A FINITE ELEMENT METHOD FOR SURFACE PDES: MATRIX PROPERTIES

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Abstract. We consider a recently introduced new finite element approach for the discretization of elliptic partial differential equations on surfaces. The main idea of this method is to use finite element spaces that are induced by triangulations of an "outer" domain to discretize the partial differential equation on the surface. The method is particularly suitable for problems in which there is a coupling with a flow problem in an outer domain that contains the surface, for example, two-phase incompressible flow problems. It has been proved that the method has optimal order of convergence both in the H^1 and in the L^2-norm. In this paper we address linear algebra aspects of this new finite element method. In particular the conditioning of the mass and stiffness matrix is investigated. For the two-dimensional case we present an analysis which proves that the (effective) spectral condition number of both the diagonally scaled mass matrix and the diagonally scalled stiffness matrix behaves like h^{-2}, where h is the mesh size of the outer triangulation.

Key words. Surface, interface, finite element, level set method, two-phase flow

AMS subject classifications. 58J32, 65N15, 65N30, 76D45, 76T99

1. Introduction.

Certain mathematical models involve elliptic partial differential equations posed on surfaces. This occurs, for example, in multiphase fluids if one takes so-called surface active agents (surfactants) into account. These surfactants induce tangential surface tension forces and thus cause Marangoni phenomena [5, 6]. In mathematical models surface equations are often coupled with other equations that are formulated in a (fixed) domain which contains the surface. In such a setting a common approach is to use a splitting scheme that allows to solve at each time step a sequence of simpler (decoupled) equations. Doing so one has to solve numerically at each time step an elliptic type of equation on a surface. *The surface may vary from one time step to another* and usually only some discrete approximation of the surface is (implicitly) available. A well-known finite element method for solving elliptic equations on surfaces, initiated by the paper [4], consists of approximating the surface by a piecewise polygonal surface and using a finite element space on a triangulation of this discrete surface, cf. [2, 5]. If the surface is changing in time, then this approach leads to time-dependent triangulations and time-dependent finite element spaces. Implementing this requires substantial data handling and programming effort.

In the recent paper [7] we introduced a new technique for the numerical solution of an elliptic equation posed on a hypersurface. The main idea is to use time-*in*dependent finite element spaces that are induced by triangulations of an "outer" domain to discretize the partial differential equation on the surface. This method is particularly suitable for problems in which the surface is given implicitly by a level set or VOF function and in which there is a coupling with a flow problem in a fixed outer domain. If in such problems one uses finite element techniques for the discetization of the flow equations in the outer domain, this setting immediately results in an easy to implement discretization method for the surface equation. If the surface varies in

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time, one has to recompute the surface mass and stiffness matrix using the same data structures each time. Moreover, quadrature routines that are needed for these computations are often available already, since they are needed in other surface related calculations, for example surface tension forces.

In [7] it is shown that this new method has optimal order of convergence in H^1 and L^2 norms. The analysis requires shape regularity of the outer triangulation, but *does not* require any type of shape regularity for discrete surface elements.

In the present paper we address linear algebra aspects of this new finite element method. In particular the conditioning of the mass and stiffness matrix is investigated. Numerical experiments in two- and three-dimensional examples (treated in section 2.2) clearly indicate and h^{-2} behaviour of the (effective) spectral condition number *both for the diagonally scaled mass and stiffness matrix*. Here h denotes the mesh size of the *outer* triangulation, which is assumed to be quasi-uniform (in a small neighbourhood of the surface). For the two-dimensional case we present an analysis which proves this h^{-2} conditioning property under reasonable assumptions. We believe that this analysis can be extended to the three-dimensional, but would require a lot of additional technical manipulations.

The remainder of the paper is organized as follows. In section 2.1 we describe the finite element method that is introduced in [7]. In section 2.2 we give results of some numerical experiments. These results illustrate the optimal order of convergence of the method and show the h^{-2} conditioning property. In section 3 we present an analysis of conditioning properties for the two-dimensional case. We start with an elementary introductory example (section 3.1). In section 3.2 we collect some preliminaries for the analysis. A condition number bound for the diagonally scaled mass matrix is derived in section 3.3. Finally, the stiffness matrix is treated in section 3.4.

2. Surface Finite Element method.

2.1. Descripton of the method. In this section we describe the finite element method from [7] for the three-dimensional case. The modifications needed for the two-dimensional case are obvious.

We assume that Ω is an open subset in $\mathbb{R}^3$ and Γ a connected C^2 compact hypersurface contained in Ω. For a sufficiently smooth function $g : \Omega \to \mathbb{R}$ the tangential derivative (along Γ) is defined by

$$\nabla_\Gamma g = \nabla g - \nabla g \cdot \mathbf{n}_\Gamma \, \mathbf{n}_\Gamma. \tag{2.1}$$

The *Laplace-Beltrami operator* on Γ is defined by

$$\Delta_\Gamma g := \nabla_\Gamma \cdot \nabla_\Gamma g.$$

We consider the Laplace-Beltrami problem in weak form: For given $f \in L^2(\Gamma)$ with $\int_\Gamma f \, d\mathbf{s} = 0$, determine $u \in H^1(\Gamma)$ with $\int_\Gamma u \, d\mathbf{s} = 0$ such that

$$\int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \, d\mathbf{s} = \int_\Gamma f v \, d\mathbf{s} \qquad \text{for all} \quad v \in H^1(\Gamma). \tag{2.2}$$

The solution u is unique and satisfies $u \in H^2(\Gamma)$ with $\|u\|_{H^2(\Gamma)} \le c \|f\|_{L^2(\Gamma)}$ and a constant c independent of f, cf. [4].

For the discretization of this problem one needs an approximation Γ_h of Γ. We assume that this approximate manifold is constructed as follows. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of tetrahedral triangulations of a *fixed* domain $\Omega \subset \mathbb{R}^3$ that contains Γ. These triangulations are assumed to be regular, consistent and stable. Take $\mathcal{T}_h \in \{\mathcal{T}_h\}_{h>0}$ and denote the set of tetrahedra that form $\mathcal{T}_h$ by $\{S\}$. We assume that Γ_h is sufficiently close to Γ (cf. (2.8), (2.9) below) and such that

- Γ_h can be decomposed as

$$\Gamma_h = \cup_{T\in\mathcal{F}_h} T, \tag{2.3}$$

  where for each T there is a corresponding tetrahedron $S_T \in \mathcal{T}_h$ with $T = S_T \cap \Gamma_h$ and $\text{meas}_2(T) > 0$. To avoid technical complications we assume that this S_T is unique, i.e., T does not coincide with a face of a tetrahedron in $\mathcal{T}_h$.
- Each T from the decomposition in (2.3) is *planar*, i.e., either a triangle or a quadrilateral.

Each quadrilateral $T \in \mathcal{F}_h$ can be subdivided into two triangles and thus we obtain a family of triangular subdivisions $\{\mathcal{F}_h\}_{h>0}$ of $(\Gamma_h)_{h>0}$. We emphasize that although the family $\{\mathcal{T}_h\}_{h>0}$ is *shape-regular* the family $\{\mathcal{F}_h\}_{h>0}$ in general is *not shape-regular*. In our examples $\mathcal{F}_h$ contains strongly deteriorated triangles that have very small angles and neighboring triangles can have very different areas, cf. Fig. 2.1.

The main idea of the method from [7] is that for discretization of the problem (2.2) we use a finite element space induced by the continuous linear finite elements on $\mathcal{T}_h$. This is done as follows. We define a subdomain that contains Γ_h:

$$\omega_h := \cup_{T\in\mathcal{F}_h} S_T. \tag{2.4}$$

This subdomain in $\mathbb{R}^3$ is partitioned in tetrahedra that form a subset of $\mathcal{T}_h$. We introduce the finite element space

$$V_h := \{\, v_h \in C(\omega_h) \mid v_{|S_T} \in P_1 \ \text{ for all } \ T \in \mathcal{F}_h \,\}. \tag{2.5}$$

This space induces the following space on Γ_h:

$$V_h^\Gamma := \{\psi_h \in H^1(\Gamma_h) \mid \exists\, v_h \in V_h : \, \psi_h = v_h|_{\Gamma_h} \,\}. \tag{2.6}$$

This space is used for a Galerkin discretization of (2.2): determine $u_h \in V_h^\Gamma$ with $\int_{\Gamma_h} u_h \mathrm{d}\mathbf{s}_h = 0$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \psi_h \, \mathrm{d}\mathbf{s}_h = \int_{\Gamma_h} f_h \psi_h \, \mathrm{d}\mathbf{s}_h \qquad \text{for all } \ \psi_h \in V_h^\Gamma, \tag{2.7}$$

with f_h an extension of f such that $\int_{\Gamma_h} f_h \mathrm{d}\mathbf{s}_h = 0$ (cf. [7] for details). Due the Lax-Milgram lemma this problem has a unique solution u_h. In [7] we analyze the discretization quality of this method. In this analysis we assume Γ_h to be sufficiently close to Γ in the following sense. Let $U \subset \mathbb{R}^3$ be a neighborhood of Γ and $d : U \to \mathbb{R}$ the signed distance function: $|d(x)| = \text{dist}(x, \Gamma)|$. We assume that

$$\text{ess sup}_{x\in\Gamma_h} |d(x)| \le c_0 h^2, \tag{2.8}$$
$$\text{ess sup}_{x\in\Gamma_h} \|\nabla d(x) - \mathbf{n}_h(x)\| \le \tilde{c}_0 h, \tag{2.9}$$

hold, with $\mathbf{n}_h(x)$ the outward pointing normal to Γ_h at $x \in \Gamma_h$. Under these assumptions the following optimal discretization error bounds are proven:

$$\|\nabla_{\Gamma_h}(u^e - u_h)\|_{L^2(\Gamma_h)} \le C\, h \|f\|_{L^2(\Gamma)}, \tag{2.10}$$
$$\|u^e - u_h\|_{L^2(\Gamma_h)} \le C\, h^2 \|f\|_{L^2(\Gamma)}, \tag{2.11}$$

with u^e a suitable extension of u and with a constant C independent of f and h.

2.2. Results of numerical experiments. In this section we present results of a few numerical experiments. As a first test problem we consider the Laplace-Beltrami equation

$$-\Delta_\Gamma u + u = f \quad \text{on } \Gamma,$$

with $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|_2 = 1\}$ and $\Omega = (-2,2)^3 + \mathbf{b}$ with $\mathbf{b} = (29^{-1}, 31^{-1}, 37^{-1})^T$. This example is taken from [1]. The shift over $\mathbf{b}$ is introduced for the following reason. The grids we use are obtained by regular (local) refinement as explained below. For the case $\mathbf{b} = 0$ there are grid points of the outer triangulation that lie *exactly* in Γ. To avoid this special case we introduce the shift. The zero order term is added to guarantee a unique solution. The source term f is taken such that the solution is given by

$$u(\mathbf{x}) = a\frac{\|\mathbf{x}\|^2}{12 + \|\mathbf{x}\|^2}\left(3x_1^2 x_2 - x_2^3\right), \quad \mathbf{x} = (x_1, x_2, x_3) \in \Omega,$$

with $a = -\frac{13}{8}\sqrt{\frac{35}{\pi}}$. A family $\{\mathcal{T}_l\}_{l\geq 0}$ of tetrahedral triangulations of Ω is constructed as follows. We triangulate Ω by starting with a uniform subdivision into 48 tetrahedra with mesh size $h_0 = \sqrt{3}$. Then we apply an adaptive red-green refinement-algorithm (implemented in the software package DROPS [3]) in which in each refinement step the tetrahedra that contain Γ are refined such that on level $l = 1, 2, \ldots$ we have

$$h_T \leq \sqrt{3}\, 2^{-l} \quad \text{for all } T \in \mathcal{T}_l \quad \text{with} \quad T \cap \Gamma \neq \emptyset.$$

The family $\{\mathcal{T}_l\}_{l\geq 0}$ is consistent and shape-regular. The interface Γ is the zero-level of $\varphi(\mathbf{x}) := \|\mathbf{x}\|^2 - 1$. Let $\varphi_l := I(\varphi)$ where I is the standard nodal interpolation operator on $\mathcal{T}_l$. The discrete interface is given by $\Gamma_{h_l} := \{\, \mathbf{x} \in \Omega \mid I(\varphi_l)(\mathbf{x}) = 0 \,\}$. Let $\{\phi_i\}_{1\leq i\leq m}$ be the nodal basis functions corresponding to the vertices of the tetrahedra in ω_h, cf. (2.4). The entries $\int_{\Gamma_h} \nabla_{\Gamma_h}\phi_i \cdot \nabla_{\Gamma_h}\phi_j + \phi_i\phi_j \mathrm{d}\mathbf{s}$ of the stiffness matrix are computed within machine accuracy. For the right-handside we use a quadrature-rule that is exact up to order five. The discrete problem is solved using a standard CG method with symmetric Gauss-Seidel preconditioner to a relative tolerance of 10^{-6}. The number of iterations needed on level $l = 1, 2, \ldots, 7$, is 14, 26, 53, 104, 201, 435, 849, respectively.

In [7] a discretization error analysis of this method is presented, which shows that it has optimal order of convergence, both in the H^1- and L^2-norm. The discretization errors in the $L^2(\Gamma_h)$-norm are given in table 2.1 (from [7]).

level l	$\|u - u_h\|_{L^2(\Gamma_h)}$	factor
1	0.1124	–
2	0.03244	3.47
3	0.008843	3.67
4	0.002186	4.05
5	0.0005483	3.99
6	0.0001365	4.02
7	3.411e-05	4.00

TABLE 2.1
Discretization errors and error reduction.

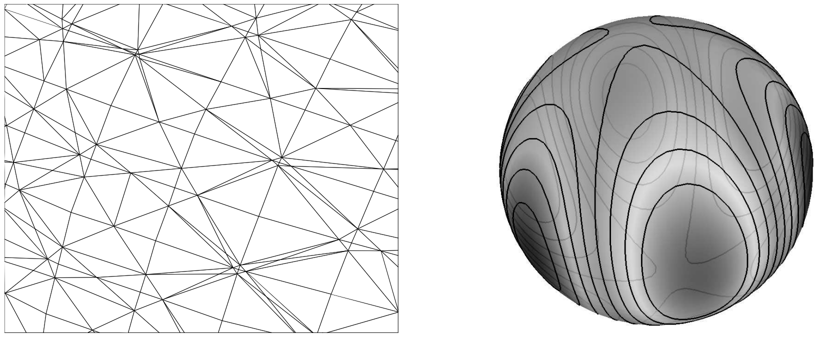

FIG. 2.1. *Detail of the induced triangulation of Γ_h (left) and level lines of the discrete solution u_h*

These results clearly show the h_l^2 behaviour as predicted by the analysis given in [7], cf. (2.11). To illustrate the fact that in this approach the triangulation of the approximate manifold Γ_h is strongly shape-irregular we show a part of this triangulation in Figure 2.1. The discrete solution is visualized in Fig 2.1.

The mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{A}$ have entries

$$M_{i,j} = \int_{\Gamma_h} \phi_i \phi_j \, \mathrm{d}\mathbf{s}_h, \quad A_{i,j} = \int_{\Gamma_h} \nabla_{\Gamma_h} \phi_i \cdot \nabla_{\Gamma_h} \phi_j \, \mathrm{d}\mathbf{s}_h, \quad 1 \leq i,j \leq m.$$

Define $\mathbf{D}_M := \mathrm{diag}(\mathbf{M})$, $\mathbf{D}_A := \mathrm{diag}(\mathbf{A})$ and the scaled matrices

$$\tilde{\mathbf{M}} := \mathbf{D}_M^{-\frac{1}{2}} \mathbf{M} \mathbf{D}_M^{-\frac{1}{2}}, \ \tilde{\mathbf{A}} := \mathbf{D}_A^{-\frac{1}{2}} \mathbf{A} \mathbf{D}_A^{-\frac{1}{2}}.$$

for different refinement levels we computed the largest and smallest eigenvalues of $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{A}}$. The results are given in Table 2.2 and Table 2.3.

level l	m	factor	λ_1	λ_2	λ_m	λ_m/λ_2	factor
1	112	-	3.8 e-17	0.0261	2.86	109	-
2	472	4.2	4.0 e-17	0.0058	2.83	488	4.5
3	1922	4.1	0	0.0012	2.83	2358	4.8
4	7646	4.0	0	0.00029	2.83	9759	4.1

TABLE 2.2
Eigenvalues of scaled mass matrix $\tilde{\mathbf{M}}$

level l	m	factor	λ_1	λ_2	λ_3	λ_m	λ_m/λ_3	factor
1	112	-	0	0	0.055	2.17	39.5	-
2	472	4.2	0	0	0.013	2.26	174	4.4
3	1922	4.1	0	0	0.0028	2.47	882	5.0
4	7646	4.0	0	0	0.00069	2.61	3783	4.3

TABLE 2.3
Eigenvalues of scaled stiffness matrix $\tilde{\mathbf{A}}$

These results show that for the scaled mass matrix there is one eigenvalue very close to or equal to zero and for the effective condition number we have $\frac{\lambda_m}{\lambda_2} \sim m \sim h_l^{-2}$. For the scaled stiffness matrix we observe that there are two eigenvalues close to or

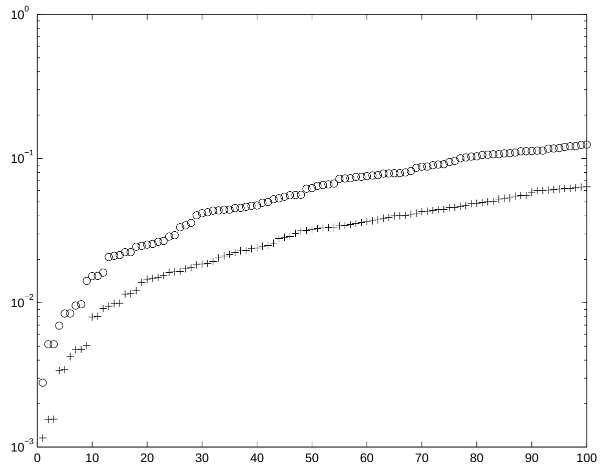

FIG. 2.2. *100 smallest nonzero eigenvalues of* $\tilde{\mathbf{M}}$ (+) *and* $\tilde{\mathbf{A}}$ (o) *on level* $l = 3$.

equal to zero and an effective condition number $\frac{\lambda_m}{\lambda_3} \sim m \sim h_l^{-2}$. In Fig. 2.2 for both matrices the 100 smallest eigenvalues away from zero are shown.

We also performed a numerical experiment with a very structured two-dimensional triangulation as illustrated in Fig. 3.3. The number of vertices is denoted by n_V ($n_V = 11$ in Fig. 3.3). The interface is given by $\Gamma = [0, 1] = [m_1, m_{n_V-1}]$. The mesh size of the triangulation is $h = \frac{2}{n_V-3}$. The vertices $v_1, v_3, \ldots, v_{n_V-2}$ and $v_0, v_2, \ldots, v_{n_V-1}$ are on lines parallel to Γ and the distances of the upper and lower lines to Γ are given by $\frac{\delta}{2}h$ and $\frac{1-\delta}{2}h$, respectively, with a parameter $\delta \in (0, 1)$ ($\delta = \frac{1}{2}$ in Fig 3.3). In this case a dimension argument immediately yields that both the mass and stiffness matrix are singular. For different values of n_V and of δ we computed the eigenvalues of the scaled mass and stiffness matrix. The results are given in tables 2.4 and 2.5. These results clearly suggest that the condition numbers of both the diagonally scaled mass and the diagonally scaled stiffness matrix behave like h^{-2} for $h \to 0$. Moreover, one observes for this particular example that the conditioning is insensitive to the distance of the interface Γ to the nodes of the outer triangulation.

δ	n_V	λ_1	λ_2	λ_{n_V}	λ_{n_V}/λ_2	factor
0.3	17	0	1.01e-2	2.42	239	-
	33	0	2.20e-3	2.42	1.10e+3	4.60
	65	0	5.14e-4	2.42	4.70e+3	4.27
	129	0	1.24e-4	2.42	1.95e+4	4.13
	257	0	3.06e-5	2.42	7.89e+4	4.06
0.5	65	0	5.14e-4	2.40	4.72e+3	-
0.1		0	5.14e-4	2.46	4.79e+3	
0.01		0	5.14e-4	2.50	4.86e+3	
0.001		0	5.14e-4	2.50	4.86e+3	

TABLE 2.4
Eigenvalues of scaled mass matrix $\tilde{\mathbf{M}}$

3. Analysis.

δ	n_V	λ_1	λ_2	λ_3	λ_{n_V}	λ_{n_V}/λ_3	factor
0.3	17	0	0	5.25e-2	2.0	38.1	-
	33	0	0	1.54e-2	2.0	130	3.41
	65	0	0	4.27e-3	2.0	468	3.60
	129	0	0	1.13e-3	2.0	1.77e+3	3.77
	257	0	0	2.92e-4	2.0	6.85e+3	3.88
0.5	65	0	0	4.27e-3	2.0	468	-
0.1		0	0	4.27e-3	2.0	468	
0.01		0	0	4.27e-3	2.0	468	
0.001		0	0	4.27e-3	2.0	468	

Table 2.5

Eigenvalues of scaled stiffness matrix $\tilde{\mathbf{A}}$

3.1. Mass and stiffness matrices and an introductory example. We take $\Gamma = [0,1]$ and consider a family of quasi-uniform triangulations $\{\mathcal{T}_h\}_{h>0}$ as illustrated in Fig 3.1, i.e., for each $T \in \mathcal{T}_h$ we have $\mathrm{meas}_1(\Gamma \cap T) > 0$ and the endpoints $x = 0$ and $x = 1$ of Γ lie on an edge of some $T \in \mathcal{T}_h$. The numbering of vertices v_i and intersection points m_i is as indicated in Fig. 3.1. We distinguish between the set of leafs L with corresponding index set ℓ and the set of nodes N (= vertices that are not leafs) with corresponding index set $\{1, 2, \ldots, n\}$. In the example in Fig. 3.1 we have $L = \{v_{1,1}, v_{6,1}, v_{9,1}, v_{9,2}, v_{13,1}\}$, $\ell = \{(1,1), (6,1), (9,1), (9,2), (13,1)\}$, $N = \{v_1, v_2, \ldots, v_{13}\}$. Note that for $i = (i_1, i_2) \in \ell$ we have $1 \le i_1 \le n$. The set of all vertices is denoted by $V = L \cup N$, and $|V| = n_V$. The corresponding index set is denoted by $\mathcal{I} = \{1, 2, \ldots, n\} \cup \ell$. This distinction between leafs and nodes is more clear, if in the triangulation we delete all edges between vertices that are on the same side of Γ. For the example in Fig. 3.1 this results in a directed graph shown in Fig. 3.2. For each node $v_i \in N$ the number of leafs attached to v_i is denoted by l_i (in our example: $l_1 = l_6 = l_{13} = 1,\ l_9 = 2,\ l_j = 0$ for all other j). The intersection points m_j are numbered as indicated in Fig. 3.1. In the analysis it is convenient to use the following notation: if $v_i, v_{i+1} \in N$ we define $m_{i,0} := m_i,\ m_{i,l_i+1} := m_{i+1}$, and $m_{1,0} := m_{1,1},\ m_{n,l_n+1} := m_{n,l_n}$. Using this the subdivision of Γ into the intersections with the triangles $T \in \mathcal{T}_h$ can be written as

$$\Gamma = \cup_{1 \le i \le n} \cup_{1 \le j \le l_i+1} [m_{i,j-1}, m_{i,j}]. \tag{3.1}$$

We define $h := \sup\{\, \mathrm{diam}(T) \mid T \in \mathcal{T}_h \,\}$, $\omega_h := \cup\{\, T \mid T \in \mathcal{T}_h \,\}$, the linear finite element space $V_h = \{\, v \in C(\Omega_h) \mid v_{|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h \,\}$ of dimension n_V, and the induced finite element space $V_h^\Gamma = \{\, w \in C(\Gamma) \mid w = v_{|\Gamma} \text{ for some } v \in V_h \,\}$ as in (2.5) and (2.6), respectively. These spaces V_h and V_h^Γ are called *outer* and *interface* finite element spaces, respectively.

For the implementation it is very convenient to *use the nodal basis functions of the outer finite element space for representing functions in the interface finite element space.* Let $\{\, \phi_i \mid i \in \mathcal{I} \,\}$ be the set of standard nodal basis functions in V_h, i.e., ϕ_i has value one at node v_i and zero values at all other $v \in V,\ v \ne v_i$. Clearly

$$V_h^\Gamma = \mathrm{span}\{\, (\phi_i)_{|\Gamma} \mid i \in \mathcal{I} \,\}$$

holds. A dimension argument shows that these functions are not independent and thus do not form a basis V_h^Γ. This set of generating functions is used for the implementation of a finite element discretization of scalar elliptic partial differential equations on Γ,

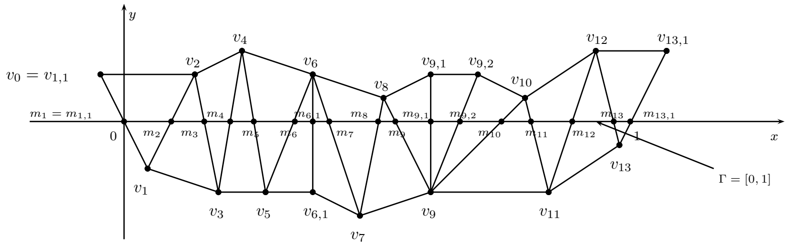

FIG. 3.1.

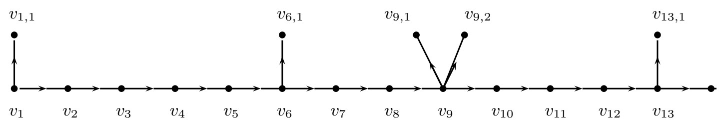

FIG. 3.2.

using the interface space V_h^Γ. The corresponding mass and stiffness matrices are given by

$$\begin{aligned} &\langle \mathbf{Mu}, \mathbf{u} \rangle = \int_0^1 u_h(x)^2\, dx, \quad \langle \mathbf{Au}, \mathbf{u} \rangle = \int_0^1 u_h'(x)^2\, dx, \\ &\text{with} \;\; u_h = \sum_{i \in \mathcal{I}} u_i (\phi_i)_{|\Gamma}, \quad \mathbf{u} := (u_i)_{i \in \mathcal{I}} \in \mathbb{R}^{n_V}. \end{aligned} \tag{3.2}$$

Both matrices are singular. The effective condition number of $\mathbf{M}$ (or $\mathbf{A}$) is defined as the ratio of the largest and smallest *nonzero* eigenvalue of $\mathbf{M}$ (or $\mathbf{A}$). Below we derive bounds for the effective condition of diagonally scaled mass and stiffness matrices .

An introductory example. First we consider a simple example with a uniform triangulation as shown in Fig. 3.3. The number of vertices is denoted by n_V ($n_V = 11$ in Fig. 3.3) and $h := \frac{2}{n_V - 3}$ is a measure for the mesh size of the triangulation. The interface $\Gamma = [0, 1] = [m_1, m_{n_V - 1}]$ is located in the middle between the upper and lower line of the outer triangulation. The nodal basis function corresponding to v_i is denoted by ϕ_i, $i = 0, 1, \ldots, n_V - 1$. We represent $u_h \in V_h^\Gamma$ as $u_h = \sum_{i=0}^{n_V - 1} u_i (\phi_i)_{|\Gamma}$. The vector representation is given by $\mathbf{u} = (u_0, u_1, \ldots, u_{n_V - 1})^T \in \mathbb{R}^{n_V}$. Now note that

$$\begin{aligned} \int_0^1 u_h(x)^2\, dx &= \sum_{i=1}^{n_V - 2} \int_{m_i}^{m_{i+1}} u_h(x)^2\, dx \\ &\sim h \sum_{i=1}^{n_V - 2} \big(u_h(m_i)^2 + u_h(m_{i+1})^2 \big) \sim h \sum_{i=1}^{n_V - 1} u_h(m_i)^2 \\ &= \frac{h}{4} \sum_{i=1}^{n_V - 1} \big(u_h(v_{i-1}) + u_h(v_i) \big)^2 = \frac{h}{4} \sum_{i=1}^{n_V - 1} \big(u_{i-1} + u_i \big)^2 = \frac{h}{4} \langle \mathbf{Lu}, \mathbf{Lu} \rangle, \end{aligned}$$

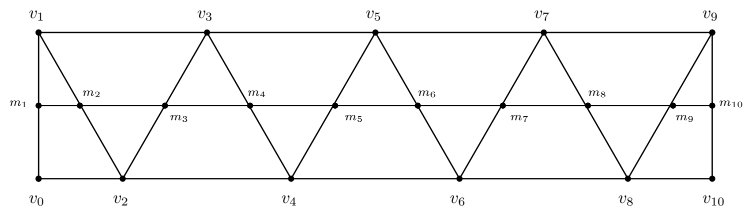

FIG. 3.3. *Example with a uniform triangulation.*

with

$$\mathbf{L} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & \emptyset & \\ & \emptyset & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix} \in \mathbb{R}^{(n_V-1)\times n_V}.$$

Thus the diagonally scaled mass matrix is spectrally equivalent to $\mathbf{L}^T\mathbf{L}$. This matrix has one zero eigenvalue $\lambda_1 = 0$, with corresponding eigenvector $(1,-1,1,-1,\ldots)^T$. The smallest nonzero eigenvalue is $\lambda_2 \sim h^2$, and thus for the effective condition number we obtain $\frac{\lambda_{n_V}}{\lambda_2} \sim h^{-2}$.

For the stiffness matrix we obtain the following:

$$\begin{aligned}\int_0^1 u_h'(x)^2\,dx &= \sum_{i=1}^{n_V-2}\int_{m_i}^{m_{i+1}} u_h'(x)^2\,dx \sim h\sum_{i=1}^{n_V-2}\Big(\frac{u_h(m_{i+1})-u_h(m_i)}{m_{i+1}-m_i}\Big)^2 \\ &\sim \frac{1}{h}\sum_{i=1}^{n_V-2}\big((u_h(v_i)+u_h(v_{i+1}))-(u_h(v_{i-1})+u_h(v_i))\big)^2 \\ &= \frac{1}{h}\sum_{i=1}^{n_V-2}\big(u_{i+1}-u_{i-1}\big)^2 = \frac{1}{h}\langle\hat{\mathbf{L}}\mathbf{u},\hat{\mathbf{L}}\mathbf{u}\rangle,\end{aligned}$$

with

$$\hat{\mathbf{L}} = \begin{pmatrix} -1 & 0 & 1 & & & \emptyset \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ \emptyset & & & -1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(n_V-2)\times n_V}.$$

Thus the diagonally scaled stiffness matrix is spectrally equivalent to $\hat{\mathbf{L}}^T\hat{\mathbf{L}}$. This matrix has two zero eigenvalues $\lambda_1 = \lambda_2 = 0$, with corresponding eigenvectors $(1,-1,1,-1,\ldots)^T$, $(1,1,\ldots,1)^T$. The smallest nonzero eigenvalue is $\lambda_3 \sim h^2$, and thus for the effective condition number we obtain $\frac{\lambda_{n_V}}{\lambda_3} \sim h^{-2}$.

We now consider a case as illustrated in Fig. 3.4 in which there is one vertex v_k ($k = 4$ in Fig. 3.4) for which $\mathrm{dist}(v_k,\Gamma) = \epsilon = \delta\frac{h}{2}$, with $\delta \in (0,1]$ and k such that $\frac{n_V}{k}$ is a fixed number if the mesh is refined.

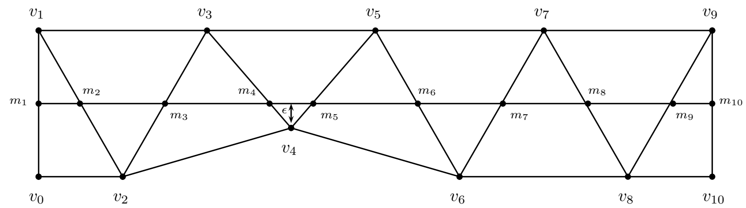

FIG. 3.4. *Degenerated case with a small distance.*

For this case we obtain:

$$\begin{aligned}\int_0^1 u_h(x)^2\,dx &\sim h\sum_{i=1}^{n_V-1} u_h(m_i)^2\\ &\sim h\sum_{i=1}^{k-1}\big(u_{i-1}+u_i\big)^2 + h(\delta u_{k-1}+u_k)^2 + h(u_k+\delta u_{k+1})^2\\ &\quad + h\sum_{i=k+2}^{n_V-1}\big(u_{i-1}+u_i\big)^2 \sim h\langle\tilde{\mathbf{L}}\mathbf{u},\tilde{\mathbf{L}}\mathbf{u}\rangle,\end{aligned}$$

with

$$\tilde{\mathbf{L}} = \begin{pmatrix} 1 & 1 & & & & & & \\ & \ddots & \ddots & & & \emptyset & & \\ & & 1 & 1 & & & & \\ & & & \delta & 1 & & & \\ & & & & 1 & \delta & & \\ & & \emptyset & & & 1 & 1 & \\ & & & & & & \ddots & \ddots \end{pmatrix} \in \mathbb{R}^{(n_V-1)\times n_V}.$$

Thus the diagonally scaled mass matrix is spectrally equivalent to $\tilde{\mathbf{L}}^T\tilde{\mathbf{L}}$. A straightforward calculation yields that this matrix has one zero eigenvalue $\lambda_1 = 0$ and for $\delta \le \sqrt{h}$ the first nonzero eigenvalue is of size $\lambda_2 \sim h\delta^2$. Hence for the effective condition number of the scaled mass matrix we obtain $\frac{\lambda_{n_V}}{\lambda_2} \sim h^{-1}\delta^{-2}$. Comparing this with the results of the 2D numerical experiment in the previous section, cf. Table 2.4, we see that the dependence of the effective spectral condition number on the distances of the vertices of the outer triangulation to Γ is a delicate issue and that in the analysis the variation of these distances should play a role .

3.2. Preliminaries. In this section we derive some results that will be used in the analyses of the mass- and stiffness matrix in the following sections.

The following identities hold for $u \in V_h$:

$$u(m_i) \;= \phi_{i-1}(m_i)u(v_{i-1}) + \phi_i(m_i)u(v_i) \quad \text{for} \;\; 1 \le i \le n, \tag{3.3}$$

$$u(m_i) = \phi_{i_1}(m_i)u(v_{i_1}) + \phi_i(m_i)u(v_i) \quad \text{for} \;\; i=(i_1,i_2)\in\ell. \tag{3.4}$$

We introduce the notation

$$
\begin{aligned}
&\tilde{u}_i := \phi_i(m_i)u(v_i) \quad \text{for} \ \ i \in \mathcal{I}, \\
&\psi_i := u(m_i) \quad \text{for} \ \ i \in \mathcal{I}, \\
&\xi_i := \begin{cases} \frac{\phi_i(m_{i+1})}{\phi_i(m_i)} & \text{for} \ \ 1 \le i \le n-1, \\ \frac{\phi_{i_1}(m_i)}{\phi_{i_1}(m_{i_1})} & \text{for} \ \ i = (i_1, i_2) \in \ell, \end{cases}
\end{aligned} \tag{3.5}
$$

and obtain the relations

$$
\psi_i = \xi_{i-1}\tilde{u}_{i-1} + \tilde{u}_i \quad \text{for} \ \ 2 \le i \le n, \tag{3.6}
$$

$$
\psi_i = \xi_i \tilde{u}_{i_1} + \tilde{u}_i \quad \text{for} \ \ i = (i_1, i_2) \in \ell. \tag{3.7}
$$

For $v_i = (x_i, y_i) \in V$ we denote the distance of v_i to the x-axis by $|y_i| =: d(v_i)$. We introduce the following assumption on the triangulations $\{\mathcal{T}_h\}_{h>0}$: For $v_i \in N$ let $v_j, v_r \in V$ be such that $v_i v_j$ and $v_i v_r$ intersect Γ. We assume:

$$
\frac{d(v_j)}{d(v_r)} \le c_1, \quad \text{with } c_1 \text{ independent of } i, j, r \text{ and } h. \tag{3.8}
$$

Remark 1. If $d(v) > c_0 h$ is satisfied for *all* $v \in V$ this implies that (3.8) holds. The condition $d(v) > c_0 h$ for all $v \in V$ implies that for each triangle $T \in \mathcal{T}_h$ the two parts of T on each side of Γ have a size that is uniformly (for $h \downarrow 0$) proportional to the size of T. Furthermore it implies that the subdivision of Γ into subintervals $[m_{i,j-1}, m_{i,j}]$ as in (3.1) is quasi-uniform. In our applications (where Γ is an approximation of the zero level of a level set function, cf. section 2.2) this is not very realistic. The assumption in (3.8) allows that Γ separates a triangle $T \in \mathcal{T}_h$ into two parts such that one of them has arbitrarily small size.

In the remainder of the paper, to simplify the notation, we use $f \sim g$ iff there are generic constants $c_1 > 0$ and c_2 independent of h, such that $c_1 g \le f \le c_2 g$.

Lemma 3.1. *For ξ_i as in (3.5) we have*

$$
\Pi_{k=j}^{i} \xi_k = \Big(\frac{1}{d(v_{j-1})} + \frac{1}{d(v_j)} \Big) \frac{1}{\frac{1}{d(v_i)} + \frac{1}{d(v_{i+1})}} \quad \textit{for} \ \ 1 \le j \le i \le n-1. \tag{3.9}
$$

Furthermore, if (3.8) is satisfied we have

$$
\xi_i \sim 1 \quad \textit{for} \ \ 1 \le i \le n-1, \ i \in \ell. \tag{3.10}
$$

Proof. From geometric properties we get

$$
\phi_i(m_i) = \frac{d(v_{i-1})}{d(v_i) + d(v_{i-1})} \quad \text{for} \ \ 1 \le i \le n, \tag{3.11}
$$

$$
\phi_{i_1}(m_i) = \frac{d(v_i)}{d(v_{i_1}) + d(v_i)} \quad \text{for} \ \ i = (i_1, i_2) \in \ell. \tag{3.12}
$$

Using this in the definition of ξ_i we obtain

$$
\xi_i = \begin{cases} \frac{d(v_{i+1})}{d(v_{i-1})} \frac{d(v_{i-1}) + d(v_i)}{d(v_i) + d(v_{i+1})} & \text{for} \ \ 1 \le i \le n-1, \\ \frac{d(v_i)}{d(v_{i_1-1})} \frac{d(v_{i_1-1}) + d(v_{i_1})}{d(v_{i_1}) + d(v_i)} & \text{for } i = (i_1, i_2) \in \ell. \end{cases} \tag{3.13}
$$

In both cases ξ_i is of the form

$$\xi_i = a\Big(\frac{\frac{1}{a}+z}{1+z}\Big),$$

namely with $a = \frac{d(v_{i+1})}{d(v_{i-1})}$, $z = \frac{d(v_i)}{d(v_{i+1})}$ if $1 \leq i \leq n-1$, and $a = \frac{d(v_i)}{d(v_{i_1-1})}$, $z = \frac{d(v_{i_1})}{d(v_i)}$ if $i \in \ell$. Note that $z > 0$ and from (3.8) it follows that $a \sim 1$. Furthermore:

$$\begin{aligned} \frac{1}{a} \leq \frac{\frac{1}{a}+z}{1+z} \leq 1 \quad &\text{for } z \geq 0,\ a \geq 1, \\ 1 \leq \frac{\frac{1}{a}+z}{1+z} \leq \frac{1}{a} \quad &\text{for } z \geq 0,\ 0 < a \leq 1. \end{aligned}$$

This yields $\min\{a,1\} \leq \xi_i \leq \max\{1,a\}$ and thus the result in (3.10) is proved.

For $1 \leq i \leq n-1$ the representation of ξ_i in (3.13) can be rewritten as

$$\xi_i = \Big(\frac{1}{d(v_{i-1})} + \frac{1}{d(v_i)}\Big)\frac{1}{\frac{1}{d(v_i)} + \frac{1}{d(v_{i+1})}}.$$

Using this the result in (3.9) immediately follows. $\square$

We introduce the notation: $\Delta_i := m_{i+1} - m_i$ $(= m_{i,j_1+1} - m_{i,0})$ for $i = 1,\dots,n$, and $\Delta_0 := \Delta_1,\ \Delta_{n+1} := \Delta_n$. Due to quasi-uniformity of $\{\mathcal{T}_h\}_{h>0}$ the following holds:

$$\begin{aligned} &|\mathrm{supp}(\phi_i) \cap \Gamma| = \Delta_{i_1} \quad \text{for } i = (i_1,i_2) \in \ell, \\ &|\mathrm{supp}(\phi_i) \cap \Gamma| = \Delta_{i-1} + \Delta_i + \Delta_{i+1} \sim h \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

Lemma 3.2. *Assume that* (3.8) *holds. Then we have*

$$\|\phi_i\|_\Gamma^2 := \int_0^1 \phi_i(x)^2\,dx \sim \Delta_{i_1}\phi_i(m_i)^2 \quad \textit{for all} \ \ i = (i_1,i_2) \in \ell, \tag{3.14}$$

$$\|\phi_i\|_\Gamma^2 \sim h\,\phi_i(m_i)^2 \quad \textit{for} \ \ 1 \leq i \leq n, \tag{3.15}$$

$$\|(\phi_i)_x\|_\Gamma^2 := \int_0^1 \phi_i'(x)^2\,dx \sim \frac{1}{\Delta_{i_1}}\phi_i(m_i)^2 \quad \textit{for all} \ \ i = (i_1,i_2) \in \ell, \tag{3.16}$$

$$\|(\phi_i)_x\|_\Gamma^2 \sim \Big(\frac{1}{\Delta_{i-1}} + \frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}\big(\xi_{i,j} - \xi_{i,j-1}\big)^2 + \frac{1}{\Delta_{i+1}}\Big)\phi_i(m_i)^2 \quad \textit{for} \ \ 1 \leq i \leq n. \tag{3.17}$$

Proof. First we consider $i = (i_1,i_2) =: (p,q) \in \ell$. Note that $\mathrm{supp}(\phi_i) \cap \Gamma = [m_{p,q-1}, m_{p,q+1}]$ and that $\phi_i(m_{p,q-1}) = \phi_i(m_{p,q+1}) = 0$. For a *linear* function g we have $\int_a^b g(x)^2\,dx \sim (b-a)\big(g(a)^2 + g(b)^2\big)$. Thus we get

$$\begin{aligned} \int_0^1 \phi_i(x)^2\,dx &= \int_{m_{p,q-1}}^{m_{p,q}} \phi_i(x)^2\,dx + \int_{m_{p,q}}^{m_{p,q+1}} \phi_i(x)^2\,dx \\ &\sim \phi_i(m_{p,q})^2\big(m_{p,q} - m_{p,q-1}\big) + \phi_i(m_{p,q})^2\big(m_{p,q+1} - m_{p,q}\big) \\ &= \phi_i(m_i)^2\big(m_{p,q+1} - m_{p,q-1}\big) = \phi_i(m_i)^2|\mathrm{supp}(\phi_i) \cap \Gamma| \sim \Delta_{i_1}\phi_i(m_i)^2. \end{aligned}$$

This proves the result in (3.14). Furthermore:

$$\int_0^1 \phi_i'(x)^2\,dx = \int_{m_{p,q-1}}^{m_{p,q}} \phi_i'(x)^2\,dx + \int_{m_{p,q}}^{m_{p,q+1}} \phi_i'(x)^2\,dx$$
$$\sim \phi_i(m_{p,q})^2\big(\frac{1}{m_{p,q}-m_{p,q-1}} + \frac{1}{m_{p,q+1}-m_{p,q}}\big) \sim \frac{1}{\Delta_{i_1}}\phi_i(m_{p,q})^2,$$

which proves the result in (3.16).

We now consider $1 \le i \le n$. We use the notation $m_{0,j} = 0$ for all j and $m_{n+1,j} = 1$ for all j. The support $\mathrm{supp}(\phi_i) \cap \Gamma = [m_{i-1,l_{i-1}}, m_{i+1,1}]$ is split into subintervals (cf. (3.1)) as:

$$[m_{i-1,l_{i-1}}, m_{i-1,l_{i-1}+1}] \cup \big(\cup_{1\le j\le l_i+1}[m_{i,j-1}, m_{i,j}]\big) \cup [m_{i+1,0}, m_{i+1,1}].$$

Note that $\phi_i(m_{i-1,l_{i-1}}) = \phi_i(m_{i+1,1}) = 0$ and $m_{i-1,l_{i-1}+1} = m_i,\ m_{i+1,0} = m_{i+1}$. We obtain

$$\begin{aligned}\int_0^1 \phi_i(x)^2\,dx &= \int_{m_{i-1,l_{i-1}}}^{m_{i-1,l_{i-1}+1}} \phi_i(x)^2\,dx + \sum_{j=1}^{l_i+1}\int_{m_{i,j-1}}^{m_{i,j}} \phi_i(x)^2\,dx + \int_{m_{i+1,0}}^{m_{i+1,1}} \phi_i(x)^2\,dx\\
&\sim \big(m_{i-1,l_{i-1}+1} - m_{i-1,l_{i-1}}\big)\phi_i(m_i)^2\\
&\quad + \sum_{j=1}^{l_i+1}\big(m_{i,j}-m_{i,j-1}\big)\big(\phi_i(m_{i,j})^2 + \phi_i(m_{i,j-1})^2\big)\\
&\quad + \big(m_{i+1,1}-m_{i+1,0}\big)\phi_i(m_{i+1})^2\\
&=\phi_i(m_i)^2\big[m_{i-1,l_{i-1}+1} - m_{i-1,l_{i-1}} + \sum_{j=1}^{l_i+1}\big(m_{i,j}-m_{i,j-1}\big)\big(\xi_{i,j}^2+\xi_{i,j-1}^2\big)\\
&\quad + (m_{i+1,1}-m_{i+1,0})\xi_i^2\big],\end{aligned}$$

with $\xi_{i,j}$, ξ_i as in (3.5), $\xi_{i,0} = \frac{\phi_i(m_{i,0})}{\phi_i(m_i)} = 1$, and for $i < n$, $\xi_{i,l_i+1} = \frac{\phi_i(m_{i,l_i+1})}{\phi_i(m_i)} = \frac{\phi_i(m_{i+1})}{\phi_i(m_i)} = \xi_i$. Using (3.10) we get

$$\begin{aligned}\int_0^1 \phi_i(x)^2\,dx &\sim \phi_i(m_i)^2\big[m_{i-1,l_{i-1}+1} - m_{i-1,l_{i-1}}\\
&\quad + \sum_{j=1}^{l_i+1}\big(m_{i,j}-m_{i,j-1}\big) + (m_{i+1,1}-m_{i+1,0})\big]\\
&= \phi_i(m_i)^2|\mathrm{supp}(\phi_i)\cap\Gamma| \sim h\phi_i(m_i)^2.\end{aligned}$$

Hence the result in (3.15) holds. We also have:

$$\begin{aligned}\int_0^1 \phi_i'(x)^2\,dx &= \int_{m_{i-1,l_{i-1}}}^{m_{i-1,l_{i-1}+1}} \phi_i'(x)^2\,dx + \sum_{j=1}^{l_i+1}\int_{m_{i,j-1}}^{m_{i,j}} \phi_i'(x)^2\,dx + \int_{m_{i+1,0}}^{m_{i+1,1}} \phi_i'(x)^2\,dx\\
&\sim \frac{\phi_i(m_i)^2}{\Delta_{i-1}} + \sum_{j=1}^{l_i+1}\frac{\big(\phi_i(m_{i,j})-\phi_i(m_{i,j-1})\big)^2}{\Delta_i} + \frac{\phi_i(m_{i+1})^2}{\Delta_{i+1}}\\
&= \phi_i(m_i)^2\Big(\frac{1}{\Delta_{i-1}} + \frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}\big(\xi_{i,j}-\xi_{i,j-1}\big)^2 + \frac{\xi_i}{\Delta_{i+1}}\Big).\end{aligned}$$

Using $\xi_i \sim 1$ this proves the result in (3.17). ∎

3.3. Analysis for the mass matrix. In this section we derive bounds for the (effective) condition number of the mass matrix $\mathbf{M}$ defined in (3.2). We define $\mathbf{D}_M := \mathrm{diag}(\mathbf{M}) = \mathrm{diag}(\|\phi_i\|_\Gamma^2)_{i\in\mathcal{I}}$. By $\langle\cdot,\cdot\rangle$ we denote the Euclidean inner product.

LEMMA 3.3. *Assume that* (3.8) *is satisfied. For all* $\mathbf{u} = (u_i)_{i\in\mathcal{I}} \in \mathbb{R}^{n_V}$, $\mathbf{u} \neq 0$, *we have*

$$\frac{\langle \mathbf{M}\mathbf{u},\mathbf{u}\rangle}{\langle \mathbf{D}_M\mathbf{u},\mathbf{u}\rangle} \sim \frac{h\sum_{i=2}^n \psi_i^2 + \sum_{i=(i_1,i_2)\in\ell}\Delta_{i_1}\psi_i^2}{h\sum_{i=1}^n \tilde{u}_i^2 + \sum_{i=(i_1,i_2)\in\ell}\Delta_{i_1}\tilde{u}_i^2} \tag{3.18}$$

with $\psi_i = u(m_i)$, $u := \sum_{i\in\mathcal{I}} u_i\phi_i$, $\tilde{u}_i = \phi_i(m_i)u_i$.

Proof. The identity $\langle \mathbf{D}_M\mathbf{u},\mathbf{u}\rangle = \sum_{i\in\mathcal{I}}\|\phi_i\|_\Gamma^2 u_i^2$ follows directly from the definition of $\mathbf{D}_M$. Furthermore, using lemma 3.2 we obtain:

$$\begin{aligned}\sum_{i\in\mathcal{I}}\|\phi_i\|_\Gamma^2 u_i^2 &= \sum_{i=1}^n \|\phi_i\|_\Gamma^2 u_i^2 + \sum_{i\in\ell}\|\phi_i\|_\Gamma^2 u_i^2 \sim h\sum_{i=1}^n \phi_i(m_i)^2u_i^2 + \sum_{i\in\ell}\Delta_{i_1}\phi_i(m_i)^2u_i^2\\ &= h\sum_{i=1}^n \tilde{u}_i^2 + \sum_{i\in\ell}\Delta_{i_1}\tilde{u}_i^2.\end{aligned}$$

We now consider the nominator. For two neighboring point m_p and m_q we introduce the mesh sizes $h_p^- := m_p - m_q$ if $m_q < m_p$, $h_p^+ := m_q - m_p$ if $m_q > m_p$ and $h_p := h_p^- + h_p^+$. Furthermore, $h_1 := h_1^+$, $h_{n,1} := h_{n,1}^-$. Using this we get

$$\begin{aligned}\langle \mathbf{M}\mathbf{u},\mathbf{u}\rangle &= \int_0^1 u(x)^2\,dx = \sum_{i=1}^n\sum_{j=1}^{l_i+1}\int_{m_{i,j-1}}^{m_{i,j}} u(x)^2\,dx\\ &\sim \sum_{i=1}^n\sum_{j=1}^{l_i+1}\big(m_{i,j}-m_{i,j-1}\big)\big(u(m_{i,j})^2 + u(m_{i,j-1})^2\big)\\ &= \sum_{i=1}^n\sum_{j=1}^{l_i+1} h_{i,j}^-\big(\psi_{i,j}^2+\psi_{i,j-1}^2\big) \sim \sum_{i=1}^n\sum_{j=0}^{l_i} h_{i,j}\psi_{i,j}^2\\ &= \sum_{i=2}^n h_i\psi_i^2 + \sum_{i\in\ell} h_i\psi_i^2 \sim h\sum_{i=2}^n\psi_i^2 + \sum_{i\in\ell}h_i\psi_i^2.\end{aligned}$$

From this and $h_i \sim \Delta_{i_1}$ for $i = (i_1,i_2)\in\ell$ the result in (3.18) follows. ∎

THEOREM 3.4. *Assume that* (3.8) *is satisfied. There exists a constant C independent of h such that*

$$\frac{\langle \mathbf{M}\mathbf{u},\mathbf{u}\rangle}{\langle \mathbf{D}_M\mathbf{u},\mathbf{u}\rangle} \le C \quad \textit{for all} \ \ \mathbf{u}\in\mathbb{R}^{n_V},\ \mathbf{u}\neq 0.$$

Proof. Using (3.6) and (3.10) we obtain, for $2 \le i \le n$,

$$\psi_i^2 \le c(\tilde{u}_{i-1}^2 + \tilde{u}_i^2).$$

Hence,

$$h\sum_{i=2}^n \psi_i^2 \le c\,h\sum_{i=1}^n \tilde{u}_i^2. \tag{3.19}$$

For $i=(i_1,i_2)\in \ell$ we have, using (3.7) and (3.10),

$$\Delta_{i_1}\psi_i^2 \le c\Delta_{i_1}(\tilde u_{i_1}^2+\tilde u_i^2)\le c(h\tilde u_{i_1}^2+\Delta_{i_1}\tilde u_i^2).$$

This yields

$$\sum_{i\in\ell}\Delta_{i_1}\psi_i^2 \le c\big(h\sum_{i=1}^n \tilde u_i^2+\sum_{i\in\ell}\Delta_{i_1}\tilde u_i^2\big). \tag{3.20}$$

Combination of (3.19), (3.20) and the result in lemma 3.3 proves the result. ▯

For the derivation of a lower bound we will need a further assumption on the triangulations $\{\mathcal{T}_h\}_{h>0}$ which is as follows
Assumption 2. Assume that there exists a constant $c_0>0$ such that $d(v_j)\ge c_0\,h^{\frac12}\max_{i=j,j+2,\ldots} d(v_i)$ for all j. Define, for $\alpha\in[0,\frac12]$:

$$N(\alpha):=\{\,v_j\in N \mid d(v_j)\le h^\alpha \max_{i=j,j+2,\ldots} d(v_i)\,\}, \tag{3.21}$$

and assume that there is a constant c_1 such that $|N(\alpha)|\le c_1h^{2\alpha-1}$ for all $\alpha\in[0,\frac12]$.

Remark 2. Note that $N(\alpha_2)\subset N(\alpha_1)$ for $0\le\alpha_1\le\alpha_2\le\frac12$ and $|N(\frac12)|=O(1)$. The condition $|N(\alpha)|\le c_1h^{2\alpha-1}$ means that the set of nodes having a certain (maximal) distance to Γ (as specified in (3.21)) becomes smaller if this distance gets smaller. In the 3D experiment in section 2.2 we observe that nodes with (very) small (i.e., $\ll h$) distances occur but that the size of this set decreases if this distance decreases. In the 2D experiment in section 2.2 we can have many nodes (namely $\sim\frac12 n$) with arbitrarily small distances to Γ. In that experiment, however, we have $d(v_j)=\max_{i=j,j+2,\ldots} d(v_i)$ for all j (the triangulation is "parallel" to Γ). Thus we have $N(0)=N$, $N(\alpha)=\emptyset$ for all $\alpha\in(0,\frac12]$ and assmption 2 is fulfilled.

Theorem 3.5. *Assume that* (3.8) *and Assumption 2 are satisfied. There exists a constant $C>0$ independent of h such that*

$$\frac{\langle \mathbf{M}\mathbf{u},\mathbf{u}\rangle}{\langle \mathbf{D}_M\mathbf{u},\mathbf{u}\rangle}\ge Ch^2|\ln h|^{-1}\quad \textit{for all}\ \ \mathbf{u}=(u_i)_{i\in\mathcal{I}}\in\mathbb{R}^{n_V},\ \mathbf{u}\ne 0,\ \textit{with}\ \ u_1=0.$$

Proof. For $2\le i\le n$ we have, using (3.6), (3.9) and $u_1=0$:

$$|\tilde u_i|\le|\psi_i|+\xi_{i-1}|\tilde u_{i-1}|\le\sum_{j=2}^i\big(\Pi_{k=j}^{i-1}\xi_k\big)|\psi_j|.$$

From this we get

$$\sum_{i=2}^n\tilde u_i^2\le\Big(\sum_{i=2}^n\sum_{j=2}^i\big(\Pi_{k=j}^{i-1}\xi_k\big)^2\Big)\sum_{j=2}^n\psi_j^2.$$

Using Assumption 2 the factor $\sum_{i=2}^n\sum_{j=2}^i\big(\Pi_{k=j}^{i-1}\xi_k\big)^2$ can be estimated as follows. To shorten notation we write $d_i:=d(v_i)$. Using the result in (3.9) we obtain

$$\big(\Pi_{k=j}^{i-1}\xi_k\big)^2\le\min\{d_{i-1},d_i\}^2\Big(\frac{1}{d_{j-1}}+\frac{1}{d_j}\Big)^2,$$

hence,

$$\sum_{j=2}^{i} \left(\Pi_{k=j}^{i-1} \xi_k\right)^2 \leq 4 \sum_{j=1}^{i} \frac{\min\{d_{i-1}, d_i\}^2}{d_j^2},$$

and

$$\begin{aligned}
\sum_{i=2}^{n} \sum_{j=2}^{i} \left(\Pi_{k=j}^{i-1} \xi_k\right)^2 &\leq 4 \sum_{i=2}^{n} \sum_{j=1}^{i} \frac{\min\{d_{i-1}, d_i\}^2}{d_j^2} \leq 4 \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{\min\{d_{i-1}, d_i\}^2}{d_j^2} \\
&\leq 8 \sum_{j=1}^{n} \sum_{i=j,j+2,j+4,\ldots}^{n} \frac{d_i^2}{d_j^2} \leq 8n \sum_{j=1}^{n} \left(\frac{\max_{i=j,j+2,\ldots} d_i}{d_j}\right)^2 =: 8n \sum_{j=1}^{n} \beta_j^2.
\end{aligned}$$

Note that for $\beta_j = \frac{\max_{i=j,j+2,\ldots} d_i}{d_j}$ we have $\beta_j \leq c_0^{-1}$ if $j \in N(\frac{1}{2})$ and $\beta_j \in [1, h^{-\frac{1}{2}})$ if $j \notin N(\frac{1}{2})$. Furthermore, for $0 \leq \alpha_1 \leq \alpha_2 \leq \frac{1}{2}$ we have $\#\{\, \beta_j \mid \beta_j \in [h^{-\alpha_1}, h^{-\alpha_2})\,\} = |N(\alpha_1)| - |N(\alpha_2)|$. Using this and Assumption 2 we obtain:

$$\begin{aligned}
8n \sum_{j=1}^{n} \beta_j^2 = 8n \sum_{j \in N(\frac{1}{2})} \beta_j^2 + 8n \sum_{j \notin N(\frac{1}{2})} \beta_j^2 &\leq ch^{-2} + ch^{-1} \int_0^{\frac{1}{2}} h^{-2\alpha} \, \mathrm{d}|N(\alpha)| \\
&\leq ch^{-2} + c\, h^{-2} \int_0^{\frac{1}{2}} h^{-2\alpha} \, \mathrm{d}h^{2\alpha} \sim h^{-2} |\ln h|.
\end{aligned}$$

Thus we obtain

$$\sum_{i=2}^{n} \tilde{u}_i^2 \leq ch^{-2} |\ln h| \sum_{j=2}^{n} \psi_j^2. \tag{3.22}$$

For $i = (i_1, i_2) \in \ell$ we get, using (3.7) and (3.10):

$$|\tilde{u}_i| \leq c|\tilde{u}_{i_1}| + |\psi_i|,$$

hence,

$$\Delta_{i_1} \tilde{u}_i^2 \leq c\big(h \tilde{u}_{i_1}^2 + \Delta_{i_1} \psi_i^2\big),$$

which yields, using (3.22),

$$\sum_{i \in \ell} \Delta_{i_1} \tilde{u}_i^2 \leq c\big(h \sum_{i=2}^{n} \tilde{u}_i^2 + \sum_{i \in \ell} \Delta_{i_1} \psi_i^2\big) \leq ch^{-2} |\ln h| \big(h \sum_{i=2}^{n} \psi_i^2 + \sum_{i \in \ell} \Delta_{i_1} \psi_i^2\big). \tag{3.23}$$

Combination of (3.22) and (3.23) with the result in lemma 3.3 completes the proof. ◻

Theorem 3.6. *Let $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n_V}$ be the eigenvalues of $\mathbf{D}_M^{-1}\mathbf{M}$. Assume that* (3.8) *and Assumption 2 are satisfied. Then*

$$\lambda_1 = 0, \quad \frac{\lambda_{n_V}}{\lambda_2} \leq Ch^{-2} |\ln h|$$

holds, with a constant C independent of h

Proof. The matrix $\mathbf{M}$ has dimension $n_V \times n_V$. The number of intersection points m_j is $n_V - 1$ and thus $\dim(V_h^\Gamma) \le n_V - 1$ holds. This implies

$$\dim(\text{range}(\mathbf{M})) = \dim(V_h^\Gamma) \le n_V - 1$$

and thus $\dim(\ker(\mathbf{M})) \ge 1$, which implies $\lambda_1 = 0$. From the Courant-Fischer representation and theorem 3.5 we obtain, with W_1 the family of 1-dimensional subspaces of $\mathbb{R}^{n_V}$,

$$\lambda_2 = \sup_{S \in W_1} \inf_{\mathbf{u} \in S^\perp} \frac{\langle \mathbf{Mu}, \mathbf{u} \rangle}{\langle \mathbf{D}_M \mathbf{u}, \mathbf{u} \rangle} \ge \inf_{\mathbf{u} \in \mathbb{R}^{n_V},\, u_1 = 0} \frac{\langle \mathbf{Mu}, \mathbf{u} \rangle}{\langle \mathbf{D}_M \mathbf{u}, \mathbf{u} \rangle} \ge Ch^2 |\ln h|^{-1}.$$

In combination with the result in theorem 3.4 this yields $\frac{\lambda_{n_V}}{\lambda_2} \le Ch^{-2} |\ln h|$. ∎

3.4. Analysis for the stiffness matrix. In this section we derive bounds for the (effective) condition number of the stiffness matrix $\mathbf{A}$ defined in (3.2).

Let $\mathbf{D}_A = \text{diag}(\mathbf{A})$ be the diagonal of the stiffness matrix.

LEMMA 3.7. *Assume that* (3.8) *holds. For all* $\mathbf{u} = (u_i)_{i \in \mathcal{I}} \in \mathbb{R}^{n_V}$, $\mathbf{u} \ne 0$, *we have*

$$\frac{\langle \mathbf{Au}, \mathbf{u} \rangle}{\langle \mathbf{D}_A \mathbf{u}, \mathbf{u} \rangle} \sim \frac{\sum_{i=1}^n \frac{1}{\Delta_i} \sum_{j=1}^{l_i+1} (\psi_{i,j} - \psi_{i,j-1})^2}{\sum_{i=1}^n \left(\frac{1}{\Delta_{i-1}} + \frac{1}{\Delta_i} \sum_{j=1}^{l_i+1} \left(\xi_{i,j} - \xi_{i,j-1} \right)^2 + \frac{1}{\Delta_{i+1}} \right) \tilde{u}_i^2 + \sum_{i \in \ell} \frac{1}{\Delta_{i_1}} \tilde{u}_i^2}\ , \tag{3.24}$$

with $\psi_i = u(m_i)$, $u := \sum_{i \in \mathcal{I}} u_i \phi_i$, $\tilde{u}_i = \phi_i(m_i) u_i$.

Proof. The identity $\langle \mathbf{D}_A \mathbf{u}, \mathbf{u} \rangle = \sum_{i \in \mathcal{I}} \|(\phi_i)_x\|_\Gamma^2 u_i^2$ follows directly from the definition of $\mathbf{D}_A$. Furthermore, using lemma 3.2 we obtain, with $g_i := \sum_{j=1}^{l_i+1} \left(\xi_{i,j} - \xi_{i,j-1} \right)^2$:

$$\begin{aligned}
\sum_{i \in \mathcal{I}} \|(\phi_i)_x\|_\Gamma^2 u_i^2 &= \sum_{i=1}^n \|(\phi_i)_x\|_\Gamma^2 u_i^2 + \sum_{i \in \ell} \|(\phi_i)_x\|_\Gamma^2 u_i^2 \\
&\sim \sum_{i=1}^n \left(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}} \right) \phi_i(m_i)^2 u_i^2 + \sum_{i \in \ell} \frac{1}{\Delta_{i_1}} \phi_i(m_i)^2 u_i^2 \\
&= \sum_{i=1}^n \left(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}} \right) \tilde{u}_i^2 + \sum_{i \in \ell} \frac{1}{\Delta_{i_1}} \tilde{u}_i^2.
\end{aligned}$$

For the nominator we have:

$$\begin{aligned}
\langle \mathbf{Au}, \mathbf{u} \rangle &= \int_0^1 u'(x)^2 \, dx = \sum_{i=1}^n \sum_{j=1}^{l_i+1} \int_{m_{i,j-1}}^{m_{i,j}} u'(x)^2 \, dx \\
&= \sum_{i=1}^n \sum_{j=1}^{l_i+1} \frac{\left(u(m_{i,j}) - u(m_{i,j-1}) \right)^2}{m_{i,j} - m_{i,j-1}} \sim \sum_{i=1}^n \frac{1}{\Delta_i} \sum_{j=1}^{l_i+1} (\psi_{i,j} - \psi_{i,j-1})^2.
\end{aligned}$$

This completes the proof. ∎

THEOREM 3.8. *Assume that* (3.8) *holds. There exists a constant* C *independent of* h *such that*

$$\frac{\langle \mathbf{Au}, \mathbf{u} \rangle}{\langle \mathbf{D}_A \mathbf{u}, \mathbf{u} \rangle} \le C \quad \textit{for all} \quad \mathbf{u} \in \mathbb{R}^{n_V},\ \mathbf{u} \ne 0.$$

Proof. We use lemma 3.7 . Using (3.6) and (3.7) we obtain

$$\begin{aligned}\psi_{i,1} - \psi_{i,0} &= \psi_{i,1} - \psi_i = \tilde{u}_{i,1} - \xi_{i-1}\tilde{u}_{i-1} + (\xi_{i,1} - 1)\tilde{u}_i \\ &= \tilde{u}_{i,1} - \xi_{i-1}\tilde{u}_{i-1} + (\xi_{i,1} - \xi_{i,0})\tilde{u}_i\end{aligned}$$

and for $2 \leq j \leq l_i + 1$

$$\psi_{i,j} - \psi_{i,j-1} = \tilde{u}_{i,j} - \tilde{u}_{i,j-1} + (\xi_{i,j} - \xi_{i,j-1})\tilde{u}_i.$$

Using $\xi_i \sim 1$ this yields, with $\tilde{u}_{i,0} := \tilde{u}_{i-1}$,

$$(\psi_{i,j} - \psi_{i,j-1})^2 \leq c\Big(\tilde{u}_{i,j}^2 + \tilde{u}_{i,j-1}^2 + (\xi_{i,j} - \xi_{i,j-1})^2\tilde{u}_i^2\Big) \quad \text{for } 1 \leq j \leq l_i + 1.$$

Hence, with $g_i := \sum_{j=1}^{l_i+1}(\xi_{i,j} - \xi_{i,j-1})^2$ we obtain

$$\sum_{j=1}^{l_i+1}(\psi_{i,j} - \psi_{i,j-1})^2 \leq c\big(\tilde{u}_{i-1}^2 + \tilde{u}_{i+1}^2 + g_i\tilde{u}_i^2 + \sum_{j=1}^{l_i}\tilde{u}_{i,j}^2\big)$$

and thus

$$\begin{aligned}\sum_{i=1}^{n}\frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j} - \psi_{i,j-1})^2 &\leq c\sum_{i=1}^{n}\frac{1}{\Delta_i}\big(\tilde{u}_{i-1}^2 + \tilde{u}_{i+1}^2 + g_i\tilde{u}_i^2\big) + c\sum_{i=(i_1,i_2)\in\ell}\frac{1}{\Delta_{i_1}}\tilde{u}_i^2 \\ &\leq c\sum_{i=1}^{n}\big(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}}\big)\tilde{u}_i^2 + \sum_{i=(i_1,i_2)\in\ell}\frac{1}{\Delta_{i_1}}\tilde{u}_i^2,\end{aligned}$$

which completes the proof. ∎

We now derive a lower bound for the smallest nonzero eigenvalue of $\mathbf{D}_A^{-1}\mathbf{A}$. For this it turns out to be more convenient to consider $u_i := u(v_i) = \phi_i(m_i)^{-1}\tilde{u}_i$ instead of $\tilde{u}_i$.

LEMMA 3.9. *For $u_i = u(v_i)$ we have the recursion*

$$u_i = (1 - \alpha_i)u_{i-1} + \alpha_i u_{i-2} + \frac{1}{\phi_i(m_i)}(\psi_i - \psi_{i-1}), \quad i = 2, \ldots, n, \tag{3.25}$$

with

$$\alpha_i := \frac{d(v_{i-1}) + d(v_i)}{d(v_{i-2}) + d(v_{i-1})}.$$

For $u_0 = u_1 := 0$ the solution of this recursion is given by

$$u_i = \sum_{j=1}^{i-1}\big(d(v_j) + (-1)^{i-j-1}d(v_i)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j), \quad i = 2, \ldots, n. \tag{3.26}$$

Proof. From (3.3) we get

$$\begin{aligned}\psi_i &= \phi_{i-1}(m_i)u_{i-1} + \phi_i(m_i)u_i \\ \psi_{i-1} &= \phi_{i-2}(m_{i-1})u_{i-2} + \phi_{i-1}(m_{i-1})u_{i-1},\end{aligned}$$

and thus, using $\phi_{j-1}(m_j) = 1 - \phi_j(m_j)$, we have

$$\begin{aligned}
u_i &= \big(1 + \frac{\phi_{i-1}(m_{i-1}) - 1}{\phi_i(m_i)}\big)u_{i-1} + \frac{1 - \phi_{i-1}(m_{i-1})}{\phi_i(m_i)}u_{i-2} + \frac{1}{\phi_i(m_i)}(\psi_i - \psi_{i-1}) \\
&= (1 - \alpha_i)u_{i-1} + \alpha_i u_{i-2} + \frac{1}{\phi_i(m_i)}(\psi_i - \psi_{i-1})
\end{aligned}$$

with $\alpha_i := \frac{1-\phi_{i-1}(m_{i-1})}{\phi_i(m_i)}$. Using the formula in (3.11) we get

$$\alpha_i = \frac{d(v_{i-1}) + d(v_i)}{d(v_{i-2}) + d(v_{i-1})}.$$

The representation

$$u_i = \sum_{k=2}^{i}\sum_{j=1}^{k-1}(-1)^{k+1-j}\big(d(v_{k-1}) + d(v_k)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j) \tag{3.27}$$

can be shown by induction as follows. For $i = 2$ we get (using (3.11)),

$$u_2 = \big(d(v_1) + d(v_2)\big)\frac{1}{d(v_1)}(\psi_2 - \psi_1) = \frac{1}{\phi_2(m_2)}(\psi_2 - \psi_1),$$

which also follows from the recursion formula if we take $u_0 = u_1 = 0$. Assume that the representation formula (3.27) is correct for indices less than or equal to $i - 1$. We then obtain

$$\begin{aligned}
&(1 - \alpha_i)u_{i-1} + \alpha_i u_{i-2} + \frac{1}{\phi_i(m_i)}(\psi_i - \psi_{i-1}) \\
&= -\alpha_i(u_{i-1} - u_{i-2}) + u_{i-1} + \frac{1}{\phi_i(m_i)}(\psi_i - \psi_{i-1}) \\
&= -\alpha_i\sum_{j=1}^{i-2}(-1)^{i-j}\big(d(v_{i-2}) + d(v_{i-1})\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j) \\
&\quad + \sum_{k=2}^{i-1}\sum_{j=1}^{k-1}(-1)^{k+1-j}\big(d(v_{k-1}) + d(v_k)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j) + \frac{d(v_{i-1}) + d(v_i)}{d(v_{i-1})}(\psi_i - \psi_{i-1}) \\
&= \sum_{j=1}^{i-1}(-1)^{i+1-j}\big(d(v_{i-1}) + d(v_i)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j) \\
&\quad + \sum_{k=2}^{i-1}\sum_{j=1}^{k-1}(-1)^{k+1-j}\big(d(v_{k-1}) + d(v_k)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j) \\
&= \sum_{k=2}^{i}\sum_{j=1}^{k-1}(-1)^{k+1-j}\big(d(v_{k-1}) + d(v_k)\big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j),
\end{aligned}$$

and thus the representation for u_i in (3.27). From this we obtain, by changing the order of summation:

$$u_i = \sum_{j=1}^{i-1}\Big(\sum_{k=j+1}^{i}(-1)^{k+1-j}\big(d(v_{k-1}) + d(v_k)\big)\Big)\frac{1}{d(v_j)}(\psi_{j+1} - \psi_j).$$

The representation in (3.26) immediately follows from this one. $\square$

For the derivation of a lower bound we will need further assumptions on the triangulations $\{\mathcal{T}_h\}_{h>0}$:

Assumption 3. We assume that the angles between $\Gamma = [0,1]$ and all sides of the triangles that intersect Γ are uniformly (w.r.t. h) bounded away from zero. Assume that there exists a constant $c_0 > 0$ such that $d(v_j) \geq c_0\, h \max\{hg_j, \max_{i=j,j+2,\dots} d(v_i)\,\}$ for all j, with $g_j := \sum_{k=1}^{j_k+1} \big(\xi_{j,k} - \xi_{j,k-1}\big)^2$ as in lemma 3.7. Define (cf. Assumption2), for $\alpha \in [0,1]$:

$$\hat{N}(\alpha) := \{\, v_j \in N \mid d(v_j) \leq h^\alpha \max\{hg_j, \max_{i=j,j+2,\dots} d(v_i)\,\}. \tag{3.28}$$

Assume that there is a constant c_1 such that $|\hat{N}(\alpha)| \leq c_1 h^{\alpha-1}$ for all $\alpha \in [0,1]$.

Remark 3. Note that $|\hat{N}(1)| = O(1)$. The condition $|\hat{N}(\alpha)| \leq c_1 h^{\alpha-1}$ means that the set of nodes having a certain (maximal) distance to Γ (as specified in (3.28)) becomes smaller if this distance gets smaller, cf. remark 2. In the 2D experiment in section 2.2 we have $d(v_j) = \max_{i=j,j+2,\dots} d(v_i)$ and $g_j = 0$ for all j (the triangulation is "parallel" to Γ). Thus we have $\hat{N}(0) = N$, $\hat{N}(\alpha) = \emptyset$ for all $\alpha \in (0,1]$ and Assumption 3 is fulfilled.

Theorem 3.10. *Assume that* (3.8) *and Assumption 3 hold. There exists a constant $C > 0$ independent of h such that*

$$\frac{\langle \mathbf{A}\mathbf{u}, \mathbf{u}\rangle}{\langle \mathbf{D}_A \mathbf{u}, \mathbf{u}\rangle} \geq Ch^2 |\ln h|^{-1} \quad \textit{for all} \;\; \mathbf{u} = (u_i)_{i\in\mathcal{I}} \in \mathbb{R}^{n_V},\; \mathbf{u} \neq 0,\; \textit{with} \;\; u_0 = u_1 = 0.$$

Proof. We use the notation $d_i := d(v_i)$, $u_i := u(v_i)$. We use the representation in lemma 3.7 and first consider the term $\sum_{i=1}^n \big(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}}\big)\tilde{u}_i^2$ in the denominator. Due to the angle condition in Assumption 3 we have $d_i \sim \Delta_i$ $(1 \leq i \leq n)$, and $\tilde{u}_i = \phi_i(m_i)u_i \sim \frac{d_{i-1}}{h} u_i$ $(1 \leq i \leq n)$. Using this and the result in (3.26) we get

$$\tilde{u}_i^2 \leq c\frac{d_{i-1}^2}{h^2} u_i^2 \leq c\frac{d_{i-1}^2}{h^2}\Big(\sum_{j=1}^{i-1} (d_j^2 + d_i^2)\frac{1}{d_j}\Big) \sum_{j=1}^n \frac{1}{d_j}(\psi_{j+1} - \psi_j)^2.$$

For the last term we have

$$\sum_{i=1}^n \frac{1}{d_i}(\psi_{i+1} - \psi_i)^2 \leq c\sum_{i=1}^n \frac{1}{\Delta_i}\sum_{j=1}^{l_i+1} (\psi_{i,j} - \psi_{i,j-1})^2,$$

and thus, using $d_{i-1} \sim \Delta_{i-1} \sim \Delta_{i+1}$, we get

$$\begin{aligned}
&\sum_{i=1}^n \big(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}}\big)\tilde{u}_i^2 \leq c\sum_{i=1}^n \big(\frac{1}{d_{i-1}} + \frac{g_i}{d_i}\big)\tilde{u}_i^2 \\
&\leq \Big[\frac{c}{h^2}\sum_{i=1}^n \big(d_{i-1} + \frac{g_i d_{i-1}^2}{d_i}\big)\sum_{j=1}^{i-1}\big(d_j + \frac{d_i^2}{d_j}\big)\Big]\sum_{i=1}^n \frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j} - \psi_{i,j-1})^2.
\end{aligned}$$

We estimate the factor in the square brackets as follows. Using $d_{i-1}d_i \le c\min\{d_{i-1}, d_i\}h$ we get:

$$
\begin{aligned}
&\frac{c}{h^2}\sum_{i=1}^{n}\big(d_{i-1} + \frac{g_i d_{i-1}^2}{d_i}\big)\sum_{j=1}^{i-1}\big(d_j + \frac{d_i^2}{d_j}\big) \\
&\le \frac{c}{h^2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{d_{i-1}d_i^2 + d_{i-1}^2 d_i}{d_j} + \frac{c}{h^2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} d_{i-1}d_j + \frac{c}{h^2}\sum_{i=1}^{n}\frac{g_i d_{i-1}^2}{d_i}\sum_{j=1}^{i-1} d_j \\
&\le c\sum_{i=1}^{n}\sum_{j=1}^{i-1}\frac{\min\{d_{i-1}, d_i\}}{d_j} + ch^{-2} + \frac{c}{h}\sum_{i=1}^{n}\frac{hg_i}{d_i}.
\end{aligned}
$$

The first term on the right handside can be bounded by $ch^{-2}|\ln h|$ using the same arguments as in the proof of Theorem 3.5. The third term can be treated in a similar way as follows. With $\tilde{N}(\alpha) = \{\, v_j \in N \mid d(v_j) \le h^{\alpha+1}g_j \,\} \subset \hat{N}(\alpha)$ and $\beta_j := \frac{hg_j}{d_j}$ we have $\beta_j \le c_0 h^{-1}$ if $j \in \tilde{N}(1)$ and $\beta_j \in [1, h^{-1})$ if $j \notin \tilde{N}(1)$. For $0 \le \alpha_1 \le \alpha_2 \le 1$ we have $\#\{\, \beta_j \mid \beta_j \in (h^{-\alpha_1}, h^{-\alpha_2}]\,\} = |\tilde{N}(\alpha_1)| - |\tilde{N}(\alpha_2)|$. Using this and Assumption 3 we obtain:

$$
\begin{aligned}
\frac{c}{h}\sum_{i=1}^{n}\frac{hg_i}{d_i} &= \frac{c}{h}\sum_{j\in\tilde{N}(1)}\beta_i + \frac{c}{h}\sum_{j\notin\tilde{N}(1)}\beta_i \le ch^{-2} + ch^{-1}\int_0^1 h^{-\alpha}\mathrm{d}|\tilde{N}(\alpha)| \\
&\le ch^{-2} + ch^{-2}\int_0^1 h^{-\alpha}\mathrm{d}h^{\alpha} \le ch^{-2}|\ln h|.
\end{aligned}
$$

Collecting these results we get

$$
\sum_{i=1}^{n}\big(\frac{1}{\Delta_{i-1}} + \frac{g_i}{\Delta_i} + \frac{1}{\Delta_{i+1}}\big)\tilde{u}_i^2 \le ch^{-2}|\ln h|\sum_{i=1}^{n}\frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j} - \psi_{i,j-1})^2. \tag{3.29}
$$

We now treat the term $\sum_{i\in\ell}\frac{1}{\Delta_{i_1}}\tilde{u}_i^2$ in the denominator in lemma 3.7. Note that

$$
\sum_{i\in\ell}\frac{1}{\Delta_{i_1}}\tilde{u}_i^2 = \sum_{1\le i\le n,\ l_i>0}\ \sum_{j=1}^{l_i}\frac{1}{\Delta_i}\tilde{u}_{i,j}^2.
$$

Using (3.7) we get, for an i with $l_i \ge 2$:

$$
\tilde{u}_{i,j} - \tilde{u}_{i,j-1} = \psi_{i,j} - \psi_{i,j-1} - (\xi_{i,j} - \xi_{i,j-1})\tilde{u}_i,
$$

and with (3.6) and $\psi_{i,0} := \psi_i$, $\xi_{i,0} := 1$:

$$
\tilde{u}_{i,1} - \xi_{i-1}\tilde{u}_{i-1} = \psi_{i,1} - \psi_{i,0} - (\xi_{i,1} - \xi_{i,0})\tilde{u}_i.
$$

This yields, for $1 \le j \le l_i$:

$$
\begin{aligned}
\tilde{u}_{i,j}^2 &\le c\Big(\tilde{u}_{i-1}^2 + \sum_{j=1}^{l_i}(\psi_{i,j} - \psi_{i,j-1})^2 + \sum_{j=1}^{l_i}(\xi_{i,j} - \xi_{i,j-1})^2\tilde{u}_i^2\Big) \\
&\le c\Big(\tilde{u}_{i-1}^2 + \sum_{j=1}^{l_i+1}(\psi_{i,j} - \psi_{i,j-1})^2 + g_i\tilde{u}_i^2\Big).
\end{aligned}
$$

Thus we get

$$\sum_{1\le i\le n,\ l_i>0}\sum_{j=1}^{l_i}\frac{1}{\Delta_i}\tilde{u}_{i,j}^2 \le c\sum_{i=1}^{n}\frac{1}{\Delta_i}\tilde{u}_{i-1}^2 + c\sum_{i=1}^{n}\frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j}-\psi_{i,j-1})^2 + c\sum_{i=1}^{n}\frac{g_i}{\Delta_i}\tilde{u}_i^2$$
$$\le c\sum_{i=1}^{n}\left(\frac{g_i}{\Delta_i}+\frac{1}{\Delta_{i+1}}\right)\tilde{u}_i^2 + c\sum_{i=1}^{n}\frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j}-\psi_{i,j-1})^2$$

Using the bound in (3.29) we obtain

$$\sum_{1\le i\le n,\ l_i>0}\sum_{j=1}^{l_i}\frac{1}{\Delta_i}\tilde{u}_{i,j}^2 \le ch^{-2}|\ln h|\sum_{i=1}^{n}\frac{1}{\Delta_i}\sum_{j=1}^{l_i+1}(\psi_{i,j}-\psi_{i,j-1})^2,$$

and combination of this with the result in (3.29) completes the proof. ∎

THEOREM 3.11. *Let* $0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{n_V}$ *be the eigenvalues of* $\mathbf{D}_A^{-1}\mathbf{A}$. *Assume that* (3.8) *and Assumption 3 are satisfied. Then*

$$\lambda_1 = 0, \quad \frac{\lambda_{n_V}}{\lambda_3} \le Ch^{-2}|\ln h|$$

holds, with a constant C *independent of* h.

Proof. A dimension argument as in the proof of theorem 3.6 yields $\lambda_1 = 0$. From the Courant-Fischer representation and theorem 3.10 we obtain, with W_2 the family of 2-dimensional subspaces of $\mathbb{R}^{n_V}$,

$$\lambda_3 = \sup_{S\in W_2}\inf_{\mathbf{u}\in S^\perp}\frac{\langle \mathbf{A}\mathbf{u},\mathbf{u}\rangle}{\langle \mathbf{D}_A\mathbf{u},\mathbf{u}\rangle} \ge \inf_{\mathbf{u}\in\mathbb{R}^{n_V},\, u_0=u_1=0}\frac{\langle \mathbf{A}\mathbf{u},\mathbf{u}\rangle}{\langle \mathbf{D}_A\mathbf{u},\mathbf{u}\rangle} \ge Ch^2|\ln h|^{-1}.$$

In combination with the result in theorem 3.8 this yields $\frac{\lambda_{n_V}}{\lambda_3} \le Ch^{-2}|\ln h|$. ∎

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