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A FINITE ELEMENT ANALYSIS FOR THE SIGNORINI  
 PROBLEM IN PLANE ELASTOSTATICS

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INTRODUCTION

If an elastic body rests upon a rigid frictionless support, the equilibrium can be formulated by means of the Signorini unilateral problem (cf. [1]). A systematic mathematical analysis of the problem was given by Fichera ([2], [3]).

For numerical solution of the Signorini problem, one can employ the finite element technique — see [8], [9]. It is the aim of our paper to present some a priori asymptotic error estimates for the finite element procedure, provided the solution is sufficiently regular. We also prove the convergence without any regularity assumptions.

1. FORMULATION OF THE SIGNORINI PROBLEM

In this section we shall introduce a variational formulation of the Signorini problem within the range of plane elastostatics, involving linear stress-strain relations and small deformations of a non-homogeneous, anisotropic body. Let  $\Omega \subset R^2$  be a bounded plane domain with Lipschitz boundary<sup>1)</sup>, occupied by an elastic body and let  $\mathbf{x} = (x_1, x_2)$  be a Cartesian coordinate system. Let  $\mathbf{n} = (n_1, n_2)$  denote the unit outward normal to the boundary  $\Gamma$ . We shall use the Sobolev spaces  $H^k(\Omega)$ ,  $k = 1, 2, \dots$  of functions, the generalized derivatives of which up to the order  $k$  exist and are square-integrable in  $\Omega$ . The usual norm of  $u$  in  $H^k(\Omega)$  will be denoted by  $\|u\|_k$ ,  $H^0(\Omega) = L_2(\Omega)$ ,<sup>2)</sup>

$$(f, g)_0 = \int_{\Omega} fg \, dx.$$

<sup>1)</sup> See [5] for the definition of Lipschitz boundary.

<sup>2)</sup> The same notation  $\|u\|_k$  will be used for vector-functions in  $[H^k(\Omega)]^2$  and the corresponding euclidean norms.

Let the displacement vector  $\mathbf{u} = (u_1, u_2) \in [H^1(\Omega)]^2$ . The strain-displacement relations are

$$(1.1) \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (i, j = 1, 2)$$

and the stress-strain relations

$$(1.2) \quad \tau_{ij} = c_{ijkl} \varepsilon_{kl} \quad (i, j = 1, 2)$$

hold, where the coefficients satisfy:

$$(1.3) \quad c_{ijkl} \in L_\infty(\Omega), \quad c_{ijkl} = c_{klij} = c_{jikl},$$

$$(1.4) \quad \exists c_0 = \text{const} > 0, \quad c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq c_0 \varepsilon_{ij} \varepsilon_{ij} \quad \forall \varepsilon_{ij} = \varepsilon_{ji}.$$

A repeated index implies summation over the range 1, 2.

The stress field satisfies the following equilibrium equations

$$(1.5) \quad \frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad (i = 1, 2),$$

where  $\mathbf{F}$  denotes the vector of body forces.

The traction-vector on the boundary

$$T_i = \tau_{ij} n_j$$

can be decomposed into the normal component

$$T_n = T_i n_i = \tau_{ij} n_i n_j$$

and the tangential component

$$T_t = T_i t_i = \tau_{ij} t_i n_j,$$

where  $\mathbf{t} = (t_1, t_2) = (-n_2, n_1)$  is the unit tangential vector.

The displacement vector can be decomposed similarly:

$$u_n = u_i n_i, \quad u_t = u_i t_i.$$

Suppose that the boundary  $\Gamma$  consists of three mutually disjoint parts

$$\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\tau \cup \bar{\Gamma}_a$$

and on  $\Gamma_u$  and  $\Gamma_\tau$  the displacements or tractions are prescribed, i.e.

$$(1.6) \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_u,$$

$$(1.7) \quad \mathbf{T} = \bar{\mathbf{T}} \quad \text{on} \quad \Gamma_\tau,$$

whereas on  $\Gamma_a$  the Signorini's conditions

$$(1.8) \quad u_n \leq 0, \quad T_n \leq 0, \quad u_n T_n = 0, \quad T_t = 0$$

hold.

Assume that  $\mathbf{F} \in [L_2(\Omega)]^2$ ,  $\bar{\mathbf{T}} \in [L_2(\Gamma_t)]^2$  are given and that both  $\Gamma_u$  and  $\Gamma_a$  contain sets open in  $\Gamma$ .

Introducing

$$\begin{aligned} A(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} c_{ijkl} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{kl}(\mathbf{v}) \, dx, \\ L(\mathbf{v}) &= \int_{\Omega} F_i v_i \, dx + \int_{\Gamma_t} \bar{T}_i v_i \, ds, \end{aligned}$$

the functional of potential energy can be defined as

$$\mathcal{L}(\mathbf{v}) = \frac{1}{2} A(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}).$$

Let

$$V = \{ \mathbf{v} \mid \mathbf{v} \in [H^1(\Omega)]^2, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u \},$$

and

$$K = \{ \mathbf{v} \mid \mathbf{v} \in V, \quad v_n \leq 0 \text{ on } \Gamma_a \}$$

be the subspace of virtual displacements and the convex cone of admissible virtual displacements, respectively.

The problem (1.1), (1.2), (1.5)–(1.8) can be formulated as follows: to find  $\mathbf{u} \in K$  such that

$$(1.9) \quad \mathcal{L}(\mathbf{u}) \leq \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K.$$

There is a close relation between the “classical” solution of (1.1), (1.2), (1.5)–(1.8) and a solution of (1.9). In fact, it holds

**Lemma 1.1.** *Any “classical” solution of (1.1), (1.2), (1.5)–(1.8) satisfies (1.9). On the contrary, if a solution of (1.9) is sufficiently regular, then it is a “classical” solution, as well.*

*Proof.* First we recall that  $\mathbf{u} \in K$  is a solution of (1.9) if and only if

$$(1.10) \quad A(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K.$$

Let  $\mathbf{u}$  be a “classical” solution. Multiplying (1.5) by a vector  $\mathbf{w} \in V$ , integrating by parts and using (1.6), (1.7), (1.8), we obtain

$$\begin{aligned} (1.11) \quad 0 &= \int_{\Omega} \left( -\tau_{ij}(\mathbf{u}) \frac{\partial w_i}{\partial x_j} + F_i w_i \right) dx + \int_{\Gamma_t \cup \Gamma_a} \tau_{ij}(\mathbf{u}) n_j w_i \, ds = \\ &= -A(\mathbf{u}, \mathbf{w}) + L(\mathbf{w}) + \int_{\Gamma_a} T_n(\mathbf{u}) w_n \, ds. \end{aligned}$$

Choose an arbitrary  $\mathbf{v} \in K$  and set  $\mathbf{w} = \mathbf{v} - \mathbf{u}$ . If  $u_n(x) < 0$ , then  $T_n(\mathbf{u})(x) = 0$ . If  $u_n(x) = 0$ , then  $w_n(x) = v_n(x) \leq 0$  and  $T_n(\mathbf{u})(x) \leq 0$ . Altogether, the last integral is non-negative and we have

$$A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) \geq 0,$$

i.e., (1.10).

On the contrary, let  $\mathbf{u} \in K$  be a sufficiently regular solution of (1.10). Denoting  $\mathbf{v} - \mathbf{u} = \mathbf{w} \in V$  and integrating by parts, we obtain

$$(1.12) \quad A(\mathbf{u}, \mathbf{w}) - L(\mathbf{w}) = - \int_{\Omega} \left( \frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j} + F_i \right) w_i dx + \\ + \int_{\Gamma} \tau_{ij}(\mathbf{u}) n_j w_i ds - \int_{\Gamma_\tau} \bar{T}_i w_i ds.$$

Choosing  $\mathbf{w} = \pm \varphi \in [\mathcal{D}(\Omega)]^2$  (where  $\mathcal{D}(\Omega)$  is the set of infinitely differentiable functions with compact support in  $\Omega$ ), we obtain the equilibrium equations (1.5). Consequently, from (1.12) and (1.10) it follows that

$$(1.13) \quad 0 \leq \int_{\Gamma_\tau} (\tau_{ij}(\mathbf{u}) n_j - \bar{T}_i) w_i ds + \int_{\Gamma_a} (T_n(\mathbf{u}) w_n + T_t(\mathbf{u}) w_t) ds.$$

The choice of  $\mathbf{w} \in V$  such that the traces of  $w_i$  vanish on  $\Gamma_a$  leads to the boundary conditions (1.7) on  $\Gamma_\tau$ . Thus in (1.13) only the last integral remains. Next choosing  $\mathbf{w}$  on  $\Gamma_a$  such that  $w_n = 0$ ,  $w_t = \pm \psi$ , we obtain  $T_t(\mathbf{u}) = 0$  on  $\Gamma_a$ .

Let  $w_n \leq 0$  be arbitrary. Then from (1.13)  $T_n(\mathbf{u}) \leq 0$  on  $\Gamma_a$  follows.

Finally,

$$(1.14) \quad A(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}) = 0$$

can be deduced from (1.10), inserting  $\mathbf{v} = \mathbf{0}$  and  $\mathbf{v} = 2\mathbf{u}$ .

Consequently, repeating the above procedure for  $\mathbf{w} = \mathbf{u}$ , we obtain

$$0 = \int_{\Gamma_a} T_n(\mathbf{u}) u_n ds.$$

As the product  $T_n(\mathbf{u}) u_n$  is non-negative, it must vanish on  $\Gamma_a$ .

**Proposition 1.1.** *There exists a unique solution of the problem (1.9).*

*Proof.* The set  $K$ , being closed and convex in  $[H^1(\Omega)]^2$ , is weakly closed. For the second Gâteaux differential of  $\mathcal{L}$  we may write

$$(1.15) \quad D^2 \mathcal{L}(\mathbf{u}; \mathbf{v}, \mathbf{v}) = A(\mathbf{v}, \mathbf{v}) \geq c_0 \int_{\Omega} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) dx \geq c_1 \|\mathbf{v}\|_1^2 \\ \forall \mathbf{u} \in [H^1(\Omega)]^2, \quad \mathbf{v} \in V,$$

(where the Korn's inequality has been used in the last step). Therefore  $\mathcal{L}$  is weakly lower semi-continuous and coercive and the existence and uniqueness of a solution follows.

## 2. FINITE ELEMENT APPROXIMATIONS

For simplicity, we restrict ourselves to polygonal domains. (For domains with smooth boundary, we refer the reader to the paper by Scarpini and Vivaldi [6], whose technique could be extended to the above problem).

Let  $\Omega \subset \mathbb{R}^2$  be a polygonal bounded domain (multiply connected, in general). We carve it into triangles  $T$  generating a triangulation  $\mathcal{T}_h$ . Denote  $h$  the maximal side of all triangles in  $\mathcal{T}_h$ . Let  $V_h$  be the space of continuous piecewise linear functions on the triangulation  $\mathcal{T}_h$ , vanishing on  $\Gamma_u$ .<sup>1)</sup>

We say that a family of triangulations  $\{\mathcal{T}_h\}$ ,  $0 < h \leq 1$ , is  $\alpha$ - $\beta$ -regular, if there exist positive  $\alpha$  and  $\beta$ , independent of  $h$  and such that (i) the minimal angle of all triangles in  $\mathcal{T}_h$  is not less than  $\alpha$  and (ii) the ratio between any two sides of  $\mathcal{T}_h$  is less than  $\beta$ .

For any  $h \in (0, 1)$  we define

$$K_h = \{ \mathbf{v} \mid \mathbf{v} \in [V_h]^2, v_n \leq 0 \text{ on } \Gamma_u \} .$$

Obviously,  $K_h \subset K \forall h \in (0, 1)$ . We say that  $\mathbf{u}_h \in K_h$  is a *finite element approximation to the problem (1.9)* if

$$(2.1) \quad \mathcal{L}(\mathbf{u}_h) \leq \mathcal{L}(\mathbf{v}) \quad \forall \mathbf{v} \in K_h .$$

It is readily seen that there exists a unique finite element approximation. This assertion can be verified by following the proof of Proposition 1.1.

We focus our attention to the estimate of the error  $\mathbf{u} - \mathbf{u}_h$  between the solutions of the problem (1.9) and (2.1), respectively. To this end we shall use the idea proposed by Mosco and Strang [10], like in [4] – Sect. 2. Let us recall the

**Lemma 2.1.** *Let  $\mathcal{J}$  be the functional defined on a closed convex subset  $K$  of a Banach reflexive space  $B$ . Assume that  $\mathcal{J}$  is twice differentiable in  $B$  and the second differential satisfies the following inequalities*

$$(2.2) \quad \alpha_0 \|z\|^2 \leq D^2 \mathcal{J}(u; z, z) \leq c \|z\|^2 \quad \forall u \in K, \quad z \in B .$$

*Let  $K_h \subset K$  be a closed convex set. Denote the minimizing element of  $\mathcal{J}$  over  $K$  and  $K_h$  by  $u$  and  $u_h$ , respectively. Assume that a  $w_h \in K_h$  exists such that  $2u - w_h \in K$ . Then it holds*

$$(2.3) \quad \|u - u_h\| \leq (c/\alpha_0)^{1/2} \|u - w_h\| .$$

For the proof – see [4] – Lemma 2.1.

<sup>1)</sup> The end-points of  $\Gamma_u$  coincide with the vertices of  $\mathcal{T}_h$ .

Hence the problem is to find a  $w_h \in K_h$  sufficiently close to  $u$  and such that  $2u - w_h \in K$ . We can prove the following

**Theorem 2.1.** *Assume that  $\mathbf{u} \in [H^2(\Omega)]^2$  and  $u_n \in H^2(\Gamma_a \cap \Gamma_m)$ , where  $\Gamma_m$ ,  $m = 1, 2, \dots, G$ , denotes any side of the polygonal boundary.*

*Then there exists a  $\mathbf{w}_h \in [V_h]^2$  such that*

$$(2.4) \quad 0 \geq w_{hm} \geq u_n \quad \text{on} \quad \Gamma_a,$$

*and, if the triangulations are  $\alpha$ - $\beta$ -regular, it achieves the optimal order of approximation, i.e.*

$$(2.5) \quad \|\mathbf{u} - \mathbf{w}_h\|_1 \leq Ch \left\{ \|\mathbf{u}\|_2 + \max_{m=1, \dots, G} \|u_n\|_{H^2(\Gamma_a \cap \Gamma_m)} \right\}$$

*(with  $C$  independent of  $h$  and  $\mathbf{u}$ ).*

Proof is based on two lemmas.

**Lemma 2.2.** *(One-sided approximation of  $u_n$  on the boundary). Let  $u_n \in H^2(\Gamma_a \cap \Gamma_m)$ ,  $m = 1, \dots, G$ . Then there exist linear spline functions  $\psi_h^{(m)} \in C(\Gamma_m)$ , (with nodes determined by the vertices of the triangulation  $\mathcal{T}_h$ ), such that*

$$(2.6) \quad 0 \geq \psi_h^{(m)} \geq u_n \quad \text{on} \quad \Gamma_m \cap \Gamma_a,$$

$$(2.7) \quad \|u_{nI} - \psi_h^{(m)}\|_{\Gamma_m}^2 \leq h^3 \int_{\Gamma_m \cap \Gamma_a} [d^2 u_n / ds^2]^2 ds$$

*holds for any  $m = 1, 2, \dots, G$ , where  $u_{nI}$  is the linear Lagrange interpolate of  $u_n$  on  $\bar{\Gamma}_m$  (with the same nodes) and*

$$\|\varphi\|_{\Gamma_m} = \sup_{s \in \Gamma_m} |\varphi(s)|.$$

*Proof is parallel to that of Lemma 2.2 in [4], where  $\Gamma$  is replaced by  $\bar{\Gamma}_m \cap \bar{\Gamma}_a$ . We set  $\psi_h^{(m)} = 0$  on  $\bar{\Gamma}_a$  and  $\psi_h^{(m)} = u_{nI}$  at the vertices of  $\bar{\Gamma}_m$  except the points  $\bar{\Gamma}_a \cap \bar{\Gamma}_m$ , where the one-sided approximation on  $\bar{\Gamma}_a$  is defined.*

**Lemma 2.3.** *Let  $\varphi_m \in C(\bar{\Gamma}_m)$ ,  $m = 1, 2, \dots, G$  be linear spline-functions with the nodes determined by a  $\alpha$ - $\beta$ -regular triangulation  $\mathcal{T}_h$ ,  $\varphi_m = 0$  on  $\bar{\Gamma}_a$ .*

*Then there exists a  $\mathbf{v}_h \in [V_h]^2$  such that  $v_{hm} = \varphi_m$  on  $\Gamma_m$  for  $m = 1, \dots, G$  and*

$$(2.8) \quad \|\mathbf{v}_h\|_1 \leq Ch^{-1/2} \max_{m=1, \dots, G} \|\varphi_m\|_{\Gamma_m}.$$

Proof is analogous to that of Lemma 3.2 in [4].

The proof of Theorem 2.1. Let  $\psi_h^{(m)}$  be the one-sided approximations of  $u_n$  defined in Lemma 2.2. Introducing

$$(2.9) \quad \varphi_m = u_{nI} - \psi_h^{(m)}, \quad m = 1, \dots, G,$$

we construct the vector-function  $\mathbf{v}_h \in [V_h]^2$  according to Lemma 2.3. Then the function

$$\mathbf{w}_h = \mathbf{u}_I - \mathbf{v}_h,$$

(where  $\mathbf{u}_I = (u_{1I}, u_{2I})$  denotes the Lagrange linear interpolate of  $\mathbf{u}$  over the triangulation  $\mathcal{T}_h$ ) satisfies (2.4), (2.5). In fact, on every  $\Gamma_m$  it holds

$$w_{hn} = u_{In} - v_{hn} = u_{nI} - \varphi_m = \psi_h^{(m)}$$

and (2.6) implies (2.4).

Furthermore, it is well-known that

$$(2.10) \quad \|\mathbf{u} - \mathbf{u}_I\|_1 \leq Ch \|\mathbf{u}\|_2.$$

Then

$$\|\mathbf{u} - \mathbf{w}_h\|_1 \leq \|\mathbf{u} - \mathbf{u}_I\|_1 + \|\mathbf{u}_I - \mathbf{w}_h\|_1 \leq \|\mathbf{u} - \mathbf{u}_I\|_1 + \|\mathbf{v}_h\|_1$$

holds and from (2.8), (2.9), (2.7) it follows

$$(2.11) \quad \|\mathbf{v}_h\|_1 \leq Ch \max_{m=1, \dots, G} \|u_n\|_{H^2(\Gamma_a \cap \Gamma_m)}.$$

Hence (2.10) and (2.11) yield the estimate (2.5).

**Corollary 2.1.** *Let  $\mathbf{u}$  and  $\mathbf{u}_h$  be the solutions of (1.9) and (2.1), respectively. If the assumptions of Theorem 2.1 are satisfied, then*

$$(2.12) \quad \|\mathbf{u} - \mathbf{u}_h\|_1 = O(h).$$

*Proof.* With regard to Lemma 2.1 and (2.5), it suffices to verify that (i) the functional  $\mathcal{L}$  satisfies (2.2) and (ii)  $\mathbf{w}_h \in K_h$ ,  $2\mathbf{u} - \mathbf{w}_h \in K$ . The positive-definiteness of (2.2) is an immediate consequence of the Korn's inequality (1.15), (for  $B \equiv V$ ). From (2.4) it follows that  $\mathbf{w}_h \in K_h$ . Obviously,  $2\mathbf{u} - \mathbf{w}_h \in V$  and on  $\Gamma_a$  we have

$$2u_n - w_{hn} \leq u_n - w_{hn} \leq 0$$

as a consequence of (2.4). Hence  $2\mathbf{u} - \mathbf{w}_h \in K$  follows and the proof is complete.

### 3. CONVERGENCE WITHOUT ANY REGULARITY ASSUMPTIONS

From the results developed by Fichera (cf. [3]) one concludes that the regularity of the solution  $\mathbf{u}$ , assumed in Theorem 2.1, cannot be expected, in general. Therefore we study the convergence of the finite element approximations without any regularity assumptions. To this end, we employ the following abstract



**Theorem 3.1.** (cf. [7] – chpt. 4). Let  $V$  be a Hilbert space with the norm  $\|\cdot\|$ ,  $K \subset V$  a convex closed subset,  $h \in (0, 1)$  a real parameter,  $K_h \subset K$  convex closed sets for any  $h$ .

Let a differentiable functional  $\mathcal{J}$  on  $V$  be given, the second differential of which exists and satisfies the inequalities (2.2) for any  $u \in K$  and  $z \in V$ .

Denote  $u$  and  $u_h$  the minimizing elements of  $\mathcal{J}$  over the sets  $K$  and  $K_h$ , respectively. Assume that  $v_h \in K_h$  exist such that

$$(3.1) \quad \lim_{h \rightarrow 0} \|u - v_h\| = 0.$$

Then it holds

$$(3.2) \quad \lim_{h \rightarrow 0} \|u - u_h\| = 0.$$

*Proof.* From (2.2) the existence and uniqueness of  $u$  and  $u_h$  follows. Let  $v_h \in K_h$  satisfy (3.1). Using the Taylor's theorem we may write

$$\mathcal{J}(v_h) = \mathcal{J}(u) + D\mathcal{J}(u, v_h - u) + \frac{1}{2}D^2\mathcal{J}(u + \vartheta_h(v_h - u); v_h - u, v_h - u).$$

Consequently, by virtue of (2.2), we conclude

$$(3.3) \quad \lim \mathcal{J}(v_h) = \mathcal{J}(u).$$

From the definition of  $u_h$  it follows

$$(3.4) \quad \mathcal{J}(u_h) \leq \mathcal{J}(v_h),$$

consequently,

$$\mathcal{J}(u_h) \leq c < +\infty \quad \forall h.$$

Since  $\mathcal{J}$  is coercive, it holds

$$\|u_h\| \leq c_1 < +\infty \quad \forall h.$$

Thus we can choose a subsequence (denoted again by  $\{u_h\}$ ) such that  $u_h \in K_h$ ,  $u_h$  tends to  $u^*$  weakly. As  $K$  is weakly closed,  $u^* \in K$ . We have

$$\mathcal{J}(u^*) \leq \lim \mathcal{J}(u_h) = \mathcal{J}(u),$$

consequently  $u^* = u$ .

There exist  $\lambda_h \in (0, 1)$  such that

$$\mathcal{J}(u_h) = \mathcal{J}(u) + D\mathcal{J}(u, u_h - u) + \frac{1}{2}D^2\mathcal{J}(u + \lambda_h(u_h - u); u_h - u, u_h - u)$$

and by virtue of (2.2)

$$\mathcal{J}(u_h) - \mathcal{J}(u) - D\mathcal{J}(u, u_h - u) \geq \frac{1}{2}\alpha_0\|u_h - u\|^2.$$

From (3.3), (3.4) and the weak convergence  $u_h \rightarrow u$  (3.2) follows for the subsequence. Since  $u$  is a unique solution, the whole sequence converges to  $u$ .

**Theorem 3.2.** Assume that there is only a finite number of "end-points"  $\bar{\Gamma}_a \cap \bar{\Gamma}_v$ ,  $\bar{\Gamma}_u \cap \bar{\Gamma}_v$ ,  $\bar{\Gamma}_u \cap \bar{\Gamma}_a$ . Then the set

$$K \cap [C^\infty(\bar{\Omega})]^2$$

is dense in  $K$ .

*Proof.* Let  $\mathbf{u} \in K$  be a fixed vector-function. Consider a system of open domains  $\{B_i\}$ ,  $i = 0, 1, \dots, r$ , which cover  $\bar{\Omega}$  and denote  $\{\varphi_i\}$  the corresponding decomposition of unity (i.e.  $\varphi_i \in \mathcal{D}(B_i)$ ,  $0 \leq \varphi_i \leq 1$ ,  $\sum_{i=0}^r \varphi_i(x) = 1 \forall x \in \bar{\Omega}$ ). Assume that  $\bar{B}_0 \subset \Omega$  and  $\bigcup_{i=1}^r B_i$  covers the boundary  $\Gamma$ . Denoting  $\mathbf{u}^j = \mathbf{u}\varphi_j$ , we have

$$\mathbf{u} = \sum_{j=0}^r \mathbf{u}^j, \quad \mathbf{u}^j \in [H^1(\Omega)]^2, \quad \text{supp } \mathbf{u}^j \subset B_j \quad \forall j.$$

We say that  $P \in \Gamma$  is a singular point, if it is a vertex of  $\Gamma$  or an "end-point"  $\bar{\Gamma}_a \cap \bar{\Gamma}_v$ ,  $\bar{\Gamma}_u \cap \bar{\Gamma}_v$ ,  $\bar{\Gamma}_u \cap \bar{\Gamma}_a$ . Suppose that each  $B_j$  contains at most one singular point. For brevity, we shall omit the superscript  $j$ . The system of  $B_i$ ,  $i = 1, \dots, r$ , can be divided into eight groups as follows.

1. *group.* Let  $B_j \cap \Gamma \subset \Gamma_u$ . Then  $u_k \in H_0^1(B_j \cap \Omega)$ , ( $k = 1, 2$ ) can be approximated by functions  $u_{k\kappa} \in C_0^\infty(B_j \cap \Omega)$  such that

$$\|u_{k\kappa} - u_k\|_1 \rightarrow 0 \quad \text{for } \kappa \rightarrow 0.$$

2. *group.* Let  $B_j \cap \Gamma \subset \Gamma_a$  and does not contain any vertex of  $\Gamma$ . Consider the local cartesian coordinate system  $(\xi, \eta)$  such that the  $\xi$ -axis coincides with  $\Gamma$ . Then we may write  $\mathbf{u} = u_\xi \mathbf{e}_\xi + u_\eta \mathbf{e}_\eta$ , where  $\mathbf{e}_\xi, \mathbf{e}_\eta$  are unit basis vectors, and  $u_\eta = -u_n \geq \geq 0$  for  $\eta = 0$ .

There exists a function  $v \in H^1(B_j \cap \Omega)$  such that  $v \geq 0$  in  $B_j \cap \Omega$ ,  $\text{supp } v \subset B_j$ ,  $v = u_\eta$  on  $\Gamma$  (see [5], chpt. 2. Th. 5.7 for the construction of that function). Let us define the extension  $Pv$  of  $v$  by means of the relation

$$(3.6) \quad Pv(\xi, \eta) = Pv(\xi, -\eta).$$

Then  $Pv \in H^1(B_j)$ . Using the regularization operator  $R_\kappa$  with the kernel

$$\omega(x, \kappa) = \begin{cases} A\kappa^{-2} \exp\left(\frac{|x^2|}{|x|^2 - \kappa^2}\right) & \text{for } |x| < \kappa, \\ 0 & \text{for } |x| \geq \kappa, \end{cases}$$

where  $\kappa$  and  $A$  are positive constants,  $x \equiv (\xi, \eta)$ , we define

$$(3.7) \quad R_\kappa Pv(x) = \int_{B_j} \omega(x - x', \kappa) Pv(x') dx', \quad x' = (\xi', \eta').$$

Since both  $\omega$  and  $Pv$  are non-negative, we obtain  $R_\kappa Pv \geq 0$  on  $\Gamma$ , moreover  $R_\kappa Pv \in C^\infty(\bar{\Omega})$  and

$$(3.8) \quad \|R_\kappa Pv - v\|_1 \rightarrow 0$$

holds for  $\kappa \rightarrow 0$ . We have

$$u_\eta = v + z, \quad z \in H_0^1(B_j \cap \Omega).$$

Setting

$$u_{\eta\kappa} = R_\kappa Pv + z_\kappa,$$

where  $z_\kappa \in C_0^\infty(B_j \cap \Omega)$  is an approximation of  $z$ , we obtain

$$(3.9) \quad \|u_{\eta\kappa} - u_\eta\|_1 \leq \|R_\kappa Pv - v\|_1 + \|z_\kappa - z\|_1 \rightarrow 0$$

for  $\kappa \rightarrow 0$  and  $u_{\eta\kappa} \in C^\infty(\bar{\Omega})$ ,  $u_{\eta\kappa} \geq 0$  on  $\Gamma$ .

We extend also the component  $u_\xi$  like  $v$  in (3.6) and regularize. Then  $u_{\xi\kappa} \equiv R_\kappa Pu_\xi \in C^\infty(\bar{\Omega})$ ,

$$(3.10) \quad \|u_{\xi\kappa} - u_\xi\|_1 \rightarrow 0 \quad \text{for } \kappa \rightarrow 0.$$

From (3.9) and (3.10) we conclude that for  $\kappa \rightarrow 0$

$$(3.11) \quad \|\mathbf{u}_\kappa - \mathbf{u}\|_1^2 = \sum_{k=1}^2 \|u_{k\kappa} - u_k\|_1^2 \leq 4(\|u_{\xi\kappa} - u_\xi\|_1^2 + \|u_{\eta\kappa} - u_\eta\|_1^2) \rightarrow 0.$$

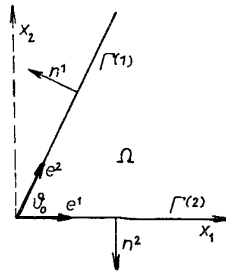


Fig. 1

3. group. Let  $B_j \cap \Gamma \subset \Gamma_a$  contain a vertex of  $\Gamma$ . In general, we use a “skew” coordinate basis for the components of  $\mathbf{u}$ . Thus let  $\mathbf{e}^1, \mathbf{e}^2$  be unit tangential and  $\mathbf{n}^1, \mathbf{n}^2$  unit outward normal vectors (see Fig. 1).

We may write

$$\mathbf{u} = \sum_{p=1}^2 u^{(p)} \mathbf{e}^p / \mathbf{e}^p \cdot \mathbf{n}^p,$$

where the dot denotes the scalar product,  $u^{(p)} = \mathbf{u} \cdot \mathbf{n}^p$ . Hence

$$u^{(p)} = u_n \leq 0 \quad \text{on } \Gamma^{(p)}.$$

Let us consider the component  $u^{(2)}$ . We transform the angular domain  $B_j \cap \Omega$  into the upper halfplane  $\{(\xi, \eta) \mid \eta > 0\}$  by means of a proper Lipschitz mapping  $T$  such that  $\Gamma^{(2)}$  is mapped into the positive  $\xi$ -axis and  $\Gamma^{(1)}$  into the negative  $\xi$ -axis. Let us extend the trace  $u^{(2)}$  from the positive onto the negative  $\xi$ -axis by means of the relation

$$Pu^{(2)}(-\xi) = Pu^{(2)}(\xi).$$

There exists a function  $\hat{v} \in H^1(\hat{B}_{j0})$ , where  $\hat{B}_{j0} = T(B_j \cap \Omega)$ , such that  $\hat{v} \leq 0$  in  $\hat{B}_{j0}$ ,  $\hat{v} = Pu^{(2)}$  on the  $\xi$ -axis,  $\text{supp } \hat{v} \subset T(B_j)$  (see again [5], Th. 2.5.7). If we define  $v(x_1, x_2) \equiv \hat{v}(T(x_1, x_2))$ , then  $v \in H^1(B_j \cap \Omega)$ ,  $v \leq 0$  in  $B_j \cap \Omega$ ,  $\text{supp } v \subset B_j$  and  $v = u^{(2)}$  on  $\Gamma^{(2)}$ . Consequently, we may write

$$u^{(2)} = v + z,$$

where  $z \in H^1(B_j \cap \Omega)$ ,  $\text{supp } z \subset B_j$ ,  $z = 0$  on  $\Gamma^{(2)}$ .

We define

$$v_\lambda = v(x + \lambda),$$

where  $\lambda \in R^2$  is a vector in the direction of the axis of the internal angle at the vertex. Then  $v_\lambda$  is an extension of  $v$  and it holds  $R_\varkappa v_\lambda \in C^\infty(\bar{\Omega})$ ,  $R_\varkappa v_\lambda \leq 0$  on  $\Gamma^{(2)}$ ,

$$\|R_\varkappa v_\lambda - v\|_1 \leq \|R_\varkappa v_\lambda - v_\lambda\|_1 + \|v_\lambda - v\|_1 \rightarrow 0$$

for  $|\lambda| \rightarrow 0$  and  $\varkappa \rightarrow 0$ ,  $\varkappa < C|\lambda|$ .

There exists a function  $w \in H^1(B_j \cap \Omega)$  such that  $w = z$  on  $\Gamma$ ,  $\text{supp } w \subset B_j$ ,  $w = 0$  in the angular domain  $0 < \vartheta < \frac{1}{2}\vartheta_0$ , where  $\vartheta_0$  denotes the internal angle and  $\vartheta$  the polar coordinate. Choosing the direction of a vector  $\lambda$  properly, we find a "shifted" function  $w_\lambda(x) = w(x + \lambda)$  such that  $R_\varkappa w_\lambda = 0$  on  $\Gamma^{(2)}$  and

$$\begin{aligned} \|R_\varkappa w_\lambda - w\|_1 &\rightarrow 0 \quad \text{for} \\ |\lambda| \rightarrow 0, \quad \varkappa \rightarrow 0, \quad \varkappa < C|\lambda|. \end{aligned}$$

Since it holds

$$z = w + z_0,$$

where  $z_0 \in H_0^1(B_j \cap \Omega)$ , we have

$$\begin{aligned} u_{\varkappa\lambda}^{(2)} &\equiv R_\varkappa v_\lambda + R_\varkappa w_\lambda + R_\varkappa z_0 \in C^\infty(\bar{\Omega}), \\ u_{\varkappa\lambda}^{(2)} &\leq 0 \end{aligned}$$

on  $\Gamma^{(2)}$  and

$$(3.12) \quad \begin{aligned} \|u_{\varkappa\lambda}^{(2)} - u^{(2)}\|_1 &\rightarrow 0 \quad \text{for} \\ \varkappa < C|\lambda|, \quad |\lambda| \rightarrow 0. \end{aligned}$$

An analogous approach can be applied to  $u^{(1)}$ . For any vector  $\mathbf{w}$  it holds  $w_k = a_1 w^{(1)} + a_2 w^{(2)}$ , where  $a_i$  are constants. Consequently, we have

$$\|w_k\|_1^2 \leq C \sum_{p=1}^2 \|w^{(p)}\|_1^2, \quad k = 1, 2.$$

Defining

$$\mathbf{u}_{\kappa\lambda} = \sum_{p=1}^2 u_{\kappa\lambda}^{(p)} \mathbf{e}^p / \mathbf{e}^p \cdot \mathbf{n}^p$$

and using (3.12) together with an analogous result for  $p = 1$ , we arrive at

$$(3.13) \quad \|\mathbf{u}_{\kappa\lambda}^j - \mathbf{u}^j\|_1 \rightarrow 0 \quad \text{for } \kappa < C|\lambda|, \quad |\lambda| \rightarrow 0.$$

4. *group.* Let  $B_j \cap \Gamma$  contain a singular point  $\bar{\Gamma}_a \cap \bar{\Gamma}_r$ , which may coincide with a vertex of  $\Gamma$ . We transform  $B_j \cap \Omega$  into the upper halfplane, mapping  $\Gamma_a$  into the positive  $\xi$ -axis. We apply the approach of the 3. group, used for  $u^{(2)}$ , setting  $\Gamma^{(2)} = \Gamma_a$ ,  $\Gamma^{(1)} = \Gamma_r$ . If  $B_j \cap \Gamma$  is straight, we use the same approach, substituting only  $-u_\eta$  for  $u^{(2)}$ .

5. *group.* Let  $B_j \cap \Gamma$  contain a point  $\bar{\Gamma}_u \cap \bar{\Gamma}_r$ , which may coincide with a vertex of  $\Gamma$ . Let  $\Gamma_u$  coincide with the positive  $\xi$ -axis of the local coordinate system. We may apply the approach used for approximating the function  $z$  in the 3. group, to both components  $u_k$ ,  $k = 1, 2$ , substituting  $\Gamma_u$  for  $\Gamma^{(2)}$ .

6. *group.* Let  $B_j \cap \Gamma$  contain a point  $\bar{\Gamma}_u \cap \bar{\Gamma}_a$  which is not a vertex. Let  $\Gamma_u$  coincide with the positive  $\xi$ -axis. The component  $u_\xi$  can be approximated as the function  $z$  in the 3. group. For  $u_\eta$ , we may write  $u_\eta = v + z$ , where  $v \in H^1(B_j \cap \Omega)$ ,  $v = u_\eta$  on  $\Gamma$ ,  $v \geq 0$  in  $B_j \cap \Omega$ ,  $\text{supp } v \subset B_j$  and  $v = 0$  in the first quadrant  $\xi > 0, \eta > 0$ ,  $z \in H_0^1(B_j \cap \Omega)$ . Defining  $v_\lambda(x) = v(x + \lambda)$ , where  $\lambda = (b, b)$ ,  $b > 0$ , and regularizing, we obtain  $R_\kappa v_\lambda = 0$  on  $\Gamma_u$ ,  $R_\kappa v_\lambda \geq 0$  on  $\Gamma_a$ . The remaining steps are obvious.

7. *group.* Let  $B_j \cap \Gamma$  contain a point  $\bar{\Gamma}_u \cap \bar{\Gamma}_a$ , coinciding with a vertex of  $\Gamma$ . Using the "skew" coordinate system, we obtain

$$u^{(1)} = u_\eta \leq 0 \quad \text{on } \Gamma^{(1)} \equiv \Gamma_a, \quad u^{(1)} = u^{(2)} = 0 \quad \text{on } \Gamma^{(2)} \equiv \Gamma_u.$$

The component  $u^{(2)}$  can be approximated like the function  $z$  in the 3. group.  $u^{(1)}$  can be written in the form  $u^{(1)} = v + z$ , where  $v \leq 0$  in  $B_j \cap \Omega$ ,  $v = 0$  for  $0 < \vartheta < \frac{1}{2}\vartheta_0$ ,  $\text{supp } v \subset B_j$  and  $v = u^{(1)}$  on  $\Gamma$ . Then "shifting"  $v$  properly (as for  $z$  in the 3. group) and regularizing, we obtain  $R_\kappa v_\lambda \leq 0$  on  $\Gamma_a$ ,  $R_\kappa v_\lambda = 0$  on  $\Gamma_u$ .

8. *group.* Let  $B_j \cap \Gamma \subset \Gamma_r$ . Since no boundary conditions are imposed, there exist approximations  $u_{k\kappa} \in C^\infty(\bar{\Omega})$ ,  $\text{supp } u_{k\kappa} \subset B_j$  such that

$$\|u_{k\kappa} - \mathbf{u}\|_1 \rightarrow 0 \quad \text{for } \kappa \rightarrow 0.$$

For  $B_0$  we define  $u_{k\kappa}^0 = R_\kappa u_k^0$ . Finally, adding  $\mathbf{u}_\kappa^j$  or  $\mathbf{u}_{\kappa\lambda}^j$ , respectively, from all the sets  $B_j$ , we are led to the assertion of the theorem.

**Theorem 3.3.** Let  $\mathbf{u}$  and  $\mathbf{u}_h$  be the solutions of the problem (1.9) and (2.1), respectively. Let the assumptions of Theorem 3.2 be satisfied. Then

$$(3.17) \quad \lim_{h \rightarrow 0} \|\mathbf{u} - \mathbf{u}_h\|_1 = 0.$$

Proof. From Theorem 3.2 it follows that a  $\mathbf{u}_\varkappa \in K \cap [C^\infty(\bar{\Omega})]^2$  exists such that

$$\|\mathbf{u} - \mathbf{u}_\varkappa\|_1 < \varepsilon/2.$$

As  $\mathbf{u}_\varkappa$  is smooth, we can define the Lagrange linear interpolate  $\mathbf{u}_{\varkappa I}$  over the triangulation  $\mathcal{T}_h$  and the estimate (cf. (2.10))

$$\|\mathbf{u}_{\varkappa I} - \mathbf{u}_\varkappa\|_1 < Ch\|\mathbf{u}_\varkappa\|_2$$

holds. We have  $\mathbf{u}_{\varkappa I} \in K_h$  and for sufficiently small  $h$

$$\|\mathbf{u} - \mathbf{u}_{\varkappa I}\|_1 \leq \|\mathbf{u} - \mathbf{u}_\varkappa\|_1 + \|\mathbf{u}_\varkappa - \mathbf{u}_{\varkappa I}\|_1 < \varepsilon.$$

Setting  $\mathbf{v}_h = \mathbf{u}_{\varkappa I}$ , the assumption (3.1) is satisfied. With  $\mathcal{J} \equiv \mathcal{L}$ , Theorem 3.1 implies the convergence of  $\mathbf{u}_h$ .

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## Souhrn

# ANALÝZA SIGNORINIHO ÚLOHY V ROVINNÉ PRUŽNOSTI METODOU KONEČNÝCH PRVKŮ

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Spočívá-li pružné těleso na dokonale tuhé a hladké opěře, pak jeho rovnováhu lze popsat pomocí Signoriniho jednostranné úlohy (viz [1]). Systematický matematický rozbor podal Fichera ([2], [3]). K numerickému řešení se hodí též metoda konečných prvků (viz [8], [9]).

V této práci se odvozují některé apriorní asymptotické odhady chyb metody konečných prvků za předpokladu jisté regularity řešení. V závěrečném odstavci je dokázána konvergence i k řešení, které není regulární.

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