A Finite Dimensional Filter with Exponential Conditional Density*

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Abstract

In this paper we consider the continuous–time nonlinear filtering problem, which has an infinite–dimensional solution in general, as proved by Chaleyat–Maurel and Michel. There are few examples of nonlinear systems for which the optimal filter is finite dimensional, in particular the Kalman, Beneš, and Daum filters. In the present paper, we construct new classes of scalar nonlinear systems admitting finite–dimensional filters. We consider a given (nonlinear) diffusion coefficient for the state equation, a given (nonlinear) observation function, and a given finite–dimensional exponential family of probability densities. We construct a drift for the state equation such that the resulting nonlinear system admits a finite–dimensional filter evolving in the prescribed exponential family, provided the coefficients of the exponential family include the observation function and its square.

1 Introduction

The nonlinear filtering problem has an infinite–dimensional solution in general. Constructing nonlinear systems for which the optimal filter is finite dimensional is a problem which has received considerable attention in the past. It turned out that such systems are quite rare. Examples were given by Beneš [1] and Daum [4]. Instead, general results on nonexistence of such systems, based on Lie–algebraic techniques, were made available by Chaleyat–Maurel and Michel [3], and related works appeared for example in Ocone and Pardoux [7], and Lévine [6]. In the present paper, we construct scalar nonlinear filtering problems admitting finite–dimensional filters, using ideas developped in Brigo [2].

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2 Problem formulation

On the probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t \,,\, 0 \leq t \leq T\}$ we consider the following scalar state and observation equations:

$$\begin{aligned} dX_t &= f_t(X_t, Y_{[0,t]})\, dt + \sigma_t(X_t)\, dW_t \,, \\ dY_t &= h(X_t)\, dt + dV_t \,, \quad Y_0 = 0 \,. \end{aligned} \tag{1}$$

We set $a_t := \sigma_t^2$. Time invariance of h is needed to simplify exposition. Notice also the presence of the observation sample path $Y_{[0,t]} = (Y_s \,,\, 0 \leq s \leq t)$ in the drift f_t. The noise processes $\{W_t \,,\, 0 \leq t \leq T\}$ and $\{V_t \,,\, 0 \leq t \leq T\}$ are two standard Brownian motions. Finally, the initial state X_0 and the noise processes $\{W_t \,,\, 0 \leq t \leq T\}$ and $\{V_t \,,\, 0 \leq t \leq T\}$ are assumed to be mutually independent.

The nonlinear filtering problem consists in finding the conditional probability distribution π_t of the state X_t given the observations up to time t, i.e. $\pi_t(dx) := P[X_t \in dx \mid \mathcal{Y}_t]$, where $\mathcal{Y}_t := \sigma(Y_s \,,\, 0 \leq s \leq t)$. We shall assume that for all $0 \leq t \leq T$, the probability distribution π_t has an unnormalized density q_t w.r.t. the Lebesgue measure. In the general case, q_t does not evolve in a finite–dimensional parametrized family of unnormalized densities. In the linear case, q_t evolves in the manifold of unnormalized Gaussian densities. Some other examples where q_t evolves in a finite–dimensional family are given in Beneš [1], and Daum [4]. Notice, however, that in these examples the drift f_t is not allowed to depend on $Y_{[0,t]}$. In the present paper we focus on (unnormalized) exponential families, according to the following

Definition 2.1 *Let $\{c_1, \cdots, c_m\}$ be scalar functions defined on* $\mathbf{R}$, *such that $\{1, c_1, \cdots, c_m\}$ are* linearly independent, *have at most polynomial growth and are twice continuously differentiable. Assume that the convex set*

$$\Theta_0 := \{\theta \in \mathbf{R}^m \,:\, \psi(\theta) = \log \int \exp[\theta^T c(x)]\, dx < \infty\} \,,$$

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has non–empty interior. *Then*

$$\mathrm{EU}(c) \;=\; \{q(\cdot,\theta,\beta)\,,\,(\theta,\beta)\in U\},$$

$$q(x,\theta,\beta) \;:=\; \exp[\theta^T c(x)+\beta]\ ,$$

where $(\theta,\beta) := [\theta_1,\cdots,\theta_m,\beta]^T$ *and* $U \subseteq \Theta_0 \times \mathbf{R}$ *is open, is called an unnormalized exponential family of probability densities.*

3 Solution and examples

Theorem 3.1 *Let* $a_t = \sigma_t^2$ *and* h *be given, and let* $\mathrm{EU}(c)$ *be a given unnormalized exponential family, with* $c_1 = h$ *and* $c_2 = h^2$. *If the drift* f_t *is defined by*

$$f_t(x,Y_{[0,t]}) = u_t(x,\theta_t) := \tfrac{1}{2}\,\frac{\partial a_t}{\partial x}(x)+\tfrac{1}{2}\,a_t(x)\,\theta_t^T\,\frac{\partial c}{\partial x}(x)\ ,$$

with

$$\theta_t^1 := \theta_0^1 + Y_t\ , \qquad \theta_t^2 := \theta_0^2 - \tfrac{1}{2}t\ ,$$

and $\theta_t^i := \theta_0^i$, *for* $i = 3,\cdots,m$, *then the optimal nonlinear filter for the system (1) with coefficients* u_t, a_t *and* h, *has the density* $q_t = q(\cdot,\theta_t,\beta_0)$ *which belongs to* $\mathrm{EU}(c)$ *for all* $t \geq 0$.

Example 3.2 [Cubic observations] The present example is inspired by the cubic sensor studied in [5], where it is proven that for the cubic sensor problem there exists no finite dimensional filter. The optimal filter for the system

$$dX_t \;=\; 3\,[\tfrac{1}{2}\,Y_t\,X_t^2 - (1+\tfrac{1}{2}\,t)\,X_t^5]\,dt + dW_t\ ,$$

$$X_0 \sim \exp[-x^6]\,dx\ ,$$

$$dY_t \;=\; X_t^3\,dt + dV_t\ , \qquad Y_0 = 0\ ,$$

is finite dimensional and has conditional law with density

$$p_{X_t|\mathcal{Y}_t}(x) \propto \exp[Y_t x^3 - (1+\tfrac{1}{2}t)x^6]\ ,$$

for all $0 \leq t \leq T$.

Example 3.3 [Linear observations] The optimal filter for the system

$$dX_t \;=\; \tfrac{1}{2}\,\{\frac{\partial a_t}{\partial x}(X_t) + a_t(X_t)\,[Y_t + \frac{\mu_0}{v_0} - (t+\frac{1}{v_0})\,X_t]\}dt$$

$$+\,\sigma_t(X_t)dW_t\ , \qquad X_0 \sim \mathcal{N}(\mu_0,v_0)\ ,$$

$$dY_t \;=\; X_t dt + dV_t\ , \qquad Y_0 = 0,$$

with $a_t = \sigma_t^2$, is finite dimensional and has conditional law with density

$$X_t|\mathcal{Y}_t \sim \mathcal{N}(\frac{\mu_0 + Y_t v_0}{1+v_0 t}, \frac{v_0}{1+v_0 t}).$$

4 Conclusion

It seems, at a first sight, that our result contradicts classical results on nonexistence of finite dimensional filters, such as for example Chaleyat–Maurel and Michel [3], and the related works Ocone and Pardoux [7], and Lévine [6]. This contradiction appears a natural consequence of the arbitrariness of σ_t and h. Nonetheless, there is no real contradiction. Indeed, since θ_t depends on the observation sample path $Y_{[0,t]} = (Y_s\,,\,0 \leq s \leq t)$, the drift itself depends on the observations. This assumption is not allowed in the works mentioned before, and indeed we cannot construct a nonlinear filtering problem with prescribed (nonlinear) σ_t and h, with drift u_t *which does not depend on the observations*, and whose solution remains finite dimensional. We have to allow for observation–dependent drift in order to prove our result.

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