A Finite Horizon Optimal Multiple Switching Problem

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Abstract

We consider the problem of optimal multiple switching in finite horizon, when the state of the system, including the switching costs, is a general adapted stochastic process. The problem is formulated as an extended impulse control problem and completely solved using probabilistic tools such as the Snell envelop of processes and reflected backward stochastic differential equations. Finally, when the state of the system is a Markov diffusion process, we show that the vector of value functions of the optimal problem is a viscosity solution to a system of variational inequalities with inter-connected obstacles.

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1 Introduction

Optimal control of multiple switching models arise naturally in many applied disciplines. The pioneering work by Brennan and Schwartz (1985), proposing a two-modes switching model for the life cycle of an investment in the natural resource industry, is probably first to apply this special case of stochastic impulse control to questions related to the structural profitability of an investment project or an industry whose production depends on the fluctuating market price of a number of underlying commodities or assets. Within this discipline, Carmona and Ludkosvki (2005) and Deng and Xia

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(2005) suggest a multiple switching model to price energy tolling agreements, where the commodity prices are modeled as continuous time processes, and the holder of the agreement exercises her managerial options by controlling the production modes of the assets. Target tracking in aerospace and electronic systems (cf. Doucet and Ristic (2002)) is another class of problems, where these models are very useful. These are often formulated as a hybrid state estimation problem characterized by a continuous time target state and a discrete time regime (mode) variables. All these applications seem agree that reformulating these problems in a multiple switching dynamic setting is a promising (if not the only) approach to fully capture the interplay between profitability, flexibility and uncertainty.

The optimal two-modes switching problem is probably the most extensively studied in the literature starting with above mentioned work by Brennan and Schwartz (1985), and Dixit (1989) who considered a similar model, but without resource extraction - see Dixit and Pindyck (1994) and Trigeorgis (1996) for an overview, extensions of these models and extensive reference lists. Brekke and Øksendal (1991) and (1994), Shirakawa (1997), Knudsen, Meister and Zervos (1998), Duckworth and Zervos (2000) and (2001), Zervos (2003) and Pham & Vath (2007) use the framework of generalized impulse control to solve several versions and extensions of this model, in the case where the decision to start and stop the production process is done over an infinite time horizon and the market price process of the underlying commodity X is a diffusion process, while Trigeorgis (1993) models the market price process of the commodity as a binomial tree. Hamadène and Jeanblanc (2007) consider a finite horizon optimal two-modes switching problem when the price processes are only adapted to the filtration generated by a Brownian motion while Hamadène and Hdhiri (2006) extend the set up of the latter paper to the case where the price processes of the underlying commodities are adapted to a filtration generated by a Brownian motion and an independent Poisson process. Porchet *et al.* (2006) also study the same problem, where they assume the payoff function to be given by an exponential utility function and allow the manager to trade on the commodities market. Finally, let us mention the work by Djehiche and Hamadène (2007) where it is shown that including the possibility of default or bankruptcy in the two-modes switching model over a finite time horizon, makes the search for an optimal strategy highly nonlinear and is not at all a trivial extension of previous results. For example, when the market price of the underlying commodities is a diffusion process, these optimal strategies are related to a system of variational inequalities with inter-connected obstacles, for which very few existence and regularity results are known in the literature.

An example of the class of multiple switching models discussed in Carmona and Ludkovski (2005) is related to the management strategies to run a power plant that converts natural gas into electricity (through a series of gas turbines) and sells it in the market. The payoff rate from running the plant is roughly given by the difference between the market price of electricity and the market price of gas needed to produce it.

Suppose that besides running the plant at full capacity or keeping it completely off (the two-modes switching model), there also exists a total of $q-2$ ($q \geq 3$) intermediate operating modes, corresponding to different subsets of turbines running.

Let ℓ_{ij} denote the switching costs from state i to state j, to cover the required extra fuel and various overhead costs. Furthermore, let $X = (X_t)_{t\geq 0}$ denote a vector of stochastic processes that stands for the market price of the underlying commodities and other financial assets that influence the production of power. The payoff rate in mode i, at time t, is then a function $\psi_i(t, X_t)$ of X_t.

A management strategy for the power plant is a combination of two sequences:

(i) a nondecreasing sequence of stopping times $(\tau_n)_{n\geq 0}$, where, at time τ_n, the manager decides to switch the production from its current mode to another one;

(ii) a sequence of indicators $(\xi_n)_{n\geq 1}$ taking values in $\{1, \ldots, q\}$ of the state the production is switched to. At τ_n the station is switched from its current mode ξ_{n-1} to ξ_n.

When the power plant is run under a strategy $(\delta, u) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$, over a finite horizon $[0, T]$, the total expected profit up to T for such a strategy is

$$J(\delta, u) = E[\int_0^T \psi_{u_s}(s, X_s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n < T]}]$$

where $u_s = \xi_n$ if $s \in [\tau_{n-1}, \tau_n[$ $(\tau_0 = 0)$.

The optimal switching problem we will investigate is to find a management strategy $(\delta^*, u^*) = ((\tau_n^*)_{n\geq 1}, (\xi_n^*)_{n\geq 1})$ such that

$$J(\delta^*, u^*) = \sup_{(\delta, u)} J(\delta, u).$$

Using purely probabilistic tools such as the Snell envelop of processes and backward SDEs, inspired by the work by Hamadène and Jeanblanc (2007), Carmona and Ludkovski (2005) suggest a powerful robust numerical scheme based on Monte Carlo regressions to solve this optimal switching problem when X is a diffusion process. They also list a number of technical challenges, such as the continuity of the associated value function etc., that prevent a rigorous proof of the existence and characterization of an optimal solution of this problem.

The objective of this work is to fill in this gap by providing a complete treatment of the optimal multiple switching problem, using the same framework. We are able to prove the existence and provide a characterization of an optimal strategy of this problem, when X and the switching costs $\ell_{i,j}$ are only adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by a Brownian motion.

We first provide a Verification Theorem that shapes the problem, via the Snell envelope of processes. We show that if the Verification Theorem is satisfied by a vector of continuous processes $(Y^1, \ldots, Y^q)$ such that, for each $i \in \{1, \ldots, q\}$,

$$Y_t^i = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{j \neq i}(-\ell_{ij}(\tau) + Y_\tau^j)1_{[\tau < T]}|\mathcal{F}_t].$$

then each Y_t^i is the value function of the optimal problem when the system is in mode i at time t:

$$Y_t^i = \text{ess sup}_{(\delta,u) \in \mathcal{D}_t} E[\int_t^T \psi_{u_s}(s, X_s)ds - \sum_{j \geq 1} \ell_{u_{\tau_{j-1}} u_{\tau_j}}(\tau_j)1\!\!1_{[\tau_j < T]}|\mathcal{F}_t].$$

where $\mathcal{D}_t$ is the set of strategies such that $\tau_1 \geq t$ a.s.

An optimal strategy (δ^*, u^*) is then given by the optimal stopping times corresponding to the Snell envelop. Moreover, it holds that $Y_0^1 = \sup_\delta J(\delta)$, provided that the system is in mode $i = 1$ at time $t = 0$.

The unique solution for the Verification Theorem is obtained as the limit of sequences of processes $(Y^{i,n})_{n \geq 0}$ where, $Y_t^{i,n}$ is the value function (or the optimal yields) from t to T, when the system is in mode i at time t and only at most n switchings after t are allowed. This sequence of value functions is defined recursively as follows.

$$Y_t^{i,0} = E[\int_t^T \psi_i(s, X_s)ds|\mathcal{F}_t]$$

and, for $n \geq 1$,

$$Y_t^{i,n} = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{j \neq i}(-\ell_{ij}(\tau) + Y_\tau^{j,n-1})1_{[\tau < T]}|\mathcal{F}_t].$$

Finally, if the process X is an Itô diffusion, with infinitesimal generator A, and each $\ell_{ij}(t)$ is a deterministic function of t, we prove existence of q deterministic continuous functions $v^1(t, x), \ldots, v^q(t, x)$ such that for any $i \in \{1, \ldots, q\}$, $Y_t^i = v^i(t, X_t)$. Moreover, the vector $(v^1, \ldots, v^q)$ is a viscosity solution of the following system of q variational inequalities with inter-connected obstacles.

$$\min\{\phi_i(t,x) - \max_{j \neq i}(-\ell_{ij}(t) + \phi_j(t,x)), -\partial_t\phi_i(t,x) - A\phi_i(t,x) - \psi_i(t,x)\} = 0, \;\; \phi_i(T,x) = 0, \quad i \in \{1, \ldots, q\}.$$

The organization of the paper is as follows. In Section 2, we give a formulation of the problem and provide some preliminary results. Sections 3 & 4 are devoted to establish the Verification Theorem and provide an optimal strategy to our problem. In Section 5, we show that, when the driving process X is an Itô diffusion, the vector of value functions of our optimal problem is a viscosity solution of a system of variational inequalities with inter-connected obstacles. Finally, in Section 6, we provide yet another numerical scheme that may be useful in simulating the value-processes satisfying the Verification Theorem.

2 Formulation of the problem and preliminary results

The finite horizon multiple switching problem can be formulated as follows. Let $\mathcal{J} := \{1, ..., q\}$ be the set of all possible activity modes of the production of the commodity. Being in mode i, a management strategy of the project consists, on the one hand, of the choice of a sequence of nondecreasing stopping times $(\tau_n)_{n\geq 1}$ (i.e. $\tau_n \leq \tau_{n+1}$ and $\tau_0 = 0$) where the manager decides to switch the activity from its current mode, i, to another one from the set $\mathcal{J}^{-i} := \{1, \ldots, i-1, i+1, \ldots, q\}$. On the other hand, it consists of the choice of the mode ξ_n to which the production is switched at τ_n from the current mode i ; ξ_n is a random variable which takes its values in $\mathcal{J}$ and is $\mathcal{F}_{\tau_n}$-measurable.
Assuming that the production activity is in mode 1 at the initial time $t = 0$, let $(u_t)_{t\leq T}$ denote the indicator of the production activity's mode at time $t \in [0, T]$:

$$u_t = 1\!\!1_{[0,\tau_1]}(t) + \sum_{n\geq 1} \xi_n 1\!\!1_{(\tau_n,\tau_{n+1}]}(t). \tag{1}$$

Note that $\delta := (\tau_n)_{n\geq 1}$ and the sequence $\xi := (\xi_n)_{n\geq 1}$ determine uniquely u and conversely, δ and u determine uniquely $(\xi_n)_{n\geq 1}$.

A strategy for our multiple switching problem will be simply denoted by (δ, u).
Finally, let $(X_t)_{0\leq t\leq T}$ denote the market price process of e.g. k underlying commodities or other financial assets that influence the profitability of the production activity.
The state of the whole economic system related to the project at time t is represented by the vector

$$(t, X_t, u_t) \in [0, T] \times I\!\!R^k \times \mathcal{J}. \tag{2}$$

Let $\psi_i(t, x)$ be the payoff rate per unit time when the system is in state (t, x, i), and for $i, j \in \mathcal{J}$ $(i \neq j)$, $\ell_{ij} := (\ell_{ij}(t))_{t\leq T}$ denotes the switching cost of the production at time t from its current mode i to another mode j.
The expected total profit of running the system with the strategy (δ, u) is given by:

$$J(\delta, u) = E[\int_0^T \psi_{u_s}(s, X_s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n<T]}].$$

Solving the optimal multi-regime starting and stopping problem turns into finding a strategy (δ^*, u^*) such that $J(\delta^*, u^*) \geq J(\delta, u)$ for any other strategies (δ, u).

2.1 Assumptions

Throughout this paper $(\Omega, \mathcal{F}, P)$ will be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$.

Let $\mathbf{F} = (\mathcal{F}_t)_{0\le t\le T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\le t\le T}$ with the P-null sets of $\mathcal{F}$. Hence $\mathbf{F}$ satisfies the usual conditions, *i.e.*, it is right continuous and complete.

Furthermore, let:

- $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathbf{F}$-progressively measurable sets ;

- $\mathcal{M}^{p,l}$ be the set of $\mathcal{P}$-measurable and $I\!R^l$-valued processes $w=(w_t)_{t\le T}$ such that $E[\int_0^T |w_s|^p ds] < \infty$ and $\mathcal{S}^p$ be the set of $\mathcal{P}$-measurable, continuous, $I\!R$-valued processes $w=(w_t)_{t\le T}$ such that $E[\sup_{0\le t\le T}|w_t|^p]<\infty$ ($p>1$ is fixed) ;

- For any stopping time $\tau\in[0,T]$, $\mathcal{T}_\tau$ denotes the set of all stopping times θ such that $\tau\le\theta\le T$, $P-a.s.$.

We now make the following assumptions on the data:

(i) The market price $X := (X_t)_{0\le t\le T}$ is $I\!R^k$-valued and each component belongs to $\mathcal{S}^p$.

(ii) The functions $\psi_i(t,x)$, $(t,x)\in[0,T]\times I\!R^k$ and $i\in\mathcal{J}$, are continuous and satisfy a linear growth condition, i.e. there exists a constant C such that $|\psi_i(t,x)|\le C(1+|x|)$ for $0\le t\le T$ and $x\in I\!R^k$.

(iii) The processes ℓ_{ij} belong to $\mathcal{S}^p$ and there exists a real constant $\gamma>0$ such P-a.s. for any $0\le t\le T$, $\min\{\ell_{ij}(t), i,j\in\mathcal{J},\, i\ne j\}\ge\gamma$.

(iv) $(\tau_n)_{n\ge1}$ are $\mathbf{F}$-stopping times and $(\xi_n)_{n\ge1}$ are random variables with values in $\mathcal{J}$ and such that for any $n\ge1$, ξ_n is $\mathcal{F}_{\tau_n}$-measurable. Additionally, we assume that for any $n\ge1$, $P[\xi_n=\xi_{n+1}]=0$. The strategies $(\delta,u)=((\tau_n)_{n\ge1},(\xi_n)_{n\ge1})$ are called admissible if they satisfy:

$$\lim_{n\to\infty}\tau_n = T \quad P-\text{a.s.}$$

The set of admissible strategies is denoted by $\mathcal{D}_a$.

Remark 1 *The above assumptions on X and ψ_i, i=1,...,q, can be modified or weakened in any way which preserves the fact that the process $(\psi_i(t,X_t);\ 0\le t\le T,\ i\in\mathcal{J})$ belongs to $\mathcal{M}^{p,1}$.*

We can now formulate the multi-regime starting and stopping problem as follows:

Problem 1 *Find a strategy $(\delta^*,u^*)=((\tau_n^*)_{n\ge1},(\xi_n^*)_{n\ge1})\in\mathcal{D}_a$ such that*

$$J(\delta^*,u^*)=\sup_{(\delta,u)\in\mathcal{D}_a} J(\delta,u). \tag{3}$$

An admissible strategy (δ, u) is called *finite* if, during the time interval $[0, T]$, it allows the manager to make only a finite number of decisions, i.e. $P[\omega, \tau_n(\omega) < T, \text{ for all } n \geq 0] = 0$. Hereafter the set of finite strategies will be denoted by $\mathcal{D}$. The next proposition tells us that the supremum of the expected total profit can only be reached over finite strategies .

Proposition 1 *The suprema over admissible strategies and finite strategies coincide:*

$$\sup_{(\delta,u)\in\mathcal{D}_a} J(\delta, u) = \sup_{(\delta,u)\in\mathcal{D}} J(\delta, u). \tag{4}$$

Proof. If (δ, u) is an admissible strategy which does not belong to $\mathcal{D}$, then $J(\delta) = -\infty$. Indeed, let $A = \{\omega, \tau_n(\omega) < T, \text{ for all } n \geq 0\}$ and A^c be its complement. Since $(\delta, u) \in \mathcal{D}_a \setminus \mathcal{D}$, then $P(A) > 0$. Since the process X belongs to $\mathcal{S}^p$ and ψ_i is of linear growth, then the processes $(\psi_i(t, X_t))_{t\leq T}$ belongs to $\mathcal{M}^{p,1}$. Therefore,

$$\begin{aligned} J(\delta, u) &\leq E[\int_0^T (\max_{i\in\mathcal{J}} |\psi_i(s, X_s)|)\ ds] \\ &-E[\{\textstyle\sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n)\}1\!\!1_A + \{\textstyle\sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n) 1\!\!1_{[\tau_n<T]}]\}1\!\!1_{A^c}] = -\infty, \end{aligned}$$

since for any $t \leq T$ and $i, j \in \mathcal{J}$, $\ell_{ij}(t) \geq \gamma 0$. This implies that $J(\delta, u) = -\infty$ and then $\sup_{(\delta,u)\in\mathcal{D}_a} J(\delta, u) = \sup_{(\delta,u)\in\mathcal{D}} J(\delta, u)$. $\Box$

We finish this section by introducing the key ingredient of the proof of the main result, namely the notion of Snell envelope and its properties. We refer to Cvitanic and Karatzas (1996) , Appendix D in Karatzas and Shreve (1998), Hamadène (2002) or El Karoui (1980) for further details.

2.2 The Snell Envelope

In the following proposition we summarize the main results on the Snell envelope of processes used in this paper.

Proposition 2 *Let $U = (U_t)_{0\leq t\leq T}$ be an $\mathbf{F}$-adapted $I\!\!R$-valued càdlàg process that belongs to the class [D], i.e. the set of random variables $\{U_\tau,\ \tau \in \mathcal{T}_0\}$ is uniformly integrable. Then, there exists an $\mathbf{F}$-adapted $I\!\!R$-valued càdlàg process $Z := (Z_t)_{0\leq t\leq T}$ such that:*

(i) Z is the smallest super-martingale which dominates U, i.e, if $(\bar{Z}_t)_{0\leq t\leq T}$ is another càdlàg supermartingale of class [D] such that for all $0 \leq t \leq T$, $\bar{Z}_t \geq U_t$ then $\bar{Z}_t \geq Z_t$ for any $0 \leq t \leq T$.

(ii) For any $\mathbf{F}$-stopping time θ we have:

$$Z_\theta = ess\ sup_{\tau\in\mathcal{T}_\theta} E[U_\tau|\mathcal{F}_\theta] \quad (and\ then\ Z_T = U_T). \tag{5}$$

The process Z is called the Snell envelope of U.

Moreover, the following properties hold:

(iii) The Dood-Meyer decomposition of Z implies the existence of a martingale $(M_t)_{0\le t\le T}$ and two nondecreasing processes $(A_t)_{0\le t\le T}$ and $(B_t)_{0\le t\le T}$ which are respectively continuous and purely discontinuous predictable such that for all $0 \le t \le T$,

$$Z_t = M_t - A_t - B_t \quad (\textit{with } A_0 = B_0 = 0).$$

Moreover, for any $0 \le t \le T$, $\{\Delta_t B > 0\} \subset \{\Delta_t U < 0\} \cap \{Z_{t-} = U_{t-}\}$.

(iv) If U has only positive jumps then Z is a continuous process. Furthermore, if θ is an $\mathbf{F}$-stopping time and $\tau_\theta^* = \inf\{s \ge \theta, Z_s = U_s\} \wedge T$ then τ_θ^* is optimal after θ, i.e.

$$Z_\theta = E[Z_{\tau_\theta^*}|\mathcal{F}_\theta] = E[U_{\tau_\theta^*}|\mathcal{F}_\theta] = ess\ sup_{\tau\ge\theta} E[U_\tau|\mathcal{F}_\theta]. \tag{6}$$

(v) If $(U^n)_{n\ge0}$ and U are càdlàg and of class [D] and such that the sequence $(U^n)_{n\ge0}$ converges increasingly and pointwisely to U then $(Z^{U^n})_{n\ge0}$ converges increasingly and pointwisely to Z^U; Z^{U_n} and Z^U are the Snell envelopes of respectively U_n and U. Finally, if U belongs to $\mathcal{S}^p$ then Z^U belongs to $\mathcal{S}^p$.

For the sake of completeness, we give a proof of the stability result (v).

Proof of (v). Since, for any $n \ge 0$, U^n converges increasingly and pointwisely to U, it follows that for all $t \in [0, T]$, $Z_t^{U_n} \le Z_t^U$ P-a.s. Therefore, $P - a.s.$, for any $t \in [0, T]$, $\lim_{n\to\infty} Z_t^{U_n} \le Z_t^U$. Note that the process $(\lim_{n\to\infty} Z_t^{U^n})_{0\le t\le T}$ is a càdlàg supermartingale of class [D], since it is a limit of a nondecreasing sequence of supermartingales (see e.g. Dellacherie and Meyer (1980), pp.86). But $U^n \le Z^{U_n}$ implies that $P - a.s.$, for all $t \in [0, T]$, $U_t \le \lim_{n\to\infty} Z_t^{U^n}$. Thus, $Z_t^U \le \lim_{n\to\infty} Z_t^{U^n}$ since the Snell envelope of U is the lowest supermartingale that dominates U. It follows that $P - a.s.$, for any $t \le 1$, $\lim_{n\to\infty} Z_t^{U^n} = Z_t^U$, whence the desired result.

Assume now that U belongs to $\mathcal{S}^p$. Since, for any $0 \le t \le T$, $-E[\sup_{0\le s\le T} |U_s||\mathcal{F}_t] \le U_t \le E[\sup_{0\le s\le T} |U_s||\mathcal{F}_t]$, using the Doob-Meyer inequality, it follows that Z^U also belongs to $\mathcal{S}^p$. □

3 A verification Theorem

In terms of a verification theorem, we show that Problem 1 is reduced to the existence of q continuous processes $Y^1, \dots, Y^q$ solutions of a system of equations expressed via the Snell envelopes. The process Y_t^i, for $i \in \mathcal{J}$, will stand for the optimal expected profit if, at time t, the production activity is in the state i.

Set

$$D_\tau(\zeta = \zeta') := \inf\{s \geq \tau, \zeta_s = \zeta'_s\} \wedge T,$$

where, τ is an $\mathbf{F}$-stopping time and $(\zeta_t)_{0\leq t\leq T}$, $(\zeta'_t)_{0\leq t\leq T}$ are two continuous $\mathbf{F}$-adapted and $I\!R-$valued processes.

Theorem 1 (*Verification Theorem*)
Assume there exist q $\mathcal{S}^p$-processes $(Y^i := (Y^i_t)_{0\leq t\leq T},\ i = 1,\dots,q)$ that satisfy

$$Y^i_t = ess\ sup_{\tau\geq t} E[\textstyle\int_t^\tau \psi_i(s,X_s)ds + \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(\tau) + Y^j_\tau)1_{[\tau<T]}|\mathcal{F}_t] \quad (\text{and then } Y^i_T = 0). \tag{7}$$

Then $Y^1,\dots,Y^q$ are unique. Furthermore :

(*i*)

$$Y^1_0 = \sup_{(\theta,v)\in\mathcal{D}} J(\theta, v). \tag{8}$$

(*ii*) *Define the sequence of $\mathbf{F}$-stopping times $(\tau_n)_{n\geq 1}$ by*

$$\tau_1 = D_0(Y^1 = \max_{j\in\mathcal{J}^{-1}}(-\ell_{1j} + Y^j)) \tag{9}$$

and, for $n \geq 2$,

$$\tau_n = D_{\tau_{n-1}}(Y^{u_{\tau_{n-1}}} = \max_{k\in\mathcal{J}^{-\tau_{n-1}}}(-\ell_{\tau_{n-1}k} + Y^k)), \tag{10}$$

where,

- $u_{\tau_1} = \sum_{j\in\mathcal{J}} j 1\!\!1_{\{\max_{k\in\mathcal{J}^{-1}}(-\ell_{1k}(\tau_1)+Y^k_{\tau_1}) = -\ell_{1j}(\tau_1)+Y^j_{\tau_1}\}}$;
- *For any $n \geq 1$ and $t \geq \tau_n$, $Y^{u_{\tau_n}}_t = \sum_{j\in\mathcal{J}} 1\!\!1_{[u_{\tau_n}=j]} Y^j_t$;*
- *For $n \geq 2$, $u_{\tau_n} = l$ on the set $\{\max_{k\in\mathcal{J}^{-u_{\tau_{n-1}}}}(-\ell_{u_{\tau_{n-1}}k}(\tau_n) + Y^k_{\tau_n}) = -\ell_{u_{\tau_{n-1}}l}(\tau_n) + Y^l_{\tau_n}\}$,*

where, $\ell_{u_{\tau_{n-1}}k}(\tau_n) = \sum_{j\in\mathcal{J}} 1\!\!1_{[\tau_{n-1}=j]}\ell_{jk}(\tau_n)$ and $\mathcal{J}^{-u_{\tau_{n-1}}} = \sum_{j\in\mathcal{J}} 1\!\!1_{[\tau_{n-1}=j]}\mathcal{J}^{-j}$.

Then, the strategy (δ, u) is optimal i.e. $J(\delta, u) \geq J(\theta, v)$ for any $(\theta, v) \in \mathcal{D}$.

Proof. The proof consists essentially in showing that each process Y^i, as defined by (7), is nothing but the expected total profit or the value function of the optimal problem, given that the system is in mode i at time t. More precisely,

$$Y^i_t = \text{ess sup}_{(\delta,u)\in\mathcal{D}_t} E[\int_t^T \psi_{u_s}(s,X_s)ds - \sum_{j\geq 1}\ell_{u_{\tau_{j-1}}u_{\tau_j}}(\tau_j)1\!\!1_{[\tau_j<T]}|\mathcal{F}_t], \tag{11}$$

where $\mathcal{D}_t$ is the set of finite strategies such that $\tau_1 \geq t$, P-a.s. if at time t the system is in the mode i. This characterization implies in particular that the processes $Y^1,\dots,Y^q$ are unique. Moreover, thanks to a repeated use of the characterization of the Snell envelope (Proposition 2, (iv)), the strategy (δ, u) defined recursively by (9) and (10), is shown to be optimal.

Indeed, since at time $t = 0$ the system is in mode 1, it holds true that, for any $0 \le t \le T$,

$$Y_t^1 + \int_0^t \psi_1(s, X_s)ds = \text{ess sup}_{\tau \ge t} E[\int_0^\tau \psi_1(s, X_s)ds + \max_{j \in \mathcal{J}^{-1}} (-\ell_{1j}(\tau) + Y_\tau^j) \mathbb{1}_{[\tau < T]} | \mathcal{F}_t]. \quad (12)$$

But, Y_0^1 is $\mathcal{F}_0$-measurable. Therefore it is $P - a.s.$ constant and then $Y_0^1 = E[Y_0^1]$.

On the other hand, according to Proposition 2, (iv), τ_1 as defined by (9) is optimal, $Y_T^1 = 0$ and

$$u_{\tau_1} = \sum_{j \in \mathcal{J}} j \mathbb{1}_{\{\max_{k \in \mathcal{J}^{-1}} (-\ell_{1k}(\tau_1) + Y_{\tau_1}^k) = -\ell_{1j}(\tau_1) + Y_{\tau_1}^j\}}.$$

Therefore,

$$\begin{aligned} Y_0^1 &= E[\int_0^{\tau_1} \psi_1(s, X_s)ds + \max_{j \in \mathcal{J}^{-1}} (-\ell_{1j}(\tau_1) + Y_{\tau_1}^j) \mathbb{1}_{[\tau_1 < T]}] \\ &= E[\int_0^{\tau_1} \psi_1(s, X_s)ds + (-\ell_{1u_{\tau_1}}(\tau_1) + Y_{\tau_1}^{u_{\tau_1}}) \mathbb{1}_{[\tau_1 < T]}]. \end{aligned} \quad (13)$$

Next, we claim that P-a.s. for every $t \in [\tau_1, T]$,

$$Y_t^{u_{\tau_1}} = \text{ess sup}_{\tau \ge t} E[\int_t^\tau \psi_{u_{\tau_1}}(s, X_s)ds + \max_{j \in \mathcal{J}^{-u_{\tau_1}}} (-\ell_{u_{\tau_1}j}(\tau) + Y_\tau^j) 1_{[\tau < T]} | \mathcal{F}_t]. \quad (14)$$

To see this, recall that for any $i \in \mathcal{J}$ and $0 \le t \le T$

$$Y_t^i = \text{ess sup}_{\tau \ge t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{j \in \mathcal{J}^{-i}} (-\ell_{ij}(\tau) + Y_\tau^j) 1_{[\tau < T]} | \mathcal{F}_t].$$

This also means that the process $\left(Y_t^i + \int_0^t \psi_i(s, X_s)ds\right)_{0 \le t \le T}$ is a supermartingale which dominates

$$\left(\int_0^t \psi_i(s, X_s)ds + \max_{j \in \mathcal{J}^{-i}} (-\ell_{ij}(t) + Y_t^j) 1_{[t < T]}\right)_{0 \le t \le T}.$$

This implies that the process $\left(\mathbb{1}_{[u_{\tau_1} = i]} (Y_t^i + \int_{\tau_1}^t \psi_i(s, X_s)ds)\right)_{t \in [\tau_1, T]}$ is a supermartingale which dominates

$$\left(\mathbb{1}_{[u_{\tau_1} = i]} (\int_{\tau_1}^t \psi_i(s, X_s)ds + \max_{j \in \mathcal{J}^{-i}} (-\ell_{ij}(t) + Y_t^j) 1_{[t < T]})\right)_{t \in [\tau_1, T]}.$$

Since $\mathcal{J}$ is finite, the process $\left(\sum_{i \in \mathcal{J}} \mathbb{1}_{[u_{\tau_1} = i]} (Y_t^i + \int_{\tau_1}^t \psi_i(s, X_s)ds)\right)_{t \in [\tau_1, T]}$ is also a supermartingale which dominates $\left(\sum_{i \in \mathcal{J}} \mathbb{1}_{[u_{\tau_1} = i]} (\int_{\tau_1}^t \psi_i(s, X_s)ds + \max_{j \in \mathcal{J}^{-i}} (-\ell_{ij}(t) + Y_t^j) 1_{[t < T]})\right)_{t \in [\tau_1, T]}$.

Thus, the process $\left(Y_t^{u_{\tau_1}} + \int_{\tau_1}^t \psi_{u_{\tau_1}}(s, X_s)ds\right)_{t \in [\tau_1, T]}$ is a supermartingale which is greater than

$$\left(\int_{\tau_1}^t \psi_{u_{\tau_1}}(s, X_s)ds + \max_{j \in \mathcal{J}^{-u_{\tau_1}}} (-\ell_{u_{\tau_1}j}(t) + Y_t^j) 1_{[t < T]}\right)_{t \in [\tau_1, T]}. \quad (15)$$

To complete the proof it remains to show that it is the smallest one which has this property and use the characterization of the Snell envelope (Proposition 2, $(i) - (ii)$).

Indeed, let $(Z_t)_{t\in[\tau_1,T]}$ be a supermartingale of class [D] such that, for any $t \in [\tau_1, T]$,

$$Z_t \geq \int_{\tau_1}^{t} \psi_{u_{\tau_1}}(s, X_s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} (-\ell_{u_{\tau_1}j}(t) + Y_t^j)1_{[t<T]}.$$

It follows that for every $t \in [\tau_1, T]$,

$$Z_t 1\!\!1_{[u_{\tau_1}=i]} \geq 1\!\!1_{[u_{\tau_1}=i]} \left(\int_{\tau_1}^{t} \psi_i(s, X_s)ds + \max_{j\in\mathcal{J}^{-i}} (-\ell_{ij}(t) + Y_t^j)1_{[t<T]} \right).$$

But, the process $(Z_t 1\!\!1_{[u_{\tau_1}=i]})_{t\in[\tau_1,T]}$ is a supermartingale and for every $t \in [\tau_1, T]$,

$$1\!\!1_{[u_{\tau_1}=i]} Y_t^i = \text{ess sup}_{\tau\geq t} E[1\!\!1_{[u_{\tau_1}=i]} (\int_{t}^{\tau} \psi_i(s, X_s)ds + \max_{j\in\mathcal{J}^{-i}} (-\ell_{ij}(\tau) + Y_\tau^j)1_{[\tau<T]})|F_t].$$

It follows that, for every $t \in [\tau_1, T]$,

$$1\!\!1_{[u_{\tau_1}=i]} Z_t \geq 1\!\!1_{[u_{\tau_1}=i]} (Y_t^i + \int_{\tau_1}^{t} \psi_i(s, X_s)ds).$$

Summing over i, we get, for every $t \in [\tau_1, T]$,

$$Z_t \geq Y_t^{u_{\tau_1}} + \int_{\tau_1}^{t} \psi_{u_{\tau_1}}(s, X_s)ds.$$

Hence, the process $\left(Y_t^{u_{\tau_1}} + \int_{\tau_1}^{t} \psi_{u_{\tau_1}}(s, X_s)ds\right)_{t\in[\tau_1,T]}$ is the Snell envelope of

$$\left(\int_{\tau_1}^{t} \psi_{u_{\tau_1}}(s, X_s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} (-\ell_{u_{\tau_1}j}(t) + Y_t^j)1_{[t<T]} \right)_{t\in[\tau_1,T]},$$

whence Eq. (14).

Now, from (14) and the definition of τ_2 in Eq. (10), we have

$$\begin{aligned} Y_{\tau_1}^{u_{\tau_1}} &= E[\textstyle\int_{\tau_1}^{\tau_2} \psi_{u_{\tau_1}}(s, X_s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} (-\ell_{u_{\tau_1}j}(\tau_2) + Y_{\tau_2}^j)1_{[\tau_2<T]}|F_{\tau_1}] \\ &= E[\textstyle\int_{\tau_1}^{\tau_2} \psi_{u_{\tau_1}}(s, X_s)ds + (-\ell_{u_{\tau_1}u_{\tau_2}}(\tau_2) + Y_{\tau_2}^{u_{\tau_2}})1_{[\tau_2<T]}|F_{\tau_1}]. \end{aligned}$$

Setting this characterization of $Y_{\tau_1}^{u_{\tau_1}}$ in (13) and noting that $1_{[\tau_1<T]}$ is F_{τ_1}-measurable, it follows that

$$\begin{aligned} Y_0^1 &= E[\int_0^{\tau_1} \psi_1(s, X_s)ds - \ell_{1u_{\tau_1}}(\tau_1)1\!\!1_{[\tau_1<T]}] \\ &\quad + E[\textstyle\int_{\tau_1}^{\tau_2} \psi_{u_{\tau_1}}(s, X_s)ds.1\!\!1_{[\tau_1<T]} - \ell_{u_{\tau_1}u_{\tau_2}}(\tau_2)1\!\!1_{[\tau_2<T]} + Y_{\tau_2}^{u_{\tau_2}}1\!\!1_{[\tau_2<T]}] \\ &= E[\int_0^{\tau_2} \psi_{u_s}(s, X_s)ds - \ell_{1u_{\tau_1}}(\tau_1)1\!\!1_{[\tau_1<T]}] - \ell_{u_{\tau_1}u_{\tau_2}}(\tau_2)1\!\!1_{[\tau_2<T]} + Y_{\tau_2}^{u_{\tau_2}}1\!\!1_{[\tau_2<T]}], \end{aligned}$$

since $[\tau_2 < T] \subset [\tau_1 < T]$.

Repeating this procedure n times, we obtain

$$Y_0^1 = E[\int_0^{\tau_n} \psi_{u_s}(s, X_s)ds - \sum_{j=1}^{n} \ell_{u_{\tau_{j-1}}u_{\tau_j}}(\tau_j)1\!\!1_{[\tau_j<T]}] + Y_{\tau_n}^{u_{\tau_n}}1\!\!1_{[\tau_n<T]}] \tag{16}$$

But, the strategy $\delta = (\tau_n)_{n\geq 1}$ is finite, otherwise Y_0^1 would be equal to $-\infty$ contradicting the assumption that the processes Y^j belong to $\mathcal{S}^p$. Therefore, taking the limit as $n \to \infty$ we obtain $Y_0^1 = J(\delta, u)$.

To complete the proof it remains to show that $J(\delta, u) \geq J(\theta, v)$ for any other finite admissible strategy $(\theta, v) := ((\theta_n)_{n\geq 1}, (\zeta_n)_{n\geq 1})$.
The definition of the Snell envelope yields

$$\begin{aligned} Y_0^1 &\geq E[\textstyle\int_0^{\theta_1} \psi_1(s, X_s)ds + \max_{j\in\mathcal{J}^{-1}}(-\ell_{1j}(\theta_1) + Y_{\theta_1}^j)1_{[\theta_1<T]}] \\ &\geq E[\textstyle\int_0^{\theta_1} \psi_1(s, X_s)ds + (-\ell_{1v_{\theta_1}}(\theta_1) + Y_{\theta_1}^{v_{\theta_1}})1_{[\theta_1<T]}]. \end{aligned}$$

But, once more using a similar characterization as (14), we get

$$\begin{aligned} Y_{\theta_1}^{v_{\theta_1}} &\geq E[\textstyle\int_{\theta_1}^{\theta_2} \psi_{v_{\theta_1}}(s, X_s)ds + \max_{j\in\mathcal{J}^{-v_{\theta_1}}}(-\ell_{v_{\theta_1}j}(\theta_2) + Y_{\theta_2}^j)1_{[\theta_2<T]}|\mathcal{F}_{\theta_1}] \\ &\geq E[\textstyle\int_{\theta_1}^{\theta_2} \psi_{v_{\theta_1}}(s, X_s)ds + (-\ell_{v_{\theta_1}v_{\theta_2}}(\theta_2) + Y_{\theta_2}^{v_{\theta_2}})1_{[\theta_2<T]}|\mathcal{F}_{\theta_1}]. \end{aligned}$$

Therefore,

$$\begin{aligned} Y_0^1 &\geq E[\textstyle\int_0^{\theta_1} \psi_1(s, X_s)ds] - \ell_{1v_{\theta_1}}(\theta_1)1_{[\theta_1<T]}] \\ &+E[1_{[\theta_1<T]}\textstyle\int_{\theta_1}^{\theta_2} \psi_{v_{\theta_1}}(s, X_s)ds - \ell_{v_{\theta_1}v_{\theta_2}}(\theta_2)1_{[\theta_2<T]} + Y_{\theta_2}^{v_{\theta_2}}1_{[\theta_2<T]}] \\ &= E[\textstyle\int_0^{\theta_2} \psi_{v_s}(s, X_s)ds - \ell_{1v_{\theta_1}}(\theta_1)1_{[\theta_1<T]} - \ell_{v_{\theta_1}v_{\theta_2}}(\theta_2)1_{[\theta_2<T]} + Y_{\theta_2}^{v_{\theta_2}}1_{[\theta_2<T]}]. \end{aligned}$$

Repeat this argument n times to obtain

$$Y_0^1 \geq E[\int_0^{\theta_n} \psi_{v_s}(s, X_s)ds - \sum_{j=1}^{n} \ell_{v_{\theta_{n-1}}v_{\theta_n}}(\theta_n)1_{[\theta_n<T]} + Y_{\theta_n}^{v_{\theta_n}}1_{[\theta_n<T]}].$$

Finally, taking the limit as $n \to \infty$ yields

$$Y_0^1 \geq E[\int_0^{T} \psi_{v_s}(s, X_s)ds - \sum_{j\geq 1} \ell_{v_{\theta_{n-1}}v_{\theta_n}}(\theta_n)1_{[\theta_n<T]}] = J(\theta, v)$$

since the strategy (θ, v) is finite. Hence, the strategy (δ, u) is optimal. The proof is now complete. □

4 Existence of the processes $(Y^1, \ldots, Y^q)$.

We will now establish existence of the processes $(Y^1, \ldots, Y^q)$. They will be obtained as a limit of a sequence of processes $(Y^{1,n}, \ldots, Y^{q,n})_{n\geq 0}$ defined recursively as follows.

For $i \in \mathcal{J}$, set, for every $0 \leq t \leq T$,

$$Y_t^{i,0} = E[\int_t^T \psi_i(s, X_s)ds|\mathcal{F}_t], \tag{17}$$

and, for $n \geq 1$,

$$Y_t^{i,n} = \text{ess sup}_{\tau\geq t}E[\int_t^\tau \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(\tau) + Y_\tau^{k,n-1})1\!\!1_{[\tau<T]}|\mathcal{F}_t]. \tag{18}$$

Set $\mathcal{D}_t^{i,n} = \{(\delta, u) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ such that $u_0 = i$, $\tau_1 \geq t$ and $\tau_{n+1} = T\}$.

Using the same arguments as the ones of the Verification Theorem, Theorem 1, the following characterization of the processes $Y^{i,n}$ holds true.

$$Y_t^{i,n} = \text{ess sup}_{(\delta,u)\in\mathcal{D}_t^{i,n}} E[\int_t^T \psi_{u_s}(s, X_s)ds - \sum_{j=1}^{n} \ell_{u_{\tau_{j-1}} u_{\tau_j}}(\tau_j) 1\!\!1_{[\tau_j < T]} |\mathcal{F}_t]. \tag{19}$$

In the next proposition we collect some useful properties of $Y^{1,n}, \ldots, Y^{q,n}$. In particular, we show that, as $n \to \infty$, the limit processes $\tilde{Y}^i := \lim_{n\to\infty} Y^{i,n}$ exist and are only càdlàg but have the same Characterization (7) as the Y^i's. Thus, the existence proof of the Y^i's will consist in showing that $\tilde{Y}^i$'s are continuous and hence satisfy the Verification Theorem. This will be done in Theorem 2, below.

Proposition 3 *(i) For each $n \geq 0$, the processes $Y^{1,n}, \ldots, Y^{q,n}$ are continuous and belong to $\mathcal{S}^p$.*

(ii) For any $i \in \mathcal{J}$, the sequence $(Y^{i,n})_{n\geq 0}$ converges increasingly and pointwisely P-a.s. for any $0 \leq t \leq T$ and in $\mathcal{M}^{p,1}$ to càdlàg processes $\tilde{Y}^i$. Moreover, these limit processes satisfy

(a)

$$E[\sup_{0\leq t\leq T} |\tilde{Y}_t^i|^p] < \infty, \qquad i \in \mathcal{J}. \tag{20}$$

(b) For any $0 \leq t \leq T$,

$$\tilde{Y}_t^i = \textit{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}} (-\ell_{ik}(\tau) + \tilde{Y}_\tau^k) 1\!\!1_{[\tau<T]} |F_t]. \tag{21}$$

Proof. (i) Let us show by induction that, for any $n \geq 0$, $Y_T^{i,n} = 0$ and $Y^{i,n} \in \mathcal{S}^p$, for every $i \in \mathcal{J}$. For $n = 0$ the property holds true since the process $(\psi_i(s, X_s))_{0\leq s\leq T}$ belongs to $\mathcal{S}^p$. Suppose now that the property is satisfied for some n. By Proposition 2, for every $i \in \mathcal{J}$ and up to a term, $Y^{i,n+1}$ is the Snell envelope of the process $\left(\int_0^t \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}} (-\ell_{ik}(t) + Y_t^{k,n}) 1\!\!1_{[t<T]}\right)_{0\leq t\leq T}$ and verifies $Y_T^{i,n+1} = 0$. Since $\max_{k\in\mathcal{J}^{-i}} (-\ell_{ik}(t) + Y_t^{k,n})\big|_{t=T} < 0$, this process is continuous on $[0, T)$ and have a positive jump at T, $Y^{i,n+1}$ is continuous and belongs to $\mathcal{S}^p$. This shows that, for every $i \in \mathcal{J}$, $Y_T^{i,n} = 0$ and $Y^{i,n} \in \mathcal{S}^p$ for any $n \geq 0$.

(ii) We show by induction on $n \geq 0$, that for each $i \in \mathcal{J}$,

$$Y^{i,n} \leq Y^{i,n+1} \leq E[\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t].$$

For $n = 0$ the property is obviously true, since it is enough to take $\tau = T$ in the definition of $Y^{i,1}$ to obtain that $Y^{i,1} \geq Y^{i,0}$. On the other hand taking into account that $\ell_{ij} \geq \gamma > 0$ we have

$$\begin{aligned} Y_t^{i,1} &= \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}} (-\ell_{ik}(\tau) + Y_\tau^{k,0}) 1\!\!1_{[\tau<T]} |\mathcal{F}_t] \\ &\leq \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s, X_s)ds + E[\int_\tau^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|F_\tau]|\mathcal{F}_t] \\ &\leq E[\textstyle\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t]. \end{aligned} \tag{22}$$

Suppose now that, for some n, we have

$$Y^{i,n} \leq Y^{i,n+1} \leq E[\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t], \quad i \in \mathcal{J}.$$

Replace $Y^{i,n+1}$ by $Y^{i,n}$ in the definition of $Y^{i,n+2}$, to obtain that $Y^{i,n+2} \geq Y^{i,n+1}$.
Finally, as is the case for $Y^{i,1}$ in (22), we also have

$$Y_t^{i,n+2} \leq E[\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Therefore, for every $i \in \mathcal{J}$, the sequence $(Y^{i,n})_{n\geq 0}$ is increasing in n and satisfies

$$Y_t^{i,n} \leq E[\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t], \quad 0 \leq, t \leq T. \tag{23}$$

Therefore, it converges to some limit $\tilde{Y}_t^i := \lim_{n\to\infty} Y_t^{i,n}$ that satisfies

$$Y_t^{i,0} \leq \tilde{Y}_t^i \leq E[\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s)|ds|\mathcal{F}_t], \quad 0 \leq t \leq T.$$

Now, using the smoothness properties of ψ_i, Doob's Maximal Inequality yields that, for each $i \in \mathcal{J}$,

$$E[\sup_{0\leq t\leq T} |\tilde{Y}_t^i|^p] < \infty.$$

By the Lebesgue Dominated Convergence Theorem, the sequence $(Y^{i,n})_{n\geq 0}$ also converges to $\tilde{Y}^i$ in $\mathcal{M}^{p,1}$.
Let us now show that $\tilde{Y}^i$ is càdlàg . We note that, for each $n \geq 1$ and $i \in \mathcal{J}$, the process $\left(Y_t^{i,n} + \int_0^t \psi_i(s, X_s)ds\right)_{0\leq t\leq T}$ is a continuous supermartingale, since, by Eq. (18), it is the Snell envelope of the continuous process $\left(\int_0^t \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(\tau) + Y_t^{k,n-1})1\!\!1_{[t<T]}\right)_{0\leq t\leq T}$. Hence, its limit process $\left(\tilde{Y}_t^i + \int_0^t \psi_i(s, X_s)ds\right)_{0\leq t\leq T}$ is càdlàg , as a limit of increasing sequence of continuous supermartingales. Therefore, $\tilde{Y}^i$ is càdlàg .
Finally, the càdlàg processes $\tilde{Y}^1, \ldots, \tilde{Y}^q$ satisfy Eq. (21), since they are limits of the increasing sequence of processes $Y^{i,n}$, $i \in \mathcal{J}$, that satisfy (18). We use Proposition 2, (v) to conclude. □

We will now prove that the processes $\tilde{Y}^1, \ldots, \tilde{Y}^q$ are continuous and satisfy the Verification Theorem, Theorem 1.

Theorem 2 *The limit processes $\tilde{Y}^1, \ldots, \tilde{Y}^q$ satisfy the Verification Theorem.*

Proof. Recall from Proposition 3 that the processes $\tilde{Y}^1, \ldots, \tilde{Y}^q$ are càdlàg , uniformly L^p-integrable and satisfy (21). It remains to prove that they are continuous.
Indeed, note that, for $i \in \mathcal{J}$, the process $\left(\tilde{Y}_t^i + \int_0^t \psi_i(s, X_s)ds\right)_{0\leq t\leq T}$ is the Snell envelope of

$$\left(\int_0^t \psi_i(s, X_s)ds + \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t) + \tilde{Y}_t^k)1\!\!1_{[t<T]}\right)_{0\leq t\leq T}.$$

Therefore, thanks to the Doob-Meyer decomposition of the Snell Envelope of processes (Proposition 2-(iii)), there exist continuous martingales $(M_t^i)_{t\le T}$ and continuous, resp. purely discontinuous, non-decreasing processes $(A_t^i)_{t\le T}$, resp. $(B_t^i)_{t\le T}$, such that, for each $i\in\mathcal{J}$, and $0\le t\le T$,

$$\int_0^t \psi_i(s,X_s)ds+\tilde{Y}_t^i=M_t^i-A_t^i-B_t^i \quad (A_0^i=B_0^i=0).$$

Moreover, the following properties for the jumps of B^i, $i\in\mathcal{J}$ hold. When there is a jump of B^i at t, there is a jump, at the same time t, of the process $(\max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t)+\tilde{Y}_t^k))_{t\le T}$. Since ℓ_{ij} are continuous, there is $j\in\mathcal{J}^{-i}$ such that $\Delta_t\tilde{Y}^j=-\Delta_t B^j<0$ and $\tilde{Y}_{t-}^i=-\ell_{ij}(t)+\tilde{Y}_{t-}^j$. Suppose now there is an index $i_1\in\mathcal{J}$ for which there exists $t\in[0,T]$ such that $\Delta_t B^{i_1}>0$. This implies that there exists another index $i_2\in\mathcal{J}^{-i_1}$ such that $\Delta_t B^{i_2}>0$ and $\tilde{Y}_{t-}^{i_1}=-\ell_{i_1i_2}(t)+\tilde{Y}_{t-}^{i_2}$. But, given i_2, there exists an index $i_3\in\mathcal{J}^{-i_2}$ such that $\Delta_t B^{i_3}>0$ and $\tilde{Y}_{t-}^{i_2}=-\ell_{i_2i_3}(t)+\tilde{Y}_{t-}^{i_3}$. Repeating this argument many times, we get a sequence of indices $i_1,\dots,i_j,\dots\in\mathcal{J}$ that have the property that $i_k\in\mathcal{J}^{-i_{k-1}}$, $\Delta_t B^{i_k}>0$ and $\tilde{Y}_{t-}^{i_{k-1}}=-\ell_{i_{k-1}i_k}(t)+\tilde{Y}_{t-}^{i_k}$.
Since $\mathcal{J}$ is finite then there exist two indices $m<r$ such that $i_m=i_r$ and $i_m,i_{m+1},...,i_{r-1}$ are mutually different. It follows that:

$$\tilde{Y}_{t-}^{i_m}=-\ell_{i_mi_{m+1}}(t)+\tilde{Y}_{t-}^{i_{m+1}}=-\ell_{i_mi_{m+1}}(t)-\ell_{i_{m+1}i_{m+2}}(t)+\tilde{Y}_{t-}^{i_{m+2}}=\dots=-\ell_{i_mi_{m+1}}(t)-\dots-\ell_{i_{r-1}i_r}(t)+\tilde{Y}_{t-}^{i_r}.$$

As $i_m=i_r$ we get

$$-\ell_{i_mi_{m+1}}(t)-\dots-\ell_{i_{r-1}i_r}(t)=0$$

which is impossible since for any $i\neq j$, all $0\le t\le T$, $\ell_{ij}(t)\ge\gamma>0$. Therefore, there is no $i\in\mathcal{J}$ for which there is a $t\in[0,T]$ such that $\Delta_t B^i>0$. This means that $B^i\equiv 0$ and the processes $\tilde{Y}^1,\dots,\tilde{Y}^q$ are continuous. Since they satisfy (21), then, by uniqueness, $Y^i=\tilde{Y}^i$, for any $i\in\mathcal{J}$. Thus, the Verification Theorem 1 is satisfied by $Y^1,\dots,Y^q$. □

We end this section by the following convergence result of the sequences $(Y^{i,n})_{n\ge0}$ to Y^i's.

Proposition 4 *It holds true that, for any $i\in\mathcal{J}$,*

$$E[\sup_{s\le T}|Y_s^{i,n}-Y_s^i|^p]\to 0 \quad as \quad n\to+\infty.$$

Proof. By Proposition 3, we know that P-a.s., for any $n\ge1$, the function $t\mapsto Y_t^{i,n}(\omega)$ is continuous and for any $0\le t\le T$ the sequence $(Y_t^{i,n}(\omega))_{n\ge1}$ converges increasingly to $Y_t^i(\omega)$. As the function $t\mapsto Y_t^i(\omega)$ is continuous then thanks to Dini's Theorem it holds true that:

$$P-\text{a.s.}\ \lim_{n\to\infty}\sup_{0\le t\le T}|Y_t^{i,n}(\omega)-Y_t^i(\omega)|=0.$$

The result now follows from the Lebesgue Dominated Convergence Theorem. □

5 Connection with systems of variational inequalities

When the underlying market price process X is Markov diffusion and the switching costs are of the form $\ell_{ij}(t, X_t)$, the classical methods of solving impulse problems (cf. Brekke and Øksendal (1994), Guo and Pham (2005)) formulate a Verification Theorem suggesting that the value function of our optimal switching problem is the unique viscosity solution the following system of quasi-variational inequalities (QVI) with inter-connected obstacles

$$\begin{cases} \min\{\phi_i(t,x) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t,x) + \phi_j(t,x)), -\partial_t\phi_i(t,x) - A\phi_i(t,x) - \psi_i(t,x)\} = 0, \\ \phi_i(T,x) = 0, \qquad i \in \mathcal{J}, \end{cases} \tag{24}$$

where A is the infinitesimal generator of the driving process X.

However, besides the technical difficulties to establish existence of a smooth solution, existence and uniqueness of a viscosity solution for such systems still remains open for most of the models discussed in the literature (See Carmona and Ludkovski (2006) for a detailed discussion).

By means of yet another characterization of the Snell envelope in terms of systems of reflected Backward SDEs, due to El Karoui et al. (1997-1)(Theorems 7.1 and 8.5), we are able to show that the vector of value processes $(Y^1, \ldots, Y^q)$ of our optimal problem is a viscosity solution of the system (24), when the switching cost functions ℓ_{ij} are only deterministic functions of *the time variable.* An example of such a family of switching costs is

$$\ell_{ij}(t) = e^{-rt}a_{ij},$$

where, a_{ij} are constant costs and $r > 0$ is some discounting rate.

We show that under mild assumptions on the coefficients $\psi_i(t,x)$ and $\ell_{ij}(t)$,

$$Y_t^i = v^i(t, X_t), \qquad 0 \le t \le T, \;\; i \in \mathcal{J},$$

where the deterministic functions $v^1(t,x), \ldots, v^q(t,x)$ are viscosity solutions of the following system of QVI with inter-connected obstacles

$$\begin{cases} \min\{v_i(t,x) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t) + v_j(t,x)), -\partial_t v_i(t,x) - Av_i(t,x) - \psi_i(t,x)\} = 0, \\ v_i(T,x) = 0, \qquad i \in \mathcal{J}. \end{cases} \tag{25}$$

For $(t,x) \in [0,T] \times I\!R^k$, let $(X_s^{tx})_{s\le T}$ be the solution of the following Itô diffusion:

$$dX_s^{tx} = b(s, X_s^{tx})ds + \sigma(s, X_s^{tx})dB_s, \quad t \le s \le T; \qquad X_s^{tx} = x \;\; \text{for } s \le t, \tag{26}$$

where, the functions b and σ, with appropriate dimensions, satisfy the following standard conditions:

There exists a constant $C \geq 0$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \quad \text{and} \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x-x'| \qquad (27)$$

for any $t \in [0,T]$ and $x, x' \in I\!\!R^k$.

These properties of σ and b imply in particular that the process $X^{tx} := (X_s^{tx})_{0\leq s\leq T}$, solution of (26), exists and is unique. Its infinitesimal generator A is given by

$$A = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma.\sigma^*)_{ij}(t,x) D_{ij} + \sum_{i=1}^{d} b_i(t,x) D_i. \qquad (28)$$

Moreover, the following estimates hold true (see e.g. Revuz and Yor (1991) for more details).

Proposition 5 *The process X^{tx} satisfies the following estimates:*

(i) For any $\theta \geq 2$, there exists a constant C such that

$$E[\sup_{0\leq s\leq T} |X_s^{tx}|^\theta] \leq C(1+|x|^\theta). \qquad (29)$$

(ii) There exists a constant C such that for any $t, t' \in [0,T]$ and $x, x' \in I\!\!R^k$,

$$E[\sup_{0\leq s\leq T} |X_s^{tx} - X_s^{t'x'}|^2] \leq C(1+|x|^2)(|x-x'|^2 + |t-t'|). \ \Box \qquad (30)$$

Let us now introduce the following assumption on the payoff rates ψ_i and the switching cost functions ℓ_{ij}:

Assumption [H].

(H1) The running costs ψ_i, $i = 1, \ldots, q$, (of Subsection 2.1) are jointly continuous and are of polynomial growth, i.e., there exist some positive constants C and δ such that for each $i \in \mathcal{J}$,

$$|\psi_i(t,x)| \leq C(1+|x|^\delta), \qquad (t,x) \in [0,T] \times I\!\!R^k.$$

(H1) For any $i, j \in \mathcal{J}$, the switching costs ℓ_{ij} are deterministic functions of t and continuous and there exists a real constant $\gamma > 0$ such for any $0 \leq t \leq T$, $\min\{\ell_{ij}(t), i, j \in \mathcal{J},\, i \neq j\} \geq \gamma$.

Taking into account Proposition 5, the processes $(\psi_i(s, X_s^{tx})_{0\leq s\leq T})_{i=1,q}$ belong to $\mathcal{M}^{2,1}$. A condition we will need to establish a characterization of the value processes of our optimal problem with a class of reflected backward SDEs. Note that the required polynomial growth condition on the ψ_i's is not contradictory with the condition listed in Assumptions 2.1 (ii), since the process X^{tx} has finite moments of all orders (see also Remark 1).

Recall the notion of viscosity solution of the system (25).

Definition 1 *Let* $(v_1,\dots,v_q)$ *be a vector of continuous functions on* $[0,T]\times \mathbb{R}^k$ *with values in* $\mathbb{R}^q$ *and such that* $(v_1,\dots,v_q)(T,x)=0$ *for any* $x\in\mathbb{R}^k$*. The vector* $(v_1,\dots,v_q)$ *is called:*

(i) A viscosity supersolution of the system (25) *if for any* $(t_0,x_0)\in[0,T]\times\mathbb{R}^k$ *and any* q*-tuplet functions* $(\varphi_1,\dots,\varphi_q)\in(C^{1,2}([0,T]\times\mathbb{R}^k))^q$ *such that* $(\varphi_1,\dots,\varphi_q)(t_0,x_0)=(v_1,\dots,v_q)(t_0,x_0)$ *and for any* $i\in\mathcal{J}$*,* (t_0,x_0) *is a maximum of* φ_i-v_i *then we have: for any* $i\in\mathcal{J}$*,*

$$\min\{v_i(t_0,x_0)-\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t_0)+v_j(t_0,x_0)),-\partial_t\varphi_i(t_0,x_0)-A\varphi_i(t_0,x_0)-\psi_i(t_0,x_0)\}\geq 0. \quad (31)$$

(ii) A viscosity subsolution of the system (25) *if for any* $(t_0,x_0)\in[0,T]\times\mathbb{R}^k$ *and any* q*-tuplet functions* $(\varphi_1,\dots,\varphi_q)\in(C^{1,2}([0,T]\times\mathbb{R}^k))^q$ *such that* $(\varphi_1,\dots,\varphi_q)(t_0,x_0)=(v_1,\dots,v_q)(t_0,x_0)$ *and for any* $i\in\mathcal{J}$*,* (t_0,x_0) *is a minimum of* φ_i-v_i *then we have: for any* $i\in\mathcal{J}$*,*

$$\min\{v_i(t_0,x_0)-\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t_0)+v_j(t_0,x_0)),-\partial_t\varphi_i(t_0,x_0)-A\varphi_i(t_0,x_0)-\psi_i(t_0,x_0)\}\leq 0. \quad (32)$$

(iii) The vector of function $(v_1,\dots,v_q)$ *is a viscosity solution of the system* (25) *if it is both a viscosity supersolution and subsolution.*

Let now $(Y_s^{1,tx},\dots,Y_s^{q,tx})_{0\leq s\leq T}$ be the vector of value processes which satisfies the Verification Theorem 1 associated with $(\psi_i(s,X_s^{tx}))_{s\leq T}$ and $\ell^{ij}(t)$. The vector $(Y^{1,tx},\dots,Y^{q,tx})$ exists through Theorem 2 combined with the estimates of X^{tx} of Proposition 5 and Assumptions [H].

The following theorem is the main result of this section.

Theorem 3 *Under Assumption* [**H**]*, there exist* q *deterministic functions* $v^1(t,x),\dots,v^q(t,x)$ *defined on* $[0,T]\times\mathbb{R}^k$ *and* $\mathbb{R}$*-valued such that:*

(i) $v^1,\dots,v^q$ *are continuous in* (t,x)*, are of polynomial growth and satisfy, for each* $t\in[0,T]$*, and for every* $s\in[t,T]$*,*

$$Y_s^{i,tx}=v^i(s,X_s^{tx}),\quad \textit{for every } i\in\mathcal{J}.$$

(ii) The vector of functions $(v^1,\dots,v^q)$ *is a viscosity solution for the system of variational inequalities (25).*

Proof. The proof is obtained through the three following steps.

Step 1. *An approximation scheme*

For $n\geq 0$, let $(Y_s^{1,n,tx})_{0\leq s\leq T},\dots,(Y_s^{q,n,tx})_{0\leq s\leq T}$ be the continuous processes defined recursively by Eqs. (17)-(18). Using Assumption [H1], the estimates (29) for X^{tx} and Proposition 3, the processes $Y^{1,n,tx}\dots,Y^{q,n,tx}$ belong to $\mathcal{S}^2$. Therefore, using a result by El Karoui et al. ((1997-1), Theorem 7.1) which characterizes a Snell envelope as a solution for a one barrier reflected BSDE, for any $n \geq 1$ and $i \in \mathcal{J}$, there exists a pair of $\mathcal{F}_t$-adapted processes $(Z^{i,n,tx}, K^{i,n,tx})$ with value in $R^d \times R^+$ such that:

$$\begin{cases} Y^{i,n,tx},\ K^{i,n,tx} \in \mathcal{S}^2 \text{ and } Z^{i,n,tx} \in \mathcal{M}^{2,d};\ K^{i,n,tx} \text{ is nondecreasing and } K_0^{i,n,tx} = 0, \\ Y_s^{i,n,tx} = \int_s^T \psi_i(u, X_u^{tx})du - \int_s^T Z_u^{i,n,tx} dB_u + K_T^{i,n,tx} - K_s^{i,n,tx}, \text{ for all } 0 \leq s \leq T, \\ Y_s^{i,n,tx} \geq \max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(s) + Y_s^{j,n-1,tx}\}, \text{ for all } 0 \leq s \leq T, \\ \int_0^T (Y_u^{i,n,tx} - \max\limits_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(u) + Y_u^{j,n-1,tx}\})dK_u^{i,n} = 0. \end{cases} \tag{33}$$

Thanks to Theorem 8.5 in El Karoui et al. (1997-1) related to the representation of solutions of reflected backward SDEs, there exist deterministic functions $v^{1,0}, \ldots, v^{q,0}$ defined on $[0,T] \times R^k$, continuous and with polynomial growth such that for every $(t,x) \in [0,T] \times R^k$ and every $i \in \mathcal{J}$,

$$Y_s^{i,0,tx} = v^{i,0}(s, X_s^{tx}), \quad t \leq s \leq T.$$

Using an induction argument, and applying Theorem 8.5 in El Karoui et al. (1997-1) at each step, yields the existence of deterministic functions $v^{1,n}, \ldots, v^{q,n}$ defined on $[0,T] \times R^k$, that are continuous and with polynomial growth such that, for every $(t,x) \in [0,T] \times R^k$ and every $i \in \mathcal{J}$,

$$Y_s^{i,n,tx} = v^{i,n}(s, X_s^{tx}), \quad t \leq s \leq T.$$

Since the sequences of processes $(Y^{i,n,tx})_{n \geq 0}$ is nondecreasing in n, then for any $i \in \mathcal{J}$, the sequences of deterministic functions $(v^{i,n})_{n \geq 0}$ is also nondecreasing.

Moreover, we have

$$\begin{aligned} v^{i,n}(t,x) &\leq Y_t^{t,x} \leq E[\textstyle\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s^{tx})|ds|\mathcal{F}_t] \\ &\leq E[\textstyle\int_t^T \max_{i=1,\ldots,q} |\psi_i(s, X_s^{tx})|ds], \end{aligned} \tag{34}$$

where, the last inequality is obtained after taking expectations, since $v^{i,n}(t,x)$ is a deterministic function. It follows that for any $i \in \mathcal{J}$, the sequence $(v^{i,n})_{n \geq 0}$ converges pointwisely to a deterministic function v^i and the last inequality in Eq. (34) implies that v^i is of polynomial growth through ψ_i and the estimates (29) for X^{tx}. Furthermore, for any $(t,x) \in [0,T] \times R^k$ we have

$$Y_s^{i,tx} = v^i(s, X_s^{tx}), \quad t \leq s \leq T. \tag{35}$$

Step 2. *$L^2(P)$-continuity of the value functions $(t,x) \longrightarrow Y^{i,tx}$.*

Let (t,x) and (t',x') be elements of $[0,T] \times I\!\!R^k$. Using the representation (11) we will show that

$$E[\sup_{0 \leq s \leq T} |Y_s^{i,t'x'} - Y_s^{i,tx}|^2] \to 0 \text{ as } (t',x') \to (t,x) \text{ for any } i \in \mathcal{J}.$$

Indeed, recall that, by (11), we have, for any $i \in \mathcal{J}$ and $s \in [0,T]$

$$Y_s^{i,tx} = \text{ess sup}_{(\delta,u)\in\mathcal{D}_s} E[\int_s^T \psi_{u_s}(s, X_s^{tx})ds - \sum_{j\geq 1} \ell_{u_{\tau_{j-1}} u_{\tau_j}}(\tau_j) 1\!\!1_{[\tau_j<T]} | \mathcal{F}_s],$$

where, $\mathcal{D}_s$ is the set of finite strategies such that $\tau_1 \geq s$, $P-a.s.$

Therefore,

$$\begin{aligned}|Y_s^{i,tx} - Y_s^{i,t'x'}| &\leq \text{ess sup}_{(\delta,u)\in\mathcal{D}_s} E[\int_s^T |\psi_{u_r}(r, X_r^{tx}) - \psi_{u_r}(r, X_r^{t'x'}, u_r)|dr|\mathcal{F}_s] \\ &\leq E[\int_0^T \{\sum_{i=1}^q |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}ds|\mathcal{F}_s].\end{aligned}$$

Now, using Doob's Maximal Inequality (see e.g. [26]) and taking expectation, there exists of a constant $C \geq 0$ such that:

$$E[\sup_{0\leq s\leq T} |Y_s^{i,tx} - Y_s^{i,t'x'}|^2] \leq CE[\int_0^T \{\sum_{i=1}^q |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}^2 ds]. \tag{36}$$

But, the right-hand side of this last inequality converges to 0 as (t', x') tends to (t, x). Indeed, for any $\varpi > 0$ it holds true that:

$$\begin{aligned}E[\int_0^T \{\sum_{i=1}^q |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}^2 ds] &\leq E[\int_0^T \{\sum_{i=1}^q |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}2 1\!\!1_{[|X_r^{tx}|+|X_r^{t'x'}|\leq\varpi]} ds] \\ &+E[\int_0^T \{\sum_{i=1}^q |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}2 1\!\!1_{[|X_r^{tx}|+|X_r^{t'x'}|>\varpi]} ds].\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, the continuity of ψ_i and Estimates (30), the first term of the right-hand side of this inequality converges to 0 as (t', x') tends to (t, x).

The second term satisfies:

$$\begin{aligned}&E[\int_0^T \{\sum_{i=1,\ldots,q} |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}2 1\!\!1_{[|X_r^{tx}|+|X_r^{t'x'}|>\varpi]} ds] \\ &\qquad \leq \{E[\int_0^T \{\sum_{i=1,\ldots,q} |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}4]\}^{\frac{1}{2}} \{E[\int_0^T 1\!\!1_{[|X_r^{tx}|+|X_r^{t'x'}|>\varpi]} ds]\}^{\frac{1}{2}} \\ &\qquad \leq \{E[\int_0^T \{\sum_{i=1,\ldots,q} |\psi_i(r, X_r^{tx}) - \psi_i(r, X_r^{t'x'})|\}4]\}^{\frac{1}{2}} \{\varpi^{-1} E[\int_0^T (|X_r^{tx}| + |X_r^{t'x'}|) ds]\}^{\frac{1}{2}}.\end{aligned}$$

Using Estimates (29) and the polynomial growth of ψ_i, it follows that, when (t', x') tends to (t, x), the supremum limit of the right-hand side of the last inequality is smaller than $\varpi^{-\frac{1}{2}} C_{tx}$ where C_{tx} is a constant. As ϖ is whatever then going back to (36) and taking the limit to obtain, for any $i \in \mathcal{J}$,

$$E[\sup_{0\leq s\leq T} |Y_s^{i,tx} - Y_s^{i,t'x'}|^2] \to 0 \qquad \text{as} \quad (t', x') \to (t, x).$$

Step 3. *the functions $v^1, \ldots, v^q$ are continuous in (t, x) and the vector of functions $(v^1, \ldots, v^q)$ is a viscosity solution of the system of variational inequalities (25).*

Thanks to the result obtained in Step 2, for any $i \in \mathcal{J}$, the function $(s,t,x) \mapsto Y_s^{i,tx}$ is continuous from $[0,T]^2 \times I\!R^k$ into $L^2(P)$. Indeed, this follows from the fact that

$$|Y_{s'}^{i,t'x'} - Y_s^{i,tx}| \leq |Y_{s'}^{i,t'x'} - Y_{s'}^{i,tx}| + |Y_{s'}^{i,tx} - Y_s^{i,tx}| \leq \sup_{s \leq T}(|Y_s^{i,t'x'} - Y_s^{i,tx}|) + |Y_{s'}^{i,tx} - Y_s^{i,tx}|.$$

Therefore, the function $(t,t,x) \mapsto Y_t^{i,tx}$ is also continuous. But, the result obtained in Step 1, implies that $Y_t^{i,tx}$ is deterministic and is equal to $v^i(t,x)$. Hence, the function v^i is continuous in (t,x). The deterministic functions v^i, $i \in \mathcal{J}$, being continuous and of polynomial growth, by Theorem 8.5 in El Karoui et al. (1997-1), these functions are viscosity solutions for the system (25). $\square$

6 Simulating the value-processes $(Y^1, ..., Y^q)$

An important issues in the optimal multiple switching problem is to provide efficient algorithms to simulate of the value-processes $(Y^1, ..., Y^q)$ solution of the Verification Theorem 1. In this section we comment on this by providing yet another approximation scheme of the value-processes $(Y^1, ..., Y^q)$ by exploiting their representation as solution for a system of BSDE with one reflecting barrier. Thanks to a result in El Karoui et al. ((1997-1), Theorem 7.1) which characterizes a Snell envelope of a process which belongs to $\mathcal{S}^2$ as a solution for a BSDE with one reflecting barrier, the vector $(Y^1, ..., Y^q)$ is the solution of the following system of reflected BSDEs:

For any $i \in \mathcal{J}$, there exists a pair of $\mathcal{F}_t$-adapted processes (Z^i, K^i) with value in $I\!R^d \times I\!R^+$ such that:

$$\begin{cases} Y^i,\ K^i \in \mathcal{S}^2 \ \text{ and } \ Z^i \in \mathcal{M}^{2,d}\ ;\ K^i \text{ is continuous nondecreasing and } K_0^i = 0, \\ Y_s^i = \displaystyle\int_s^T \psi_i(u, X_u)du - \int_s^T Z_u^i dB_u + K_T^i - K_s^i, \ \text{for all } \ 0 \leq s \leq T, \\ Y_s^i \geq \max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(s) + Y_s^j\}, \quad \text{for all } \ 0 \leq s \leq T, \\ \displaystyle\int_0^T (Y_u^i - \max_{j \in \mathcal{J}^{-i}}\{-\ell_{ij}(u) + Y_u^j\})dK_u^i = 0. \end{cases} \tag{37}$$

Note that, when $X \equiv X^{tx}$, taking the limit in (33), we obtain the solution of the system (37).

It is now well known that the solution of a reflected BSDE can be approximated, in using a penalization scheme, by solutions of standard BSDEs (see El Karoui et al. (1997-1) for more details). Indeed, for $n \geq 0$, consider the following sequence of SDEs

$$Y_t^{i,n} = \int_t^T \psi_i(s, X_s)ds + n\int_t^T (L_s^{i,n} - Y_s^{i,n})^+ ds - \int_t^T Z_s^{i,n} dB_s, \ \ i \in \mathcal{J}, \ \ t \in [0,T], \tag{38}$$

where, for every $i \in \mathcal{J}$,

$$L_t^{i,n} = \max_{k \in \mathcal{J}^{-i}}(-\ell_{ik}(t) + Y_t^{k,n}), \ \ t \in [0,T].$$

Now, if we define the generator $f_n = (f_n^1, \ldots, f_n^q) : \ [0,T] \times I\!R^q \to I\!R^q$ by

$$f_n^i(s, (y_1, ..., y_q)) = \psi_i(s, X_s) + n(\max_{k \in \mathcal{J}^{-i}}(-\ell_{ik}(s) + y_k) - y_i)^+, \ \ i \in \mathcal{J},$$

the $I\!R^q$-valued process $Y^n = (Y^{1,n}, \dots, Y^{q,n})$ satisfies the following BSDE:

$$Y_t^n = \int_t^T f_n(s, Y_s^n)ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T]. \tag{39}$$

The function f_n being Lipschitz continuous w.r.t. y, uniformly in t, therefore through a result by Gobet et al. (2005) on numerical schemes of BSDEs, this multidimensional equation can be numerically solved, at least in the case when the process X is a Markovian diffusion. Therefore, this provides a way to simulate Y^i since, as we will show it in Theorem 4 below, the sequence $(Y^{i,n})_{n\geq 0}$ converges to Y^i. Indeed, we have:

Proposition 6 *For every $i \in \mathcal{J}$ and every $t \in [0, T]$, the sequence $(Y_t^{i,n})_{n\geq 0}$ is non-decreasing and P-a.s. $Y_t^{i,n} \leq Y_t^i$.*

Proof. For $n \in I\!N$, and $k \in I\!N^*$, consider the following scheme. For every $i \in \mathcal{J}$

$$Y_t^{i,n,k} = \int_t^T \psi_i(s, X_s)ds + n\int_t^T (\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(s) + Y_s^{j,n,k-1}) - Y_s^{i,n,k})^+ds - \int_t^T Z_s^{i,n,k}dB_s, \quad t \in [0, T]$$

and

$$Y^{i,n,0} = E[\int_t^T \psi_i(s, X_s)ds|\mathcal{F}_t], \quad t \leq T.$$

¿ From a result in El Karoui et al. (1997-1), $Y^{i,n,k}$ converges to $Y^{i,n}$ when k tends to infinity. Now, let us show by induction on k that:

$$P - a.s. \quad Y_t^{i,n,k} \leq Y_t^{i,n+1,k}, \quad n \geq 0, \quad i \in \mathcal{J}, \quad t \in [0, T].$$

For $k = 0$ the property holds true. Suppose now that it is also verified for some $k - 1$ and let us show that it is valid for k. For any $n \geq 0$, $i \in \mathcal{J}$ and $t \in [0, T]$ we have:

$$Y_t^{i,n+1,k} = \int_t^T \psi_i(s, X_s)ds + (n+1)\int_t^T (\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(s) + Y_s^{j,n+1,k-1}) - Y_s^{i,n+1,k})^+ds - \int_t^T Z_s^{i,n+1,k}dB_s$$

and

$$Y_t^{i,n,k} = \int_t^T \psi_i(s, X_s)ds + n\int_t^T (\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(s) + Y_s^{j,n,k-1}) - Y_s^{i,n,k})^+ds - \int_t^T Z_s^{i,n,k}dB_s.$$

Thanks to the induction hypothesis, for any $n \geq 0$, $i \in \mathcal{J}$ and $t \leq T$, we have $Y_t^{i,n,k-1} \leq Y_t^{i,n+1,k-1}$. Therefore, using the comparison theorem of solutions of standard BSDEs (see e.g. El Karoui et al. (1997-2), Theorem 2.2) we get that

$$Y_t^{i,n+1,k} \geq Y_t^{i,n,k}, \quad t \leq T,$$

which is the desired result. Now taking the limit as k goes to $+\infty$, we obtain, for any $n \geq 0$ and $i \in \mathcal{J}$, $Y^{i,n} \leq Y^{i,n+1}$.

To finish the proof it remains to show that for any $k \geq 0$ and $n \geq 0$ we have $Y_t^{i,n,k} \leq Y_t^i$ for any $i \in \mathcal{J}$ and $t \leq T$ and then take the limit as k goes to infinity. Once more using induction on k, it hods true that, for all $k \geq 0$, $n \geq 0$ and $t \in [0,T]$,

$$Y_t^{i,n,k} \leq Y_t^i, \quad \text{for any } i \in \mathcal{J}.$$

Indeed, for $k = 0$ the property is obviously satisfied. In order to go from k to $k+1$, we note that, by Eq. (37), and since for that k, $Y_t^{i,n,k} \leq Y_t^i$, for any $i \in \mathcal{J}$, it holds that for every $t \leq T$, $(\max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(t) + Y_t^{j,n,k}) - Y_t^i)^+ = 0$.
Hence, for all $t \leq T$,

$$Y_t^i = \int_t^T \psi_i(s, X_s)ds + n\int_t^T (\max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(s) + Y_s^{j,n,k}) - Y_s^i)^+ ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s.$$

Now, taking into account that the process K^i is non-decreasing and finally and using the Comparison Theorem of solutions of standard BSDEs, we get that

$$Y_t^{i,n,k+1} \leq Y_t^i, \quad \text{for any } i \in \mathcal{J}.$$

Finally taking the limit as $k \to \infty$ we get that, for all $n \geq 0$ and $t \in [0,T]$,

$$Y_t^{i,n} \leq Y_t^i, \quad \text{for any } i \in \mathcal{J}.$$

The proof is now complete. □

Theorem 4 *For any $i \in \mathcal{J}$ it holds true that:*

$$E[\sup_{0\leq t\leq T} |Y_t^{i,n} - Y_t^i|^2] \to 0 \textit{ as } n \to \infty.$$

Proof. We have, for every $i \in \mathcal{J}$ and all $t \in [0,T]$, $Y_t^{i,n} \leq Y_t^{i,n+1}$. Therefore there exists a process $\bar{Y}^i$ such that,

$$\lim_{n\to+\infty} Y^{i,n} = \bar{Y}_t^i \leq Y_t^i, \quad t \in [0,T].$$

Moreover, from (38) we get that, for any $t \leq T$,

$$Y_t^{i,n} = \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s, X_s)ds + (L_\tau^{i,n} \wedge Y_\tau^{i,n})1_{[\tau<T]}|\mathcal{F}_t].$$

This is due to the facts that the process $n\int_0^\cdot (L_s^{i,n} - Y_s^{i,n})^+ ds$ is increasing and satisfies $\int_0^T (Y_s^{i,n} - L_s^{i,n} \wedge Y_s^{i,n})n(L_s^{i,n} - Y_s^{i,n})^+ ds = 0$. Therefore, in order to conclude, it is enough to use the representation result by El Karoui et al. (1997-1) of solution of reflected BSDEs as Snell envelopes of processes.

Now, since the process $Y_t^{i,n} + \int_0^t \psi_i(s, X_s)ds$ is a continuous supermartingale, the non-decreasing limit $\bar{Y}^i$ is a càdlàg process. Using now the result given in Proposition 2 - v, it follows that

$$\bar{Y}_t^i = \text{ess sup}_{\tau \geq t} E[\int_t^\tau \psi_i(s, X_s)ds + (\bar{L}_\tau^i \wedge \bar{Y}_\tau^i)1_{[\tau < T]}|\mathcal{F}_t], \quad t \leq T$$

with $\bar{L}_t^i = \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(t) + \bar{Y}_t^k)$ is the nondecreasing limit of $L^{i,n}$. But, from (38), taking expectation, dividing by n and taking the limit as $n \to \infty$ we obtain

$$\int_0^T (\bar{L}_s^i - \bar{Y}_s^i)^+ ds = 0$$

which implies that for any $t \leq T$, $\bar{Y}_t^i \geq \bar{L}_t^i$, since these latter processes are càdlàg . Now we can argue as in Section 4 to show that the processes $\bar{Y}^i$, $i \in \mathcal{J}$, are continuous. Therefore they satisfy the Verification Theorem whose solution is unique. Hence, for any $i \in \mathcal{J}$, we have $\bar{Y}^i = Y^i$ and the sequences $(Y^{i,n})_{n \geq 0}$ are nondecreasing and converge to the continuous processes Y^i. Finally in order to conclude we just need to use first Dini's Theorem and then the Lebesgue dominated convergence theorem. $\square$

Remark 2 *It doesn't seem easy to obtain a convergence rate of $Y^{i,n}$ to Y^i. In the two-modes case and when the switching costs are constant, Hamadène and Jeanblanc (2007)(Proposition 4.2) show that the rate of convergence is $\frac{1}{n}$. This very interesting issue will be addressed in a forthcoming work.*

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