Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (5) (2019), 1417 – 1429
DOI : 10.15672/HJMS.2018.582

Research Article

A fitted approximate method for a Volterra delay-integro-differential equation with initial layer

Gabil M. Amiraliyev*, Ömer Yapman, Mustafa Kudu

Department of Mathematics, Faculty of Arts and Sciences, Erzincan University 24100, Erzincan, Turkey

Abstract

This study is concerned with the finite-difference solution of singularly perturbed initial value problem for a linear first order Volterra integro-differential equation with delay. The method is based on the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. The emphasis is on the convergence of numerical method. It is shown that the method displays uniform convergence in respect to the perturbation parameter. Numerical results are also given.

Mathematics Subject Classification (2010). 65L10, 65L12, 65L20, 65R20

Keywords. Volterra delay-integro-differential equation, singular perturbation, finite difference, uniform convergence

1. Introduction

Volterra delay-integro-differential equations (VDIDEs) have a major influence on the field of science such as ecology, medicine, physics, biology and so on [7–9, 15, 22]. These equations play a significant role in modelling of some phenomena in engineering and sciences, and hence have led researchers to develop a theory and numerical computation and analysis for VDIDEs.

Here we shall concern with the development of fitted difference method for singularly perturbed Volterra delay-integro-differential equation (SPVDIDE):

$$Lu := \varepsilon u' + a(t)u + \int_{t-r}^{t} K(t,s)u(s)ds = f(t),\ t \in I, \tag{1.1}$$

subject to

$$u(t) = \varphi(t),\ t \in I_0, \tag{1.2}$$

where $I = (0,T] = \bigcup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \le r_p\}$, $1 \le p \le m$ and $r_s = sr$, for $0 \le s \le m$, $\bar{I} = [0,T]$ and $I_0 = [-r, 0]$. $\varepsilon \in (0,1]$ is the perturbation parameter and r is a constant delay, which is independent of ε. $a(t) \ge \alpha > 0$, $f(t)$ $(t \in \bar{I})$, $\varphi(t)$ $(t \in I_0)$ and $K(t,s)$

*Corresponding Author.
Email addresses: gabilamirali@yahoo.com (G.M. Amiraliyev), yapmanomer@gmail.com (Ö. Yapman), muskud28@yahoo.com (M. Kudu)
Received: 05.07.2017; Accepted: 17.04.2018

($(t,s) \in \overline{I} \times \overline{I}$) are assumed to be sufficiently smooth functions such that the solution, $u(t)$, has initial layer at $t = 0$ for small values of ε.

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. In general, the solutions of such equations exhibit multiscale phenomena. Within certain thin subregions of the domain, the scale of some derivatives is significantly larger than other derivatives. These thin regions of rapid change are called, boundary or interior layers, as appropriate. Such type of equations occur frequently in mathematical problems in the sciences and engineering for example, in fluid flow at high Reynold number, electrical networks, chemical reactions, control theory, the equations governing flow in porous media, the drift-diffusion equations of semi-conductor device physics, and other physical models [10, 11, 23, 25, 26]. It is well-known that standard discretization methods do not work well for these problems as they often produce oscillatory solutions which are inaccurate if the perturbed parameter ε is small. To obtain robust numerical methods it is necessary to fit the coefficients (fitted operator methods) or the mesh (fitted mesh methods) to the behavior of the exact solution [1–3, 11, 16, 23, 24, 28, 34] (see also references cited in them). For a survey of early results in the theoretical analysis of singularly perturbed Volterra integro-differential equations (VIDEs) and in the numerical analysis and implementation of various techniques for these problems we refer to the book [17]. An analysis of approximate methods when applied to singularly perturbed VIDEs can also be found in [2, 5, 6, 18, 21, 27, 29, 33].

In the last few years, a considerable amount of effort has been devoted to the numerical solution of VDIDEs. An overview of the approximate methods for VDIDEs may be obtained from [4, 12, 14, 19, 31, 32, 38–40]. Effective methods for the numerical solutions of high-order Fredholm and Volterra-Fredholm-Hammerstein integro-differential equations were proposed by Turkyilmazoglu [35, 36].

The above mentioned papers, related to VDIDEs were only concerned with the regular cases, i.e., in the absence of initial/boundary layers. SPVDIDEs also frequently arise in many scientific applications. Wu and Gan [37] investigated error behaviour of linear multistep methods applied to SPVDIDEs and derived global error estimates $A(\alpha)-$stable linear multistep methods with convergent quadrature rule. He and Xu [13] discussed the exponential stability of impulsive SPVDIDEs. Amiraliyev and Yilmaz [3] gave an exponentially fitted difference method on a uniform mesh for (1.1)-(1.2) except for a delay term in differential part and shown that the method is first-order convergent uniformly in ε. A useful discussion of uniform convergence on a fitted mesh, for another form of SPVDIDEs have been investigated in [20].

In this present paper, we analyze the numerical solution of the initial-value problem (1.1)-(1.2). The numerical method presented here comprises a fitted difference scheme on a uniform mesh. Fitted operator method is widely used to construct and analyze uniform difference methods, especially for a linear differential problems. We have derived this approach on the basis of the method of integral identities with the use of interpolating quadrature rules with the weight and remainder terms in integral form. This results in a local truncation error containing only first derivatives of exact solution and hence facilitates examination of the convergence. In Section 2, we state some important properties of the exact solution. The derivation of the difference scheme has been given in Section 3. In Section 4, we present the error analysis for the approximate solution. The method is shown to be first order uniformly convergent with respect to the singular perturbation parameter. In Section 5, we give numerical example, which validate the theoretical analysis computationally. The approach to construct discrete problem and error analysis for approximate solution is similar to those one's from [1–3, 20].

Notation. Throughout the paper C (sometimes subscripted) will denote a generic positive constant independent of the mesh and perturbation parameter ε. Also, $\|.\|_\infty$

indicates a continuous maximum norm on the corresponding closed interval, in particular we shall use $\|g\|_\infty = \max_{t\in[0,T]} |g(t)|$, for any $g \in C[0,T]$.

2. Properties of the exact solution

In this section, we will establish a priori bounds on the solution of (1.1)-(1.2) and its first derivative. These bounds will be used in the error analysis in later sections.

Lemma 2.1. ([3]) *Assume that $a, f \in C[0,T]$ such that $a(t) \geq \alpha > 0$, $|F(t)| \leq \mathcal{F}(t)$ and $\mathcal{F}(t)$ is a nondecreasing continuous function. Then the solution of the initial value problem*

$$\varepsilon v'(t) + a(t)v(t) = F(t),\ t \in I, \tag{2.1}$$

$$v(0) = A \tag{2.2}$$

satisfies

$$|v(t)| \leq |A| + \alpha^{-1}\mathcal{F}(t),\ t \in I.$$

Lemma 2.2. *For $a, f \in C^1[0,T]$,*

$$\left|\frac{\partial}{\partial t}K(t,s)\right| \leq M_0 < \infty,$$

the solution of (1.1)-(1.2) satisfies

$$\|u\|_\infty \leq C_0 \tag{2.3}$$

$$|u'(t)| \leq C\left\{1 + \frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right\},\ 0 \leq t \leq T, \tag{2.4}$$

where

$$C_0 = \left(|\varphi(0)| + \alpha^{-1}\overline{K}\,\|\varphi\|_{1,0} + \alpha^{-1}\,\|f\|_\infty\right)e^{\alpha^{-1}\overline{K}T},$$

$$\overline{K} = \max_{\overline{I}\times\overline{I}} |K(t,s)|,$$

$$\|\varphi\|_{1,0} = \int_{-r}^{0} |\varphi(t)|\,dt.$$

Proof. Since

$$\left|f(t) - \int_{t-r}^{t} K(t,s)u(s)ds\right| \leq \|f\|_\infty + \overline{K}\int_{t-r}^{t} |u(s)|\,ds$$

$$\leq \|f\|_\infty + \overline{K}\begin{cases} \int_{t-r}^{0} |\varphi(s)|\,ds + \int_{0}^{t} |u(s)|\,ds, & \text{for } t < r \\ \int_{t-r}^{t} |u(s)|\,ds, & \text{for } t > r \end{cases}$$

then we can write

$$\left|f(t) - \int_{t-r}^{t} K(t,s)u(s)ds\right| \leq \overline{K}\,\|\varphi\|_{1,0} + \|f\|_\infty + \overline{K}\int_{0}^{t} |u(s)|\,ds, \quad t > 0.$$

Now, applying the Lemma 2.1 to (1.1)-(1.2) we get

$$|u(t)| \leq |\varphi(0)| + \alpha^{-1}\overline{K}\,\|\varphi\|_{1,0} + \alpha^{-1}\,\|f\|_\infty + \alpha^{-1}\overline{K}\int_{0}^{t} |u(s)|\,ds, \quad t \in I.$$

From here, by using the Gronwall's inequality it follows that

$$|u(t)| \leq \Big(|\varphi(0)| + \alpha^{-1}\overline{K}\, \|\varphi\|_{1,0} + \alpha^{-1}\, \|f\|_{\infty} \Big) e^{\alpha^{-1}\overline{K}t}$$

which leads to (2.3).

We now prove (2.4). Differentiating (1.1) gives

$$\varepsilon v' + a(t)v = \Phi(t), \ t > 0, \tag{2.5}$$

with

$$v(t) = u'(t)$$

$$\Phi(t) = f'(t) - a'(t)u(t) - \int\limits_{t-r}^{t} \frac{\partial}{\partial t}K(t,s)u(s)ds - K(t,t)u(t) + K(t,t-r)u(t-r).$$

Under the conditions of the Lemma 2.2 we have

$$|\Phi(t)| \leq \|f'\|_{\infty} + \|a'\|_{\infty}\|u\|_{\infty} + M_0\Big(\|\varphi\|_{1,0} + \|u\|_{\infty} \Big) + \overline{K}\Big(\|u\|_{\infty} + \|\varphi\|_{\infty} \Big), \ \text{for} \ 0 < t < r$$

$$|\Phi(t)| \leq \|f'\|_{\infty} + \|a'\|_{\infty}\|u\|_{\infty} + M_0\|u\|_{\infty} + 2\overline{K}\,\|u\|_{\infty}, \ \text{for} \ t > r$$

which by virtue of (2.3) implies that

$$|\Phi(t)| \leq C, \quad 0 \leq t \leq T. \tag{2.6}$$

Next, from (1.1) we have the estimate for $u'(0)$:

$$|u'(0)| \leq \frac{1}{\varepsilon}\left| f(0) - a(0)u(0) - \int\limits_{-r}^{0} K(0,s)u(s)ds \right|$$
$$\leq \frac{1}{\varepsilon}\Big(\|f\|_{\infty} + \|a\|_{\infty} C_0 + \overline{K}\,\|\varphi\|_{1,0} \Big),$$

thereby

$$|v(0)| \leq \frac{C}{\varepsilon}. \tag{2.7}$$

From (2.5) and (2.7) it follows that

$$v(t) = v(0)e^{-\frac{1}{\varepsilon}\int\limits_0^t a(\xi)d\xi} + \frac{1}{\varepsilon}\int\limits_0^t \Phi(\xi)e^{-\frac{1}{\varepsilon}\int\limits_\xi^t a(\eta)d\eta} d\xi.$$

Therefore

$$|v(t)| \leq |v(0)|\, e^{-\frac{\alpha t}{\varepsilon}} + \frac{1}{\varepsilon}\int\limits_0^t |\Phi(\xi)|\, e^{-\frac{\alpha(t-\xi)}{\varepsilon}} d\xi$$

and after using (2.6) and (2.7)

$$|v(t)| \leq \frac{C}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}} + C\alpha^{-1}\Big(1 - e^{-\frac{\alpha t}{\varepsilon}}\Big)$$

which immediately leads to (2.4). □

3. The difference scheme

We introduce the uniform mesh on the $[0,T]$:

$$\omega_{N_0}=\left\{t_i: i\tau,\ i=1,2,...,N_0;\ \tau=\frac{T}{N_0}=\frac{r}{N}\right\}$$

which contains N mesh points at each subinterval I_p $(1\leq p\leq m)$:

$$\omega_{N,p}=\{t_i:\ (p-1)N+1\leq i\leq pN\},\ \ 1\leq p\leq m,$$

and consequently

$$\omega_{N_0}=\bigcup_{p=1}^{m}\omega_{N,p}\,.$$

To simplify the notation, we set $g_i=g(t_i)$ for any function $g(t)$ and moreover y_i denotes an approximation of $u(t)$ at t_i, also $g_{i-\frac{1}{2}}=g(t_i-\frac{\tau}{2})$. For any mesh function g_i defined on ω_{N_0} we use the backward difference and norms:

$$g_{\bar{t},i}=\frac{g_i-g_{i-1}}{\tau},\qquad \|g\|_{\infty,\omega_{N,p}}=\max_{(p-1)N\leq i\leq pN}|g_i|\,.$$

For the difference approximation to (1.1), we integrate (1.1) over (t_{i-1},t_i) :

$$\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}Lu(t)\varphi_i(t)dt=\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}f(t)\varphi_i(t)dt \tag{3.1}$$

with the exponential basis functions

$$\varphi_i(t)=e^{-\frac{a_i}{\varepsilon}(t_i-t)},\ \ i=1,2,...,N_0,$$

where

$$\mu_i=\tau^{-1}\int_{t_{i-1}}^{t_i}\varphi_i(t)dt=\frac{1-e^{-a_i\rho}}{a_i\rho},\ \ \rho=\frac{\tau}{\varepsilon}.$$

We note that the function $\varphi_i(t)$ is the solution of the problem

$$\begin{aligned}&-\varepsilon\varphi'(t)+a_i\varphi(t)=0,\ \ t_{i-1}\leq t\leq t_i,\\ &\varphi(t_i)=1.\end{aligned} \tag{3.2}$$

After applying the method of exact difference schemes (see e.g., [2, 3] and [30, pp. 207-214] for comprehensive description concerning to the second order differential equations) we obtain

$$\begin{aligned}\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}[\varepsilon u'(t)+a(t)u(t)]\varphi_i(t)dt&=\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}[\varepsilon u'(t)+a(t_i)u(t)]\varphi_i(t)dt\\ &\quad+\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}[a(t)-a(t_i)]u(t)\varphi_i(t)dt\\ &=\varepsilon\theta_i u_{\bar{t},i}+a_iu_i+R_i^{(1)}\end{aligned} \tag{3.3}$$

with

$$\theta_i=\frac{a_i\rho}{1-e^{-a_i\rho}}e^{-a_i\rho} \tag{3.4}$$

and the remainder term

$$R_i^{(1)}=\mu_i^{-1}\tau^{-1}\int_{t_{i-1}}^{t_i}[a(t)-a(t_i)]u(t)\varphi_i(t)dt. \tag{3.5}$$

Further

$$\mu_i^{-1}\tau^{-1}\int\limits_{t_{i-1}}^{t_i} f(t)\varphi_i(t)dt = f_i + R_i^{(2)} \tag{3.6}$$

with

$$R_i^{(2)} = \mu_i^{-1}\tau^{-1}\int\limits_{t_{i-1}}^{t_i} [f(t) - f(t_i)]\varphi_i(t)dt. \tag{3.7}$$

For the integral term from (3.1), after applying the appropriate quadrature rules, we have

$$\mu_i^{-1}\tau^{-1}\int\limits_{t_{i-1}}^{t_i} \varphi_i(t)\Bigg(\int\limits_{t-r}^{t} K(t,s)u(s)ds\Bigg)dt = \int\limits_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K(t_{i-\frac{1}{2}},s)u(s)ds + R_i^{(3)}$$

$$= \tau \sum_{j=i-N+1}^{i-1} K(t_{i-\frac{1}{2}},s_j)u_j + R_i^{(3)} + R_i^{(4)}, \tag{3.8}$$

where

$$R_i^{(3)} = \mu_i^{-1}\tau^{-1}\int\limits_{t_{i-1}}^{t_i} dt\varphi_i(t)\int\limits_{t_{i-1}}^{t_i} \frac{d}{dt}\Bigg(\int\limits_{\xi-r}^{\xi} K(\xi,s)u(s)ds\Bigg)\Big[T_0(t-\xi) - T_0(t_{i-\frac{1}{2}}-\xi)\Big]d\xi, \tag{3.9}$$

$$R_i^{(4)} = \sum_{j=i-N+1}^{i-1}\int\limits_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \Big(t_{j-\frac{1}{2}} - \xi - \tau T_0(t_j - \xi)\Big)\frac{d}{ds}\Big(K(t_{i-\frac{1}{2}},\xi)u(\xi)\Big)d\xi \tag{3.10}$$

and

$$T_0(\lambda) = 1,\ \lambda \geq 0; \quad T_0(\lambda) = 0,\ \lambda < 0.$$

Consequently we have the exact relation for $u(t_i)$

$$L_N u_i := \varepsilon\theta_i u_{\bar{t},i} + a_i u_i + \tau\sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j}u_j = f_i - R_i \tag{3.11}$$

with remainder term

$$R_i = R_i^{(1)} - R_i^{(2)} + R_i^{(3)} + R_i^{(4)} \tag{3.12}$$

where $R_i^{(k)}$ $(k = 1, 2, 3, 4)$ are defined by (3.5),(3.7),(3.9),(3.10), respectively.

Based on (3.11) we propose the following difference scheme for approximating (1.1)-(1.2)

$$L_N y_i := \varepsilon\theta_i y_{\bar{t},i} + a_i y_i + \tau\sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j}y_j = f_i,\ 1 \leq i \leq N_0, \tag{3.13}$$

$$y_i = \varphi_i,\ -N \leq i \leq 0, \tag{3.14}$$

where θ_i is defined by (3.4).

4. Stability bound and convergence

Lemma 4.1. *Consider the following difference problem*

$$l_N v_i := \varepsilon\theta_i v_{\bar{t},i} + a_i v_i = F_i,\ 1 \leq i \leq N_0, \tag{4.1}$$

$$v_0 = A. \tag{4.2}$$

Let $|F_i| \leq \mathfrak{F}_i$ and $\mathfrak{F}_i$ be a nondecreasing function. Then the solution of (4.1)-(4.2) satisfies

$$|v_i| \leq |A| + \alpha^{-1}\mathfrak{F}_i,\ 1 \leq i \leq N_0. \tag{4.3}$$

Proof. First, we note that for the difference operator $l_N v_i$ the maximum principle holds in the form: If for any mesh function $l_N v_i \geq 0,\ 1 \leq i \leq N$ and $v_0 \geq 0$, then $v_i \geq 0$, for $0 \leq i \leq N$.

Consider now the barrier functions

$$\psi_i^{\pm} = \pm v_i + |A| + \alpha^{-1}\mathcal{F}_i$$

and take into consideration that

$$\mathcal{F}_{\bar{t},i} = \frac{\mathcal{F}_i - \mathcal{F}_{i-1}}{\tau} \geq 0$$

since $\mathcal{F}_i$ is a nondecreasing function. Therefore,

$$\psi_0^{\pm} = \pm A + |A| + \alpha^{-1}\mathcal{F}_i \geq 0$$

and

$$l_N \psi_i^{\pm} := \pm F_i + a_i\,|A| + \varepsilon\theta_i\alpha^{-1}\mathcal{F}_{\bar{t},i} + a_i\alpha^{-1}\mathcal{F}_i \geq \pm F_i + \mathcal{F}_i \geq 0,$$

so that according to the maximum principle $\psi_i^{\pm} \geq 0$, which proves (4.3). □

Lemma 4.2. *For the solution of (3.13)-(3.14) the following inequality holds*

$$\|y\|_{\infty,\overline{\omega}_{N_0}} \leq \gamma_0\Big(\,|\varphi_0| + \alpha^{-1}\overline{K}\,\|\varphi\|_{1,0}\Big) + \gamma_1\,\|f\|_{\infty,\omega_{N_0}}\,, \tag{4.4}$$

where

$$\gamma_0 = e^{\alpha^{-1}\overline{K}T}, \quad \gamma_1 = \alpha^{-1}e^{\alpha^{-1}\overline{K}T}.$$

Proof. Since

$$\left|\tau\sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j}\,y_j\right| \leq \tau\overline{K}\sum_{j=i-N+1}^{i-1}|y_j| \leq \begin{cases} \tau\overline{K}\sum\limits_{j=i-N+1}^{0}|\varphi_j|\,, & i=1 \\ \tau\overline{K}\sum\limits_{j=i-N+1}^{0}|\varphi_j| + \tau\overline{K}\sum\limits_{j=1}^{i-1}|y_j|\,, & 1<i\leq N-1 \\ \tau\overline{K}\sum\limits_{j=i-N+1}^{i-1}|y_j|\,, & i>N-1 \end{cases}$$

then it is not hard to see that

$$\left|f_i - \tau\sum_{j=i-N+1}^{i-1}K_{i-\frac{1}{2},j}\,y_j\right| \leq \|f\|_{\infty,\overline{\omega}_{N_0}} + \overline{K}\,\|\varphi\|_{1,0} + \tau\overline{K}\sum_{j=1}^{i}|y_{j-1}|\,.$$

Next, after applying Lemma 4.1 to (3.13)-(3.14) we get

$$|y_i| \leq |\varphi_0| + \alpha^{-1}\Big(\,\|f\|_{\infty,\overline{\omega}_{N_0}} + \overline{K}\,\|\varphi\|_{1,0}\Big) + \alpha^{-1}\tau\overline{K}\sum_{j=1}^{i}|y_{j-1}|\,.$$

From here by using the difference analogue of Gronwall's inequality we arrive at

$$|y_i| \leq \bigg(\,|\varphi_0| + \alpha^{-1}\Big(\,\|f\|_{\infty,\overline{\omega}_{N_0}} + \overline{K}\,\|\varphi\|_{1,0}\Big)\bigg)e^{\alpha^{-1}\overline{K}t_i},\ \ 1\leq i\leq N_0$$

which implies the validity of (4.4).

According to (3.11),(3.13) for the error of the approximate solution $z_i = y_i - u_i$ we have

$$L_N z_i = R_i,\ \ 1 \leq i \leq N_0, \tag{4.5}$$

$$z_i = 0, \quad -N \leq i \leq 0. \tag{4.6}$$

□

Lemma 4.3. *The error of the approximate solution z_i satisfies*

$$\|z\|_{\infty,\overline{\omega}_{N_0}} \leq \gamma_1 \|R\|_{\infty,\omega_{N_0}} \tag{4.7}$$

Proof. It evidently follows from (4.4) by taking $\varphi \equiv 0$ and $f \equiv R$. □

Lemma 4.4. *Let $a,f \in C^1[0,T]$, $\varphi \in C[-r,0]$ and $K \in C^1\Big([0,T]^2\Big)$. Then for the truncation error R_i, the following estimate holds*

$$\|R\|_{\infty,\omega_{N_0}} \leq C\tau. \tag{4.8}$$

Proof. We begin with the inequality

$$|R_i| \leq \sum_{k=1}^{4} \left|R_i^{(k)}\right| \tag{4.9}$$

and estimate $R_i^{(k)}$ separately.

For $a(t)$, by the mean value theorem

$$|a(t) - a(t_i)| \leq |a'(\xi_i)|\,|t - t_i| \leq C\tau \ \text{ on } \ t_{i-1} \leq t \leq t_i.$$

Thereby for $R_i^{(1)}$, using also $|u| \leq C$ from Lemma 2.2, we have

$$\left|R_i^{(1)}\right| \leq \mu_i^{-1}\tau^{-1} \left| \int\limits_{t_{i-1}}^{t_i} [a(t) - a(t_i)]u(t)\varphi_i(t)dt \right| \leq C\tau\mu_i^{-1}\tau^{-1} \int\limits_{t_{i-1}}^{t_i} \varphi_i(t)dt = C\tau.$$

For $R_i^{(2)}$ similarly as above we get

$$\left|R_i^{(2)}\right| \leq C\tau.$$

For $R_i^{(3)}$ using also (2.3), we have

$$\left|R_i^{(3)}\right| \leq 2\mu_i^{-1}\tau^{-1} \int\limits_{t_{i-1}}^{t_i} ds\varphi_i(s) \int\limits_{t_{i-1}}^{t_i} \left\{ \int\limits_{\xi-r}^{\xi} \left|\frac{\partial}{\partial t}K(\xi,s)\right| |u(s)|\,ds + |K(\xi,\xi)| + |K(\xi,\xi-r)| \right\} d\xi$$
$$\leq 2r\Big(\overline{K}^{(1)} \|u\|_{\infty,\overline{I}} + 2\overline{K}\Big)\tau,$$

with

$$\left|\frac{\partial}{\partial t}K(t,s)\right| \leq K^{(1)}.$$

Finally for $R_i^{(4)}$ by virtue of (2.4), we obtain

$$\left|R_i^{(4)}\right| \leq 2\tau \sum_{j=i-N+1}^{i-1} \int\limits_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \Big(\overline{K}^{(2)} \|u\|_{\infty,\overline{I}} + \overline{K}\,|u'(\xi)|\Big)d\xi$$

$$\leq 2\overline{K}^{(2)} \|u\|_{\infty,\overline{I}}\, \tau^2(N-1) + 2\tau\overline{K} \sum_{j=i-N+1}^{i-1} \int\limits_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} |u'(\xi)|\,d\xi$$

$$\leq 2r\overline{K}^{(2)} \|u\|_{\infty,\overline{I}}\, \tau + 2\tau\overline{K} \int\limits_{t_{i-N+\frac{1}{2}}}^{t_{i-\frac{1}{2}}} |u'(\xi)|\,d\xi, \tag{4.10}$$

with

$$\left|\frac{\partial}{\partial s}K(t,s)\right| \le K^{(2)}.$$

Since

$$\int\limits_{t_{i-N+\frac{1}{2}}}^{t_{i-\frac{1}{2}}} |u'(\xi)|\,d\xi \le \int\limits_{t_{i-N+\frac{1}{2}}}^{0} |\varphi'(\xi)|\,d\xi + C\int\limits_{0}^{t_{i-\frac{1}{2}}} \left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha\xi}{\varepsilon}}\right)d\xi$$
$$\le \|\varphi'\|_{1,0} + C\left(r+\alpha^{-1}\left(1-e^{-\frac{\alpha t_{i-\frac{1}{2}}}{\varepsilon}}\right)\right)$$
$$\le \|\varphi'\|_{1,0} + C\left(r+\alpha^{-1}\right),\ \text{ for } 1\le i\le N \tag{4.11}$$

and

$$\int\limits_{t_{i-N+\frac{1}{2}}}^{t_{i-\frac{1}{2}}} |u'(\xi)|\,d\xi \le C\int\limits_{t_{i-N+\frac{1}{2}}}^{t_{i-\frac{1}{2}}} \left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha\xi}{\varepsilon}}\right)d\xi$$
$$\le C\left(t_{i-\frac{1}{2}} - t_{i-N+\frac{1}{2}} + \alpha^{-1}\left(e^{-\frac{\alpha t_{i-N+\frac{1}{2}}}{\varepsilon}} - e^{-\frac{\alpha t_{i-\frac{1}{2}}}{\varepsilon}}\right)\right)$$
$$\le C\left(r+\alpha^{-1}\right),\ \text{ for } i\ge N+1 \tag{4.12}$$

the inequality (4.10) along with (4.11),(4.12) implies that

$$\left|R_i^{(4)}\right| \le C\tau.$$

Consequently, from (4.9) the proof follows. □

Now, we can formulate the main result of the paper.

Theorem 4.5. *For $a,f \in C^1[0,T]$, $\varphi \in C^1[-r,0]$ and $K \in C^1\left([0,T]^2\right)$, the solution of the difference problem (3.13)-(3.14) uniformly first order convergent to the solution of (1.1)-(1.2):*

$$\|y-u\|_{\infty,\overline{\omega}_{N_0}} \le C\tau.$$

Proof. It evidently follows from (4.7) by taking into consideration (4.8). □

5. Numerical example

Example 5.1. Consider the test problem

$$\varepsilon u' + 2u - \int\limits_{t-1}^{t} u(s)ds = -1 + t - \frac{\varepsilon}{2}\left(1-e^{-\frac{2t}{\varepsilon}}\right),\ \ t\in(0,2],$$
$$u(t)=1,\ \ -1\le t\le 0.$$

The exact solution is given by

$$u(t) = \begin{cases} e^{-\frac{2t}{\varepsilon}}, & 0\le t\le 1\\ \frac{e^{-\lambda_1(t-1)}-e^{-\lambda_2(t-1)}}{\sqrt{1+\varepsilon}} - 1 + e^{-\frac{2t}{\varepsilon}} + e^{-\frac{2(t-1)}{\varepsilon}}, & 1\le t\le 2\end{cases}$$

where

$$\lambda_1 = \frac{1-\sqrt{1+\varepsilon}}{\varepsilon},\quad \lambda_2 = \frac{1+\sqrt{1+\varepsilon}}{\varepsilon}.$$

We define the exact error e_ε^N and the computed $\varepsilon-$uniform maximum pointwise error e^N as follows

$$e_\varepsilon^N = \|y-u\|_{\infty,\overline{\omega}_{N_0}},$$
$$e^N = \max_\varepsilon e_\varepsilon^N,$$

where y is the numerical approximation to u for various of N and ε. Parameter-uniform rates of convergence are computed by

$$r^N = \ln\left(e^N/e^{2N}\right)/\ln 2.$$

The values of ε and N for which we solve the test problem are $\varepsilon = 2^{-i},\ i = 0, 6, 12, 18, 24;\ N = 64, 128, 256, 512, 1024$. From Table 1 we observe that the $\varepsilon-$uniform experimental rate of convergence is monotonically increasing towards one, so in agreement with the theoretical rate given by Theorem 4.5.

Table 1 Errors e_ε^N, e^N and rates of convergence r_ε^N, r^N for Example 5.1.

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^0	0.008613	0.004680	0.002473	0.001271	0.000644
	0.88	0.92	0.96	0.98	
2^{-6}	0.008602	0.004739	0.002522	0.001305	0.000657
	0.86	0.91	0.95	0.99	
2^{-12}	0.008606	0.004741	0.002523	0.001305	0.000657
	0.86	0.91	0.95	0.99	
2^{-18}	0.008609	0.004743	0.002524	0.001306	0.000657
	0.86	0.91	0.95	0.99	
2^{-24}	0.008609	0.004743	0.002524	0.001306	0.000657
	0.86	0.91	0.95	0.99	
e^N	0.008613	0.004743	0.002524	0.001306	0.000657
r^N	0.86	0.91	0.95	0.99	

Example 5.2. Consider the initial-value problem

$$\varepsilon u' + u + \int_{1-t}^{t} su(s)ds = 5t^2 - 2,\ \ 0 \le t \le 2,$$
$$u(t) = 5 + t,\ \ -1 \le t \le 0,$$

whose exact solution is not known.

We estimate errors in the computed solution and rates of convergence using the double mesh principle. For this purpose, we calculate another approximate solution y^{2N} on a mesh that is obtained by uniformly bisecting the original mesh $\overline{\omega}_{N_0}$. We estimate the errors for different values of ε and N by

$$e_\varepsilon^N = \left\|y^N - y^{2N}\right\|_{\infty,\overline{\omega}_{N_0}}.$$

$\varepsilon-$uniform errors and $\varepsilon-$uniform rates of convergence are computed in the same way as in Example 5.1.

The values of ε and N for which we solve the test problem are $\varepsilon = 2^{-i},\ i = 0, 6, 12, 18, 24;\ N = 64, 128, 256, 512, 1024$. From Table 2 we see that the values of r^N close to one for large values of N.

Table 2 Errors e_ε^N, e^N and rates of convergence r_ε^N, r^N for Example 5.2.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^0	0.035015	0.019972	0.010703	0.005617	0.002867
	0.81	0.90	0.93	0.97	
2^{-6}	0.044512	0.025213	0.013605	0.007091	0.003594
	0.82	0.89	0.94	0.98	
2^{-12}	0.060601	0.034326	0.018522	0.009721	0.004928
	0.82	0.89	0.93	0.98	
2^{-18}	0.066050	0.037155	0.019910	0.010449	0.005297
	0.83	0.90	0.93	0.98	
2^{-24}	0.067411	0.038450	0.020462	0.010739	0.005444
	0.81	0.91	0.93	0.98	
e^N	0.067411	0.038450	0.020462	0.010739	0.005444
r^N	0.81	0.91	0.93	0.98	

6. Conclusion

In this paper, we have proposed the fitted finite-difference method for the linear first order SPVDIDE exhibiting initial layer. We have shown that the method is first order uniformly convergent with respect to perturbation parameter. The numerical results also show that the presented method is first order uniformly accurate and hence it can be strongly recommended for SPVDIDEs. The main lines for the analysis of the uniform convergence carried out here can be used for study of more complicated linear SPVDIDEs as well as quasilinear SPVDIDEs.

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