Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:2** (2011), 53–58 DOI: 10.2298/FIL1102053G

A FIXED POINT RESULT FOR MAPPINGS WITH CONTRACTIVE ITERATE AT A POINT IN G-METRIC SPACES

Ljiljana Gajić and Zagorka Lozanov-Crvenković

Abstract

Using the setting of generalized metric space, so called G-metric space, a fixed point theorems for mappings with a contractive iterate at a point are proved. This result generalize well known comparable result.

1 Introduction

In [10] V.M. Sehgal generalized a well-know Banach theorem. He proved fixed point theorem for mappings with a contractive iterate at a point.

Theorem 1.1. *Let (X, d) be a complete metric space $A : X \to X$ be a continuous mapping with property that for every $x \in X$ there exists $n(x) \in \mathbb{N}$ so that for every $y \in X$*

$$d(A^{n(x)}x, A^{n(x)}y) \leq q \cdot d(x, y), \quad \text{where } q \in [0, 1).$$

Then A has a unique fixed point in X and $\lim_k A^k(x_0) = z$, for each $x_0 \in X$.

L.F. Guseman [4] proved that in this theorem the condition that A be continuous can be dropped. Our aim in this study is to show that Guseman results are valid in more general class of spaces.

On the other hand, there have been a number of generalizations of metric spaces.

In 1963. S.Gähler introduced the notion of 2-metric spaces but different authors proved that there is no relation between these two function and there is no easy relationship between results obtained in the two settings. Because of that, B.C

2010 *Mathematics Subject Classifications.* 47H10

Key words and Phrases. Fixed point, common fixed point, G-metric space, contractive iterate at a point.

Received: September 24, 2010.

Communicated by Dejan Ilić

This work was supported by grants approved by the Ministry of Science and Technological Development, Republic of Serbia. For the first author by the grant number 144016, and for the second author by the grant number 144025.

Dhage [3] introduced a new concept of the measure of nearness between three or more object. But topological structure of so called D-metric spaces was incorrect. Finally, Z. Mustafa and B. Sims [6] introduced correct definition of generalized metric space in that sense.

Definition 1.1. *[6] Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$*;*

(G2) $0 < G(x, x, y)$*; for all* $x, y \in X$*, with* $x \neq y$*;*

(G3) $G(x, x, y) \leq G(x, y, z)$*, for all* $x, y, z \in X$*, with* $z \neq y$*;*

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$*, (symmetry in all three variables);*

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$*, for all* $x, y, z, a \in X$*.*

Then the function G is called a generalized metric*, or, more specifically, a G*-metric *on X, and the pair (X, G) is called a G*-metric space.

Definition 1.2. *[6] Let (X, G) be a G-metric space, and let $\{x_n\}$ be sequence of points of X, a point $x \in X$ is said to be the* limit *of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence $\{x_n\}$ is G*-convergent *to x*

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

Definition 1.3. *[6] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.1. *[6] Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.4. *[6] A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Proposition 1.2. *[5] Every G-metric space (X, G) will define a metric space (X, d_G) by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X.$$

Note that if (X, G) is a symmetric G-metric space, then

$$d_G(x, y) = 2G(x, y, y), \text{ for all } x, y \in X.$$

Proposition 1.3. *[5] A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

2 The main theorem

Lemma 2.1. *Let (X, G) be a G-metric space and $f : X \to X$ a mapping. Let $B \subseteq X$ with $f(B) \subseteq B$ be a subset of X. If there exist a $u \in B$ and some positive integer $n(u)$ such that $f^{n(u)}(u) = u$ and*

$$G\big(f^{n(u)}(z), f^{n(u)}(u), f^{n(u)}(u)\big) \leq q \cdot G(z, u, u), \tag{1}$$

for some $q \in [0, 1)$ and all $z \in B$ then u is the unique fixed point of f in B and $\lim\limits_{k} f^k(y_0) = u$ for each $y_0 \in B$.

Proof. By (1), u is the unique fixed point for the mapping $f^{n(u)}$ in B. Then $f(u) = f\big(f^{n(u)}(u)\big) = f^{n(u)}\big(f(u)\big)$ implies that $f(u) = u$.

Let $y_0 \in B$. For m sufficiently large write $m = k \cdot n(u) + r,\ k \geq 1$ and $1 \leq r < n(u)$. Then

$$\begin{aligned} G(f^m(y_0), u, u) &= G\big(f^{k \cdot n(u)+r}(y_0), f^{n(u)}(u), f^{n(u)}(u)\big) \\ &\leq q \cdot G\big(f^{(k-1)n(u)+r}(y_0), u, u\big) \leq \ldots \\ &\leq q^k \cdot G\big(f^r(y_0), u, u\big) \leq q^k \cdot M \end{aligned}$$

for $M = \max\big\{G\big(f^p(y_0), u, u\big) |\ 1 \leq p \leq n(u) - 1\big\}$, implies that $\lim\limits_{k} f^k(y_0) = u$.

For $f : X \to X$ the set $\mathcal{O}(f; x_0) = \big\{f^n(x_0) :\ n \in \mathbb{N}\big\}$ is called the orbit for $x_0 \in X$.

Theorem 2.1. *Let (X, G) be a complete G-metric space and let $f : X \to X$ be a mapping. Suppose that exists $B \subseteq X$ such that:*

a) $f(B) \subseteq B$;

b) For some $x_0 \in B$ limit of any sequence in $\mathcal{O}(f; x_0)$, if exists, belongs to B;

c) For some $q \in [0, 1)$ and each $x \in \overline{\mathcal{O}(f; x_0)}$ there is an integer $n(x) \geq 1$ such that

$$G\big(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\big) \leq q \cdot G(z, x, x) \tag{2}$$

for all $z \in B$.

Then there exists a unique $u \in B$ such that $f(u) = u$ and $f^k(x_0) \to u,\ k \to \infty$. Moreover, $f^k(y_0) \to u,\ k \to \infty$, for all $y_0 \in B$. Furthermore, if

$$G\big(f^{n(u)}(z), f^{n(u)}(u), f^{n(u)}(u)\big) \leq q \cdot G(z, u, u)$$

for each $z \in X$, then u is the unique fixed point in X.

Proof. At first let us show that

$$\sup_m G\big(f^m(x_0), x_0, x_0\big) = M < +\infty.$$

For any $m \in \mathbb{N}$ there exist $k, r \in \mathbb{N},\ 1 \le r \le n(x_0)-1$ such that $m = k \cdot n(x_0)+r$. Then

$$\begin{aligned} & G\big(f^m(x_0), x_0, x_0\big) \le G\big(f^{kn(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\big) + G\big(f^{n(x_0)}(x_0), x_0, x_0\big) \\ \le\ & qG\big(f^{(k-1)n(x_0)+r}(x_0), x_0, x_0\big) + G\big(f^{n(x_0)}(x_0), x_0, x_0\big) \\ \le\ & qG\big(f^{(k-1)n(x_0)+r}(x_0), f^{n(x_0)}(x_0), f^{n(x_0)}(x_0)\big) + (1+q)G\big(f^{n(x_0)}(x_0), x_0, x_0\big) \\ \le\ & q^2G\big(f^{(k-2)n(x_0)+r}(x_0), x_0, x_0\big) + (1+q)G\big(f^{n(x_0)}(x_0), x_0, x_0\big) \le ... \\ \le\ & q^kG\big(f^r(x_0), x_0, x_0\big) + (1+q+...+q^{k-1})G\big(f^{n(x_0)}(x_0), x_0, x_0\big) \\ \le\ & \frac{1}{1-q}C = M < +\infty, \end{aligned}$$

for $C = \max\{G\big(f^p(x_0), x_0, x_0\big) :\ 1 \le p \le n(x_0)\}$.

Now, let us define the sequence $x_k = f^{n(x_{k-1})}(x_{k-1}),\ k \in \mathbb{N}$. Then for each $p \in \mathbb{N}$

$$\begin{aligned} & G(x_k, x_k, x_{k+p}) = G\big(f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k+p-1})}(x_{k+p-1})\big) \\ =\ & G\big(f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k+p-1})}f^{n(x_{k+p-2})}...f^{n(x_{k-1})}(x_{k-1})\big) \\ \le\ & qG\big(x_{k-1}, x_{k-1}, f^{n(x_{k+p-1})}...f^{n(x_k)}(x_{k-1})\big) \le ... \\ \le\ & q^kG\big(x_0, x_0, f^{n(x_{k+p-1})}...f^{n(x_k)}(x_0)\big) \le q^kM, \end{aligned}$$

so $\{x_k\}$ is Cauchy sequence and there exists $u = \lim_k x_k$, for some $u \in B$.

Further, for all $k \in \mathbb{N}$,

$$G\big(f^{n(u)}(u), f^{n(u)}(u), f^{n(u)}(x_k)\big) \le qG(u, u, x_k)$$

so $\lim_k f^{n(u)}(x_k) = f^{n(u)}(u)$.
On the other side

$$\begin{aligned} & G\big(f^{n(u)}(x_k), x_k, x_k\big) = G\big(f^{n(u)}f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1}), f^{n(x_{k-1})}(x_{k-1})\big) \\ \le\ & qG\big(f^{n(u)}(x_{k-1}), x_{k-1}, x_{k-1}\big) \le ... \le q^kG\big(f^{n(u)}(x_0), x_0, x_0\big) \end{aligned}$$

which implies that $\lim_k G\big(f^{n(u)}(x_k), x_k, x_k\big) = 0$.

Since G is continuous it means that

$$G\big(f^{n(u)}(u), u, u\big) = 0.$$

Hence $f^{n(u)}(u) = u$. By Lemma 2.1, u is the unique fixed point of f in B and $\lim_k f^k(y_0) = u$ for each $y_0 \in B$. With additional condition on X uniqueness in X is obvious. This complete the proof.

For $B = X$ we have the next corollary.

Corollary 2.1. *Let (X, G) be G-metric space and $f : X \to X$ be a mapping. Suppose that exist a $q \in [0, 1)$ and a point $x_0 \in X$ with $\overline{\mathcal{O}(f; x_0)}$ complete such that for each $x \in \overline{\mathcal{O}(f; x_0)}$ there exists an integer $n(x) \geq 1$ and*

$$G\big(f^{n(x)}(z), f^{n(x)}(x), f^{n(x)}(x)\big) \leq q \cdot G(z, x, x),$$

for all $z \in X$.

Then there exists a unique $u \in X$ such that $f(u) = u$ and $f^k(y_0) \to u$, $k \to \infty$, for any $y_0 \in X$.

For $(x) = 1$, independently on x, we have the next corollary.

Corollary 2.2. *Let (X, G) be a G-metric space and $f : X \to X$ be a mapping. Suppose that there exist $q \in [0, 1)$ and a point $x_0 \in X$, with $\overline{\mathcal{O}(f; x_0)}$ complete, such that for each $x \in \overline{\mathcal{O}(f; x_0)}$*

$$G(f(z), f(z), f(z)) \leq qG(z, x, x) \tag{3}$$

for all $z \in X$. Then there exists a unique $u \in X$ such that $f(u) = u$ and $f^k(y_0) \to u$, $k \to \infty$, for any $y_0 \in X$.

Remark 1. *If G is symmetric then $d_G(x, z) = 2G(z, x, x)$, and thus inequation (3) becomes*

$$d_G(f(x), f(z)) \leq qd_G(x, z).$$

Remark 2. *Let us note in Theorem 2.1 and so in Corollary 2.2 mapping f is not necessarily continuous.*

Remark 3. *For $B = \mathcal{O}(f; x_0)$ one can see that Theorem 2.1 in fact generalized Theorem 2.1, from [1].*

Now, using the same methods as in proof of Theorem 2.1 one can prove the next statement.

Theorem 2.2. *Let (X, G) be a complete G-metric space $f : X \to X$ a mapping and $u, x_0 \in X$ with $\lim_k f^k(x_0) = u$. If f satisfies (1) for some $q \in [0, 1)$ and all $z \in X$ then u is a unique fixed point for f in X. Moreover, $\lim_k f^k(y_0) = u$ for any $y_0 \in X$.*

As an easy consequence of Corollary 2.2 we have the next common fixed point result.

Theorem 2.3. *Let $\{f_k\}$ be a sequence of selfmappings on a complete G-metric space (X, G). If for some $q \in [0, 1)$ and each $x \in X$ there exists integer $n(x) \geq 1$ such that for all $z \in X$*

$$G\big(f_i^{n(x)}(z), f_j^{n(x)}(x), f_k^{n(x)}(x)\big) \leq q \cdot G(z, x, x)$$

whenever at least two of i, j, k are equal, then there exists a unique common fixed point for sequence $\{f_k\}$.

References

[1] M. Abbas, B.E Rhoades, *Common fixed point results for noncommuting mappings without continuity in generalized metric space*, Applied Mathematics and Computation, 215(2009) 262–269.

[2] R. Chugh, T. Kadian, A. Rani, B.E. Rhoades, *Property P in G-metric spaces*, Fixed point Theory and Applications, FPTA, (2010), in press.

[3] B.C. Dhage, *Generalized Metric Spaces and Mappings with Fixed Point*, Bull. Calcutta Math. Soc. 84 (1992), no 4, 329-336.

[4] L.F. Guseman, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc., 26(1970), 615-618.

[5] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some common fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theor Appl. (2008) (Article ID 189870, 12

[6] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. 7(2)(2006), pp.289-297.

[7] Z. Mustafa, B. Sims, *Fixed point theorems for contractive mappinngs in complete G-metric spaces*, Fixed Point Theory and Aplications, Vol. 2009Article ID 917175, 10 pages doi: 10.1155/2009/917175 pages).

[8] Z. Mustafa, W. Shatanawi, M. Bataineh, *Existnce of Fixed Point Results in G-metric Spaces*, Inter.J.MathScienes, Vol. 2009(2009) Article ID 283028, 10 pages, doi:10.1155/2009/283028

[9] R. Sadati, S.M. Vaezpour, P. Vetro, B.E. Rhoades, *Fixed point theorems in generalized partialy ordered G-metric spaces*, Mathematical and Computer Modeling, (2010), doi:10.1016/j.mcm.2010.05.009

[10] V.M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc., 23(1969), 631-634.

Author 1: Ljiljana Gajić

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

E-mail: `gajic@dmi.uns.ac.rs`

Author 2: Zagorka Lozanov-Crvenković

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

E-mail: `zlc@dmi.uns.ac.rs`