at least three values from four neighboring sites have to be exchanged which amounts to a higher load of communication compared to LBM (despite the fact that LBM is based on larger stencils). Thus, LBM takes advantage of structural properties of the Navier-Stokes equations (i.e. the connection to the Boltzmann equation) which are neglected by standard schemes. In fact, LBM benefits two-fold: first, the exchange of data among grid points is reduced because the discretization of a scalar equation obviously requires less communication than the discretization of a system of equations. Secondly, a simple (and therefore fast) splitting scheme can be chosen as discretization of the Boltzmann equation without taking care of stiff terms because discretization errors have the correct structure to be incorporated in the viscous terms of the Navier-Stokes equations. Of course, these advantages can easily turn into disadvantages if the lattice Boltzmann approach should be applied to modifications of the Navier-Stokes equations for which no natural kinetic counterpart is available. (In such cases, finding a suitable Boltzmann equation which is related to the modified system can be very difficult or even impossible.)

Before we go over to the case $\omega \neq 1$, let us mention an auxiliary result which we

need later: if (40) is replaced by

$$f(\mathbf{x}+\mathbf{v}(k\Delta x),\mathbf{v},t+k\Delta t)=f^{eq}(\rho(\mathbf{x},t),\epsilon\mathbf{u}(\mathbf{x},t);\mathbf{v})$$

with some fixed $k\in\mathbb{N}$, the resulting finite difference scheme is again consistent to the incompressible Navier-Stokes system but now with Reynolds number $Re=6/k$. This is easily seen by replacing Δx and Δt with $k\Delta x$ and $k\Delta t$ in (43) and (44). Obviously, k factors out in all approximations of first order derivatives but remains in front of the second order ones. Note that the corresponding finite difference stencils have width $2k+1$.

7 LBM as combination of finite difference schemes

In the previous section, we have investigated the lattice Boltzmann evolution in the case $\omega=1$. Now, we extend the interpretation to the neighboring cases $\omega=1+\mathcal{O}(\epsilon)$. The basic idea is to write the lattice Boltzmann evolution operator (given by (28)) as linear combination of two extreme cases

$$\mathcal{L}_{\epsilon,\omega}=(1-\omega)\mathcal{L}_{\epsilon,0}+\omega\mathcal{L}_{\epsilon,1}$$

where $\mathcal{L}_{\epsilon,0}$ is defined as

$$(\mathcal{L}_{\epsilon,0}g)(\mathbf{x},\mathbf{v})=g(\mathbf{x}-\epsilon\mathbf{v},\mathbf{v}) \tag{45}$$

and $\mathcal{L}_{\epsilon,1}$ is related to the case $\omega=1$ which has been investigated in the previous section

$$\begin{gathered}(\mathcal{L}_{\epsilon,1}g)(\mathbf{x},\mathbf{w})=f^{eq}(\rho_g(\mathbf{x}-\epsilon\mathbf{w}),\epsilon\mathbf{u}_g(\mathbf{x}-\epsilon\mathbf{w});\mathbf{w}),\\ \rho_g=\langle g\rangle\,,\quad \mathbf{u}_g=\langle\mathbf{v}g\rangle/(\epsilon\rho_g).\end{gathered} \tag{46}$$

Note that $\mathcal{L}_{\epsilon,0}$ is precisely the solution operator of the collision-less Boltzmann equation in the diffusion scaling $\partial_t g+\mathbf{v}\cdot\nabla g/\epsilon=0$ (i.e. $\mathcal{L}_{\epsilon,0}g$ yields the solution of the transport equation at time $\Delta t=\epsilon^2$). Formally, $\omega=0$ corresponds to $\tau=\infty$ and the approximated system of moment equations is obtained from (23), (24), and (25) by setting $\omega=0$. In particular, the equations for $\rho,\mathbf{u}$ and Θ are given by

$$\begin{aligned}\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{u}&=\frac{1}{6}\Delta\rho\\ \frac{\partial\rho\mathbf{u}}{\partial t}+\operatorname{div}\Theta+\frac{1}{3\epsilon^2}\nabla\rho&=\frac{1}{6}(\Delta+2\nabla\operatorname{div})\rho\mathbf{u}-\frac{1}{18}\nabla\Delta\rho\\ \frac{\partial\Theta}{\partial t}+\frac{1}{3}\begin{pmatrix}\partial_{x_2}q_2 & \partial_{x_2}q_1+\partial_{x_1}q_2\\ \partial_{x_2}q_1+\partial_{x_1}q_2 & \partial_{x_1}q_1\end{pmatrix}&=-\frac{1}{3\epsilon^2}\left(2S[\rho\mathbf{u}]-\frac{1}{9}\nabla\otimes\nabla\rho\right)+B^{\Theta}.\end{aligned} \tag{47}$$

Note that, in the limit $\epsilon\to0$, the Navier-Stokes equations are *not* obtained since Θ is no longer related to $\rho\mathbf{u}\otimes\mathbf{u}$. Consequently, the lattice Boltzmann evolution (28)

can be viewed as a linear combination of two schemes with only one of them being consistent to the incompressible Navier-Stokes system. The factor $(1-\omega)$ in front of the inconsistent part $\mathcal{L}_{\epsilon,0}$ satisfies $|1-\omega| = |1-1/\tau| < 1$ if $\tau > 1/2$ which is a reasonable assumption in view of the required positivity of the Reynolds number (11).

For general $\omega \neq 1$, n steps of the lattice Boltzmann method $\mathcal{L}^n_{\epsilon,\omega}$ can be expressed in terms of the building blocks $\mathcal{L}_{\epsilon,0}$ and $\mathcal{L}_{\epsilon,1}$

$$\mathcal{L}^n_{\epsilon,\omega} = \sum_{\boldsymbol{\alpha}\in I_n} (1-\omega)^{|\boldsymbol{\alpha}|}\omega^{|\bar{\boldsymbol{\alpha}}|} \prod_{i=1}^{n} \mathcal{L}^{\alpha_i}_{\epsilon,0}\mathcal{L}^{\bar{\alpha}_i}_{\epsilon,1}$$

where I_n is the set of all vectors of length n with components in $\{0,1\}$, $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$, and the complementary vector $\bar{\boldsymbol{\alpha}}$ to $\boldsymbol{\alpha} \in I_n$ is defined by $\bar{\alpha}_i = 1-\alpha_i$ for all i. Note that $|\boldsymbol{\alpha}|$ counts the number of free flow steps so that the influence of terms with many free flow steps is small due to the damping by $(1-\omega)^{|\boldsymbol{\alpha}|}$. In particular, if $\omega > 1$, the factors $\omega^{|\bar{\boldsymbol{\alpha}}|}$ are amplifying so that the behavior of LBM is dominated by the kinetic scheme. In this case, the free transport part, which is not consistent to the Navier-Stokes equations, only acts as a correction. We remark that, in typical applications, LBM exactly operates in the range $1/2 < \tau \leq 1$ which is related to $1 \leq \omega < 2$. The simplest situation, apart from $\omega = 1$, is given by $\omega = 1 + C\epsilon$. In this case, the influence of the pure free transport contribution is damped below the truncation error already after two steps. We have

$$\mathcal{L}^2_{\epsilon,\omega} = \omega^2\mathcal{L}^2_{\epsilon,1} + (1-\omega)\omega\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} + (1-\omega)\omega\mathcal{L}_{\epsilon,1}\mathcal{L}_{\epsilon,0} + (1-\omega)^2\mathcal{L}^2_{\epsilon,0}. \tag{48}$$

To estimate the influence of $\mathcal{L}^2_{\epsilon,0}$, we apply it to a distribution function g which is consistent to our scaling in the sense that Q_i moments of g are order one quantities. Then, according to the comments above, $\langle \mathcal{L}^2_{\epsilon,0} g\rangle$ and $\langle \mathbf{v}/\epsilon\mathcal{L}^2_{\epsilon,0} g\rangle$ approximate (47) at time $t = 2\Delta t = 2\epsilon^2$ with initial values $\rho = \langle g\rangle$ and $\rho\mathbf{u} = \langle \mathbf{v}/\epsilon g\rangle$. Thus,

$$\left\langle \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} \mathcal{L}^2_{\epsilon,i} g \right\rangle = \begin{pmatrix} \rho \\ \rho\mathbf{u} \end{pmatrix} + \mathcal{O}(\Delta t), \qquad i = 0,1$$

(the case $i = 1$ has been treated in Section 6). Subtracting the two relations, we find $\mathcal{L}^2_{\epsilon,0} - \mathcal{L}^2_{\epsilon,1} = \langle \mathcal{O}\rangle(\epsilon^2)$, where $\langle \mathcal{O}\rangle(\epsilon^2)$ means that values of order ϵ^2 are obtained after application to a distribution function and integration with weights Q_0, Q_1, Q_2. Taking into account that $(1-\omega)^2 = \mathcal{O}(\epsilon^2)$ and setting $\lambda = 2(1-\omega)\omega$, equation (48) turns into

$$\mathcal{L}^2_{\epsilon,\omega} = (1-\lambda)\mathcal{L}^2_{\epsilon,1} + \lambda\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} + \frac{\lambda}{2}[\mathcal{L}_{\epsilon,1}, \mathcal{L}_{\epsilon,0}] + \langle \mathcal{O}\rangle(\epsilon^4)$$

where $[\cdot,\cdot]$ is the commutator. Due to our remark at the end of Section 6

$$(\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} g)(\mathbf{x},\mathbf{v}) = f^{eq}(\rho_g(\mathbf{x}-2\epsilon\mathbf{v}), \epsilon\mathbf{u}_g(\mathbf{x}-2\epsilon\mathbf{v}); \mathbf{v})$$

yields, after taking mass and momentum integrals, a finite difference scheme (formulated just in terms of ρ and $\mathbf{u}$) which is consistent to the Navier-Stokes equations with Reynolds number $Re = 6/2 = 3$. On the other hand, $\mathcal{L}_{\epsilon,1}^2$ corresponds to $Re = 6$ so that the combination

$$(1-\lambda)\mathcal{L}_{\epsilon,1}^2 + \lambda \mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} \tag{49}$$

describes a finite difference scheme for the Navier-Stokes equations with

$$\frac{1}{Re} = \frac{1-\lambda}{6} + \frac{\lambda}{3} = \frac{1}{3}\left(\frac{3}{2} - \omega\right) + \mathcal{O}(\epsilon^2) = \frac{1}{3}\left(\tau - \frac{1}{2}\right) + \mathcal{O}(\epsilon^2)$$

(since $\tau = 1/(1-(1-\omega)) = 1 + (1-\omega) + \mathcal{O}(\epsilon^2)$). This is the known relation (11) for LBM up to terms of the order ϵ^2 so that the difference between the scheme (49) and the lattice Boltzmann method is of the order of the truncation error

$$\mathcal{L}_{\epsilon,\omega}^2 = (1-\lambda)\mathcal{L}_{\epsilon,1}^2 + \lambda \mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} + \langle \mathcal{O} \rangle (\epsilon^4). \tag{50}$$

(This shows that $\frac{\lambda}{2}[\mathcal{L}_{\epsilon,1}, \mathcal{L}_{\epsilon,0}] = \langle \mathcal{O} \rangle (\epsilon^4)$ which can also be obtained directly.)

Relation (50) extends the result of Section 6 and leads to the following interpretation of LBM: For $\omega = 1$, LBM is an explicit finite difference scheme in the flow variables ρ and $\mathbf{u}$. For $\omega = 1 + \mathcal{O}(\epsilon)$, two steps of LBM can be identified (in the relevant ϵ-order) with the combination of two finite difference schemes which are formulated in terms of ρ and $\mathbf{u}$. Both schemes are consistent to the Navier-Stokes equations but with different Reynolds numbers. By combining the schemes, a whole range of Reynolds numbers is covered and the combination parameter is directly related to the viscosity.

For $\omega = 1 + \mathcal{O}(\epsilon^{\frac{1}{m}})$, the influence of the pure free transport contribution is damped below the truncation error after m steps. The basic structure, that LBM can be viewed as combination of schemes which are formulated directly in terms of the flow variables $\rho, \mathbf{u}$ and which approximate Navier-Stokes equations with fixed Reynolds numbers, remains the same. However, since this explanation becomes quite complicated, we use a different approach for the general case $\omega \neq 1$ in the following section.

8 LBM as relaxation scheme

In the previous sections, we have related LBM directly to the Navier-Stokes system which is possible in the case $\omega = 1$ and for $\omega = 1 + \mathcal{O}(\epsilon)$. In the generic case $\omega \neq 1$, however, equation (29) for M_i^{n+1} involves contributions

$$\frac{1}{\left\langle Q_j^2 f^* \right\rangle} \left\langle Q_i Q_j \left((1-\omega) M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle, \qquad j = 0, \ldots, 8 \tag{51}$$

from all moments. Note that

$$(1-\omega)\mathbf{M}+\omega\mathbf{M}^{eq}=(\rho,\rho u_1,\rho u_2,\Theta_{11}^{\omega},\Theta_{12}^{\omega},\Theta_{22}^{\omega},(1-\omega)q_1,(1-\omega)q_2,(1-\omega)s)^T$$

with $\Theta^{\omega}=(1-\omega)\Theta+\omega\rho\mathbf{u}\otimes\mathbf{u}$. According to Section 5, (51) represents finite difference approximations. Moreover, since LBM is ϵ^2-consistent to the moment system (23), (24), and (25), we can easily find out which of the stencils appears in relevant order. In fact, stencils are important in the update rule for M_i if they are related to expressions in the corresponding equation of the moment system. For example, the mass conservation in (23) does not involve $\Theta^{\omega}, \mathbf{q}$ or s from which we can conclude that the stencils in (51) with $i=0$ and $j=3,\ldots,8$ lead to expressions which are of the order of the truncation error. The stencils resulting from $j=0,1,2$, on the other hand, have already been introduced in Section 6. As in (43), we find

$$\frac{\rho^{n+1}-\rho^n}{\Delta t}+D_k\rho^n u_k^n=\frac{1}{6}L\rho^n+\mathcal{O}(\Delta t).$$

Similarly, the relevant stencils in the momentum equation are those given in (44)

$$\frac{(\rho u_1)^{n+1}-(\rho u_1)^n}{\Delta t}+(D_1\Theta_{11}^{\omega,n}+\tilde{D}_2\Theta_{12}^{\omega,n})+\frac{1}{3\epsilon}D_1\rho^n$$
$$=\frac{1}{6}(L\rho^n u_1^n+2D_{1k}\rho^n u_k^n)+\mathcal{O}(\Delta t)$$

and a corresponding relation holds for the second component of momentum. In view of the limit $\epsilon\to 0$, the most important aspect of the Θ-equation (24) is the discretization of the tensor $S_{kl}[\rho\mathbf{u}]=(\partial\rho u_k/\partial x_l+\partial\rho u_l/\partial x_k)/2$. Analyzing the stencils in (51) with $i=3,2,5$ and $j=1,2$ we find

$$S[\rho\mathbf{u}]\approx\begin{pmatrix} D_1\rho u_1 & \frac{1}{2}(\tilde{D}_2\rho u_1+\tilde{D}_1\rho u_2)\\ \frac{1}{2}(\tilde{D}_2\rho u_1+\tilde{D}_1\rho u_2) & D_2\rho u_2\end{pmatrix}$$

Similarly, the mixed terms $\partial_{x_1}q_2,\partial_{x_2}q_1$ in (24) are discretized with $\tilde{D}_k$ given in (35), (36) and the remaining first order derivatives $\partial_{x_1}q_1$ and $\partial_{x_2}q_2$ are treated with (37). In (25), all first order space derivatives are approximated with the help of (37) apart from $\partial_{x_i}\Theta_{12}^{\omega}$ in the equation for $\mathbf{q}$ which are based on (35) and (36). Again, the kinetic formulation (28) is a very efficient way to evaluate the stencils with very little communication.

As we have noted already in Section 4, the moment system (23), (24), and (25) is a relaxation-type system for the incompressible Navier-Stokes equations. Since the lattice Boltzmann method is equivalent to the above explicit finite difference approximation of the moment system, LBM can be regarded as a relaxation scheme. Note, however, that relaxation schemes are usually formulated in such a way that they turn into schemes for the limiting equations if the relaxation parameter is set to zero which is achieved by an implicit treatment of the stiff terms (see [14]). In the case of LBM, however, all terms are treated explicitly which forces Δt to be of order ϵ^2. In particular, setting ϵ to zero cannot lead to a new scheme. For a different approach based on the lattice Boltzmann moment system, see [13, 27].

9 Conclusions

We have investigated a frequently used lattice Boltzmann method for incompressible Navier-Stokes flows in two dimensions. The results are also applicable to other Lattice Boltzmann models because they rest on the basic observation that discrete microscopic transport together with weighted velocity averages is a reformulation of finite difference approximations. The investigation of the particular nine-velocity model implies:

- Standard LBM can be considered as a linear combination of a direct and a relaxation scheme. The discretization is based on fully explicit finite difference approximations.
- LBM can also be regarded as a linear combination of finite difference schemes for the Navier-Stokes equations with fixed viscosities.
- The kinetic aspects of LBM determine the structure of the finite difference stencils (transport+averaging=stencil).
- The kinetic formulation leads to an efficient evaluation of the stencils with comparably little communication.
- The scheme works in the stability constellation $\Delta t/\Delta x^2 = \mathcal{O}(1)$ known from the explicit scheme for advection diffusion equations.
- In the standard case, there is no special treatment of the stiff velocity-pressure coupling which arises in nearly incompressible situations.
- LBM has a high memory consumption because the kinetic variables outnumber the actual flow variables.
- For relaxation parameters close to one, the extra information in the kinetic variables is partly redundant since all occupation numbers are determined by ρ and $\mathbf{u}$ (up to corrections of the order of the truncation error).
- The formulation as relaxation system clearly shows the restricted relevance of some of the variables.
- The often claimed advantage of LBM, to be a simple algorithm which is easily parallelizable, is explained by the interpretation as fully explicit, low order finite difference method.

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