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A FINITENESS CRITERION FOR COMPACT VARIETIES OF SURJECTIVE HOLOMORPHIC MAPPINGS

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It is well known that there exist at most finitely many surjective meromorphic mappings from any compact variety X onto a Riemann surface Y of genus$\geqq 2$. There are several possibilities of generalizing this fact to higher dimensions. For instance, the same assertion holds, if Y is a variety of general type (Kobayashi-Ochiai [7]), or if Y is Carathéodory-hyperbolic (Urata [12]). In [8], S. Lang raised the question whether the statement also carries over to algebraic varieties that are hyperbolic in the sense of Kobayashi. A partial answer to this problem has been given by J. Noguchi [11] who proved the finiteness assertion for hyperbolic Kähler manifolds Y with semi-positive canonical bundle. Employing a different approach, we shall show that the semi-positivity condition for K_Y may be dropped; however, the smoothness of Y as well as the Kähler condition still remain essential in our considerations.

1. Statement of the results and preliminary remarks

Irreducible reduced complex spaces will be called *varieties; a subvariety* of a complex space X is a closed complex subspace that is a variety. For compact complex space X, Y, we let $\mathrm{Hol}(X, Y):=\{\alpha: X\to Y: \alpha \text{ holomorphic}\}$ be endowed with the Douady structure (see [1]). The subset $\mathrm{Sur}(X, Y)$ of all surjective α is open in $\mathrm{Hol}(X, Y)$ and therefore carries a complex structure with canonical properties. If X is reduced, then $\mathrm{Sur}(X, Y)$ is closed in $\mathrm{Hol}(X, Y)$, and the group of holomorphic automorphisms $\mathrm{Aut}(X)$ is closed in $\mathrm{Hol}(X, X)$.

Let X, Y be reduced and compact. If Y is hyperbolic (in the sense of Kobayashi), then $\mathrm{Hol}(X, Y)$ is compact and hence so is $\mathrm{Sur}(X, Y)$. Furthermore, if X is a torus (i.e., a compact connected complex Lie group), then $\mathrm{Hol}(X, Y)$ consists of constant maps only; in particular, Y contains no rational curves. (Compare [6]).

1.1 THEOREM. *Let Y be a compact hyperbolic Kähler manifold. Then $\mathrm{Sur}(X, Y)$ is finite for all compact varieties X.*

As an immediate consequence we obtain

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1.1.1 COROLLARY. *Let Y be as above, and let X be any compact variety. Then there exist at most finitely many surjective meromorphic mappings from X onto Y.*

In fact, by Hironaka [3], we may assume that X is smooth. Then every meromorphic map from X to Y is holomorphic, since Y contains no rational curves.

Clearly, Theorem 1.1 follows directly from

1.2 THEOREM. *Let Y be a compact Kähler manifold such that there exists no non-constant holomorphic map from any torus into Y, and let X be any compact variety.*

Then $\mathrm{Sur}(X, Y)$ *contains no positive-dimensional compact subvarieties.*

2. Compact varieties of surjective holomorphic mappings

2.1 LEMMA. *Let $\alpha: X \to Y'$, $\beta: Y' \to Y$ be surjective holomorphic mappings between compact varieties with β finite and Y' normal.*

If $Z \subset \beta \circ \mathrm{Aut}(Y') \circ \alpha := \{\beta \circ \gamma \circ \alpha : \gamma \in \mathrm{Aut}(Y')\}$, then there exists a compact connected complex subgroup T of $\mathrm{Aut}(Y')$ with $Z \subset \beta \circ T \circ \alpha$.

Proof. It suffices to show that the map $\mathrm{Aut}(Y') \to \mathrm{Hol}(X, Y)$, $\gamma \mapsto \beta \circ \gamma \circ \alpha$, is proper. Clearly, if $(\beta \circ \gamma_n \circ \alpha)$ is a convergent sequence in $\mathrm{Hol}(X, Y)$, then $(\beta \circ \gamma_n)$ converges to some g in $\mathrm{Hol}(Y', Y)$. Let $U := g^{-1}(Y \setminus B)$, where $B \subset Y$ denotes the branch locus of β. As $\beta | \beta^{-1}(Y \setminus B) \to Y \setminus B$ is finite and locally biholomorphic, the set $V := \{y' \in U : (\gamma_n(y')) \text{ is convergent}\}$ is open and closed in U. Replacing (γ_n) by a suitable subsequence, we may assume that V is non-empty, whence $V = U$. Then $\gamma' : U \to Y$, defined by $\gamma'(y') := \lim \gamma_n(y')$, is holomorphic, since so is $g | U = \beta \circ \gamma'$. As Y' is normal, γ' extends to $\gamma \in \mathrm{Aut}(Y')$ with $\gamma = \lim \gamma_n$.

For reference purposes, we list some results of [4] and [5], adapted to the present situation.

Let $\phi : X \times Z \to Y$ be a holomorphic map between compact varieties such that all $\phi(\cdot, z)$ are surjective.

2.2 There exist $f : X \to X'$ and $\phi' : X' \times Z \to Y$ such that all $\phi(\cdot, z)$ are finite and $\phi = \phi' \circ (f \times id_Z)$ (see [4], 2.2).

From now on assume that Y is normal and that all $\phi(\cdot, z)$ are finite.

2.3 If some $\phi(\cdot, z_0)$ is an unramified covering, then $\phi(\cdot, z) \in \phi(\cdot, z_0) \circ \mathrm{Aut}(X)$ for all z. (See [5], 5.1.1). Using 2.1, we conclude: There exists a connected compact complex subgroup $T \subset \mathrm{Aut}(X)$ such that $\phi(\{x\} \times Z) \subset \phi(Tx \times \{z_0\})$ for all x.

2.4 A subset S of Y is called *ϕ-invariant*, if $\phi(\{x\} \times Z) \subset S$ for all $x \in X$ with $\phi(\{x\} \times Z) \cap S \neq \emptyset$. A ϕ-invariant subvariety of Y that contains no proper ϕ-

invariant subsets will be called *ϕ-minimal*.

As was shown in ([5], 3.2), there exists a dense open ϕ-invariant $U\subset Y$ together with a proper holomorphic map $p: U\to U'$ such that all fibres of p are ϕ-minimal subvarieties of Y. If Y is smooth, then p can be chosen smooth as well.

3. The projective case

3.1 THEOREM. *Let Y be a projective manifold without rational curves, and let X be any normal compact variety.*

For every compact subvariety Z of Sur (X, Y) *there exist surjective holomorphic $\alpha: X\to Y'$, $\beta: Y'\to Y$ with β finite and Y' normal, such that $\beta\circ\alpha\in Z\subset\beta\circ$*Aut *$Y')\circ\alpha$.*

Proof. Let $\phi=(X\times Z\overset{\phi'}{\to}Y'\overset{\beta}{\to}Y)$ be the Stein factorization of the evaluation map $\phi: X\times Z\to Y$. From Mori's description of thc pseudoample cone of a projective manifold [10] we infer that the canonical bundle of Y is numerically effective, and the assertion follows from ([5], 2.2).

As an immediate consequence, we note that 3.1 together with 2.1 proves 1.2 under the additional assumption that Y be projective.

4. Proof of Theorem 1.2

Assuming the contrary, let $Z\subset$Sur (X, Y) be a closed subvariety of minimal positive dimension, and denote by $\phi: X\times Z\to Y$ the evaluation map. By 2.2, we need only consider the case that all $\phi(\cdot, z)$ are finite, and by 2.4, we may assume that Y is ϕ-minimal; moreover, X can be assumed normal. Then, by 3.1, Y can not be projective.

By 2.3, the maps $\phi(\cdot, z)$ can not be unramified coverings. Therefore, Y must contain some subvariety Y' of codimension one. As Y is ϕ-minimal, there exists $x\in X$ with $\emptyset\neq Z_0:=\phi(x, \cdot)^{-1}(Y')\neq Z$. Thus $\dim Z_0=\dim Z-1$, whence $Z=1$ by the minimality assumption on Z.

Let now $Y_0\subset Y$ be a Moišezon subvariety of maximal dimension, and fix any $z\in Z$. Then the irreducible components $X_1, \cdots, X_k$ of $\phi(\cdot, z)^{-1}(Y_0)$ are Moisezon as well, and hence so are the subvarieties $Y_\kappa:=\phi(X_\kappa\times Z)$, $\kappa=1\cdots k$. Clearly, Y_0 is contained in every Y_κ, where $Y_0=Y_1=\cdots=Y_k$ by the maximality of Y_0. Thus Y_0 is ϕ-invariant and therefore equal to Y. As Y is Kähler, we conclude that Y is projective (compare [9]), a contradiction.

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