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A Fixed Point Theorem for Generalized Weak Contractions

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Abstract. A fixed point theorem for a generalized weak contractive map in a metric space is proven by generalizing some recent results of Đorić [4, Theorem 2.2], Zhang and Song [30, Corollary 2.2] and others. The result is illustrated by examples.

1. Introduction

For the sake of brevity, we follow the following notations, wherein T is a map to be defined specifically in a particular context, while x and y are elements of some specific domain:

$$Mg(Tx, Ty) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\};$$
$$M(Tx, Ty) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

The classical Banach contraction theorem has numerous extensions and generalizations (see, for instance, [1]-[30]). The following important generalization is due to Ćirić [2].

Theorem 1.1. *Let X be a complete metric space and $T : X \to X$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(Tx, Ty) \le rM_g(Tx, Ty). \tag{1.1}$$

Then T has a unique fixed point.

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A map T satisfying (1.1) is called a generalized contraction. The following is the quasi-contraction theorem, given by Ćirić [3], and is considered the most general contraction theorem in metric fixed point theory (cf. [14], [18]-[20]).

Theorem 1.2. *Let X be a complete metric space and $T : X \to X$. Assume there exists $r \in [0, 1)$ such that for every $x, y \in X$,*

$$d(Tx, Ty) \leq rM(Tx, Ty). \tag{1.2}$$

Then T has a unique fixed point.

Notice that (1.1) implies (1.2), that is, T satisfying the condition (1.1) also satisfies (1.2). We remark that (1.1) is the condition (21′) and (1.2) is the condition (24) in a comprehensive comparison of contractive conditions listed by Rhoades [19] (see also [14]).
Recently, Suzuki [28, Theorem 2] obtained a powerful generalization of the Banach contraction theorem, and the same has been extended in various ways (see, for instance, [6], [10]-[13], [17], [25], [29]). Using the idea of the Suzuki contraction [28] (see also [27]) and the generalized contraction (1.1), Đorić and Lazović [6, Corollary 2.3] obtained the following generalization of Theorem 1.1 in the following manner.

Theorem 1.3. *Define a nonincreasing function θ from $[0, 1)$ onto $(0, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases} \tag{1.3}$$

Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rMg(Tx, Ty).$$

Then T has a unique fixed point.

The Banach contraction theorem and its several extensions have been generalized using recently developed notion of weakly contractive maps. The following basic result is due to Rhoades [18].

Theorem 1.4. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1.4}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Dutta and Choudhary [7] obtained the following generalization of Theorem 1.4.

Theorem 1.5. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \tag{1.5}$$

where

(i) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous and monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$.*
(ii) $\varphi : [0, \infty) \to [0, \infty)$ *is lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.*

Then T has a unique fixed point.

Further, the contractive condition (1.5) has been found to be equivalent to some (ψ, φ)-contractive conditions studied by Jachymski (for details, one may refer to [8, Theorem 3]).
Theorems 1.1 and 1.5 have been generalized by Đorić [4, Theorem 2.2] in the following manner.

Theorem 1.6. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\psi(d(Tx, Ty)) \le \psi(Mg(Tx, Ty)) - \varphi(Mg(Tx, Ty)), \tag{1.6}$$

where ψ and φ are defined as in Theorem 1.5. Then T has a unique fixed point.

Now, the question is, whether it is possible to further generalized Theorem 1.6. Our main result provides an answer to this question. Also we present a weakly contractive version of Theorem 1.3 and generalize Theorems 1.1, 1.4, 1.5 and 1.6.

2. Main Results

The following is the main result of this paper.

Theorem 2.1. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } \psi(d(Tx, Ty)) \le \psi(Mg(Tx, Ty)) - \varphi(Mg(Tx, Ty)), \tag{2.1}$$

where ψ and φ are defined as in Theorem 1.5. Then T has a unique fixed point.

Proof. Pick $x_0 \in X$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$.
Notice that for any n,

$$\frac{1}{2}d(x_{n-1}, x_n) \le d(x_{n-1}, x_n). \tag{2.2}$$

Therefore by (2.1), we have

$$\psi(d(Tx_n, Tx_{n-1})) \le \psi(M_g(Tx_n, Tx_{n-1})) - \varphi(M_g(Tx_n, Tx_{n-1})).$$

By (2.1) and the definition of M_g, we have

$$\begin{aligned}
\psi(d(x_{n+1}, x_n)) &\le \psi(d(Tx_n, Tx_{n-1})) \\
&\le \psi(M_g(Tx_n, Tx_{n-1})) - \varphi(M_g(Tx_n, Tx_{n-1})) \\
&= \psi\left(\max\left\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2}\right\}\right) \\
&\quad - \varphi\left(\max\left\{d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)}{2}\right\}\right) \\
&= \psi\left(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2}\right\}\right) \\
&\quad - \varphi\left(\max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2}\right\}\right)
\end{aligned}$$

So

$$\psi(d(x_{n+1}, x_n)) \le \psi(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}) - \varphi(\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}).$$

This yields

$$\psi(d(x_{n+1}, x_n)) \le \psi(d(x_n, x_{n-1})) - \varphi(d(x_n, x_{n-1})) \le \psi(d(x_n, x_{n-1})).$$

Consequently,

$$\psi(d(x_{n+1}, x_n)) \le \psi(d(x_n, x_{n-1})),$$

and by the property of ψ,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}), \tag{2.3}$$

and this is true for any n.
Hence the sequence $\{d(x_{n+1}, x_n)\}$ is monotonic nonincreasing and bounded below. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = \lim_{n\to\infty} d(x_n, x_{n-1}), \tag{2.4}$$

Therefore, by the lower semi-continuity of φ,

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(d(x_n, x_{n-1})).$$

We claim that $r = 0$. In fact taking upper limits as $n \to \infty$ on each side of the following inequality:

$$\psi(d(x_{n+1}, x_n)) \leq \psi(d(x_n, x_{n-1})) - \varphi(d(x_n, x_{n-1})),$$

and using (2.4), this gives

$$\psi(r) \leq \psi(r) - \varphi(r).$$

Consequently $\varphi(r) \leq 0$. Hence by the property of the function φ, $\varphi(r) = 0$. But $\varphi(r) = 0$ implies $r = 0$. So, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = 0. \tag{2.5}$$

Next we show that $\{x_n\}$ is a Cauchy sequence. If not, there is an $\varepsilon > 0$ and there exist integers m_k and n_k with $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{and} \quad d(x_{m_k-1}, x_{n_k}) < \varepsilon\,.$$

Now (2.5) and the inequality

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \text{ implies that } \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon\,.$$

Also (2.5) and the inequality

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k})$$

gives that $\varepsilon \leq \lim_{k\to\infty} d(x_{m_k+1}, x_{n_k})$, while (2.5) and the inequality

$$d(x_{m_k+1}, x_{n_k}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k})$$

yields $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \varepsilon$. Hence

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \varepsilon.$$

By a similar way, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k+2}) = \varepsilon.$$

Now

$$\frac{1}{2} d(x_{m_k}, Tx_{m_k}) = \frac{1}{2} d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{m_k+1}) \leq d(x_{m_k}, x_{n_k+1})$$

[since $m_k \geq n_k$ so $m_k + 1 \geq n_k + 1$ so by using (2.3)]. Therefore by (2.1),

$$\psi(d(Tx_{m_k}, Tx_{n_k+1})) \leq \psi(M_g(Tx_{m_k}, Tx_{n_k+1})) - \varphi(M_g(Tx_{m_k}, Tx_{n_k+1})).$$

Now from the definition of M_g we have

$$M_g(Tx_{m_k}, Tx_{n_k+1}) = \max\left\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), \frac{d(x_{m_k}, Tx_{n_k+1}) + d(x_{n_k+1}, Tx_{m_k})}{2}\right\}$$

Then

$$\lim_{k\to\infty} M_g(Tx_{m_k}, Tx_{n_k+1}) = \max\left\{\varepsilon,\ 0,\ 0,\ \frac{\varepsilon + \varepsilon}{2}\right\} = \varepsilon.$$

Thus

$$\begin{aligned}\psi(d(x_{m_k+1}, x_{n_k+2})) &= \psi(d(Tx_{m_k}, Tx_{n_k+1}))\\ &\leq \psi\left(\max\left\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), \frac{d(x_{m_k}, Tx_{n_k+1}) + d(x_{n_k+1}, Tx_{m_k})}{2}\right\}\right)\\ &\quad - \varphi\left(\max\left\{d(x_{m_k}, x_{n_k+1}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k+1}, Tx_{n_k+1}), \frac{d(x_{m_k}, Tx_{n_k+1}) + d(x_{n_k+1}, Tx_{m_k})}{2}\right\}\right).\end{aligned}$$

By taking limits as $k \to \infty$, it follows that $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon)$, which is a contradiction with $\varepsilon > 0$, it follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it has a limit in X. Call it z.
Now we show that z is a fixed point of T. We claim that

$$\frac{1}{2}d(x_{2n}, Tx_{2n}) \leq d(x_{2n}, z) \quad \text{or} \quad \frac{1}{2}d(x_{2n+1}, Tx_{2n+1}) \leq d(x_{2n+1}, z).$$

Otherwise, we have:

$$\begin{aligned}d(x_{2n}, x_{2n+1}) &\leq d(x_{2n}, z) + d(z, x_{2n+1})\\ &< \frac{1}{2}d(x_{2n}, Tx_{2n}) + \frac{1}{2}d(x_{2n+1}, Tx_{2n+1})\\ &\leq \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]\\ &\leq \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n}, x_{2n+1})] \qquad (\text{as } d(x_n, x_{n+1}) \text{ is nonincreasing})\\ &= d(x_{2n}, x_{2n+1}).\end{aligned}$$

Therefore, $d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n+1})$ is a contradiction. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\frac{1}{2}d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k}, z)$. So, by (2.1), one gets:

$$\begin{aligned}\psi(d(Tx_{n_k}, Tz)) &\leq \psi(M_g(Tx_{n_k}, Tz)) - \varphi(M_g(Tx_{n_k}, Tz))\\ &= \psi\left(\max\left\{d(z,\ x_{n_k}), d(z, Tz), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2}\right\}\right)\\ &\quad - \varphi\left(\max\left\{d(z, x_{n_k}), d(z, Tz), d(x_{n_k}, Tx_{n_k}), \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2}\right\}\right).\end{aligned}$$

Making $n \to \infty$,

$$\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz)).$$

This yields $z = Tz$.

In order to prove the uniqueness of the fixed point z, suppose that y is another fixed point of T. Then

$$\frac{1}{2}d(z,Tz) = 0 \leq d(y,z)$$

implies

$$\begin{aligned}\psi(d(y,z)) = \psi(d(Ty,Tz)) &\leq \psi(M_g(Ty,Tz)) - \varphi(M_g(Ty,Tz)) \\ &= \psi(d(y,z)) - \varphi(d(y,z)).\end{aligned}$$

This gives $\varphi(d(y,z)) \leq 0$. Hence $y = z$. This completes the proof. □

The following results are derived from Theorem 2.1.

Corollary 2.1. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad \psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)),$$

where ψ and φ are defined as in Theorem 1.5. Then T has a unique fixed point.

Corollary 2.2. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)),$$

where φ is defined as in Theorem 1.5. Then T has a unique fixed point.

Corollary 2.3. *Let X be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,*

$$\psi(d(Tx,Ty)) \leq \psi(M_g(Tx,Ty)) - \varphi(M_g(Tx,Ty)),$$

where ψ and φ are defined as in Theorem 1.5. Then T has a unique fixed point.

We remark that Corollary 2.3 is a particular case of [24, Corollary 2.9]. Further, a result of Zhang and Song [30, Corollary 2.2] is obtained from Corollary 2.3 when $\psi(t) = t$.

The following example shows the generality of Theorem 2.1 over Theorems 1.5. Further, it is interesting to note that the map T of Example 2.1 does not satisfy the hypotheses of Theorem 1.2.

Example 2.1. *Let $X = \{(0,0),(0,4),(4,0),(0,5),(5,0),(4,5),(5,4)\}$ be endowed with the metric d defined by*

$$d[(x_1,x_2),(y_1,y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let T be such that

$$T(x_1,x_2) = \begin{cases}(x_1,0) & \textit{if } x_1 \leq x_2 \\ (0,x_2) & \textit{if } x_1 > x_2.\end{cases}$$

Then T does not satisfy the condition (1.2) of Theorem 1.2 at $x = (4,5)$, $y = (5,4)$. Choose $\psi(t) = t$ and $\varphi(t) = \frac{1}{7}t$, it is readily verified that the condition (1.5) of Theorem 1.5 is not satisfied at $x = (4,5)$, $y = (5,4)$. However, all the hypotheses of Theorem 2.1 are easily verified for the map T. This example can also be discussed under the conditions of Corollary 3.2 of [5].

The following Example shows the generality of Theorem 2.1 over Theorems 1.6. Further, it is interesting to note that the map T of Example 2.2 does not satisfy the hypotheses of Theorem 1.3.

Example 2.2. *Let $X = \{(1,1),(1,5),(5,1),(5,6),(6,5)\}$ be endowed with the metric d defined as in Example 2.1 and let the self-mapping T on X be defined as follows:*

$$T(x_1,x_2) = \begin{cases} (x_1,1) & \text{if } x_1 \le x_2 \\ (1,x_2) & \text{if } x_1 > x_2. \end{cases}$$

Choose $\psi(t) = \frac{3}{4}t$ and $\varphi(t) = \frac{1}{8}t$. It is readily verified that T does not satisfy the condition (1.6) of Theorem 1.6 at $(x,y) = ((5,6),(6,5))$ and $((6,5),(5,6))$.
Also T does not satisfy the condition (1.3) of Theorem 1.3 at $x = (5,6)$, $y = (6,5)$. However, all the hypotheses of Theorem 2.1 are easily verified for the map T.

The following Example 2.3 shows the generality of Corollary 2.1 and 2.2 over Theorems 1.4 and 1.5.

Example 2.3. *Let $X = \{(0,0),(0,4),(4,0),(4,5),(5,4)\}$ be endowed with the metric d defined as in Example 2.1 and let the self-mapping T on X be defined as follows:*

$$T(x_1,x_2) = \begin{cases} (x_1,0) & \text{if } x_1 \le x_2 \\ (0,x_2) & \text{if } x_1 > x_2. \end{cases}$$

Choose $\psi(t) = t$ and $\varphi(t) = \frac{1}{7}t$. Then it is seen that the conditions (1.4) and (1.5) of Theorem 1.4 and Theorem 1.5 are not satisfied at $x = (4,5)$, $y = (5,4)$. However, all the hypotheses of Corollaries 2.1 and 2.2 are easily verified for the map T. Examples 2.2 and 2.3 can also be discussed under the contractive conditions of Theorem 3 of [29].

The following example shows the generalization of the main result (Theorem 2.1) of this paper as compared to Theorem 3 of [29], although it fulfils the contractive constraints of Corollary 3.2 of [5]:

Example 2.4. *Let $X = \{(1,1),(1,4),(4,1)\}$ be endowed with the metric d defined as in Example 2.1 and let the self-mapping T on X be defined as follows:*

$$T(x_1,x_2) = \begin{cases} (1,1) & \text{if } x_1 \le x_2 \\ (1,4) & \text{if } x_1 > x_2. \end{cases}$$

Choose $\psi(t) = \frac{3}{4}t$ and $\varphi(t) = \frac{1}{8}t$. It is readily verified that T does not satisfy equation (1) of Theorem 3 of [29] at $(x,y) = ((1,1),(4,1))$, since the true constraint:

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d((1,1),T(1,1)) = \frac{1}{2}d((1,1),(1,1)) = 0 < d(x,y) = d((1,1),(4,1)) = 3$$

does not imply the constraint:

$$d(Tx,Ty) = d(T(1,1),T(4,1)) = d((1,1),(1,4)) = 3 < d(x,y) = d((1,1),(4,1)) = 3,$$

which is also a contradiction. However, all the hypotheses of Theorem 2.1 of this paper are easily verified for the map T for every x, y of X. The above example agrees with the contractive constraints of Corollary 3.2 of [5].

The stability of discrete dynamic systems can be discussed under wide general conditions by using Theorem 2.1 in an "ad-hoc" way for such a purpose as it is discussed in the subsequent example:

Example 2.5. *Consider $X = \mathbf{R}$ with (X,d) being a complete metric space endowed with the Euclidean metric, which is also the Euclidean norm $\| \quad \|$ so that $(X,d) \equiv (X, \| \quad \|)$ is also a Banach space, and let $T : \mathbf{R} \to \mathbf{R}$ be a self-mapping which generates the scalar sequence $x_{n+1} = Tx_n = a_n x_n$; $\forall\ n \in \mathbf{Z}_+$ with $x_0 \neq 0$ and $\{a_n\} \subset \mathbf{R}$ being bounded. In this case the Euclidean metric and norm are defined by the absolute value, that is, $d(x,y) = |x-y|$ for any pair x, y in $\mathbf{R}$. The real subsequence $\left\{x_{\sum_{k=0}^{n-1} p_k}\right\}$ of $\{x_n\}$ of the trajectory solution of the discrete dynamic system is now considered* *under Theorem 2.1 as follows. Replace* $x_{n-1} \to x_{\sum_{k=0}^{n-1} p_k}$, $Tx_{n-1} \to a_{p_{n-1}} x_{\sum_{k=0}^{n-1} p_k}$, $x_n \to x_{\sum_{k=0}^{n} p_k}$ *and* $Tx_n \to a_{p_n} x_{\sum_{k=0}^{n} p_k}$ *in Theorem 2.1 and then replace accordingly* $M_g(Tx_n, Tx_{n-1})$ *by*

$$M_g\left(Tx_{\sum_{k=0}^{n} p_k}, Tx_{\sum_{k=0}^{n-1} p_k}\right) = \max(b(p_{n-1}, p_n), b(p_{n-1}, p_n), f(p_{n-1}), g(p_{n-1}, p_n)) \left|x_{\sum_{k=0}^{n-1} p_k}\right| \tag{2.6}$$

where

$$b(p_{n-1}, p_n) = \left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right|; \quad c(p_{n-1}, p_n) = |1 - a_{p_n}| \left|\prod_{k=p_{n-1}}^{p_n-1} [a_k]\right|; \quad f(p_{n-1}) = |1 - a_{p_{n-1}}|;$$
$$g(p_{n-1}, p_n) = \frac{1}{2}\left(\left|\prod_{k=p_{n-1}}^{p_n-1} [a_k] - a_{p_n}\right| + \left|1 - \prod_{k=p_{n-1}}^{p_n} [a_k]\right|\right). \tag{2.7}$$

Note also that

$$d\left(x_{\sum_{k=0}^{n-1} p_k} x_{\sum_{k=0}^{n} p_k}\right) = \left|x_{\sum_{k=0}^{n-1} p_k} - x_{\sum_{k=0}^{n} p_k}\right| = \left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right| \left|x_{\sum_{k=0}^{n-1} p_k}\right| = b(p_{n-1}, p_n) \left|x_{\sum_{k=0}^{n-1} p_k}\right|$$

so that the contractive conditions and (2.1) *takes the following form:*

$$\begin{aligned} &|1 - a_{p_{n-1}}| \le 2\left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right| \\ \Rightarrow\quad &\varphi(\max(b(p_{n-1}, p_n), b(p_{n-1}, p_n), f(p_{n-1}), g(p_{n-1}, p_n))) \\ &\le \psi(\max(b(p_{n-1}, p_n), b(p_{n-1}, p_n), f(p_{n-1}), g(p_{n-1}, p_n))) - \psi\left(\left|Tx_{\sum_{k=0}^{n} p_k} - Tx_{\sum_{k=0}^{n-1} p_k}\right|\right) \end{aligned} \tag{2.8}$$

subject to $1 \le p_n - p_{n-1} \le p$ *for all* $n \in \mathbf{Z}_+$ *and some fixed* $p \in \mathbf{Z}_+$.

Theorem 2.1 leads to the following result concerning Example 2.5:

Theorem 2.2. *The class of scalar discrete dynamic systems* $x_{n+1} = a_n x_n$; $\forall\ n \in \mathbf{Z}_+$ *with* $x_0 \neq 0$ *is globally asymptotically stable to the origin* $x = 0$, *for any initial condition and any bounded parameterizing sequence* $\{[a_n]\} \subset \mathbf{R}$, *provided that:*

(i) *there is some strictly increasing sequence of nonnegative integers* $\{p_n\}$, *with the bounded incremental positive sequence* $\{p_n - p_{n-1}\}$ *being subject to* $1 \le p_n - p_{n-1} \le p$ *for all* $n \in \mathbf{Z}_+$ *and some fixed* $p \in \mathbf{Z}_+$ *such that* (2.8), *under the definitions* (2.7), *holds for any given functions* $\varphi, \psi : [0, \infty) \to [0, \infty)$ *fulfilling the conditions of Theorem 1.5.*

(ii) $\limsup_{n\to\infty} \left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right| \neq 0.$

The point $x^* = 0$ *is the unique fixed point of* $T : \mathbf{R} \to \mathbf{R}$ *and also the equilibrium point of all such a class of dynamic systems.*

Proof. It follows from Theorem 2.1 that the Cauchy sequence subsequence $\left\{x_{\sum_{k=0}^{n-1} p_k}\right\} \to x = Tx$ which is the unique limit so that it has to be zero from $\limsup_{n\to\infty} \left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right| \neq 0$ and $\left|x_{\sum_{k=0}^{p_n} p_k} - x_{\sum_{k=0}^{p_n-1} p_k}\right| = \left|1 - \prod_{k=p_{n-1}}^{p_n-1} [a_k]\right| \left|x_{\sum_{k=0}^{p_n-1} p_k}\right| \to 0$ as $n \to \infty$. Now, note that the points of the sequence $\{x_n\}$ also converge to zero since $\left\{x_{\sum_{k=0}^{n-1} p_k}\right\} \to 0$ implies, since $\{[a_n]\} \subset \mathbf{R}$ is bounded:

$$x_{\sum_{k=0}^{n-1} p_k + j} = T^{j-1}\left(x_{\sum_{k=0}^{n-1} p_k}\right) = \left(\prod_{k=p_n-1}^{p_{n-1}+j-1} [a_k]\right)\left(x_{\sum_{k=0}^{n-1} p_k}\right) \to 0$$

for all $1 \le j \le p_{n-1}$ and all $n \in \mathbf{Z}_{0^+}$. Thus $x^* = 0$ is the unique equilibrium point of the dynamic system and also the unique fixed point of the self-mapping $T : \mathbf{R} \to \mathbf{R}$ which generates the sequence trajectory solution from any initial condition. □

Remark 2.1. *Note that Theorem 2.2 is also valid if the parameterizing sequence is solution-dependent so that* $a_n = a_n(x_j : 0 \le j \le n)$.

Remark 2.2. *Note also that the extension of Theorem 2.2 is direct to the case when* $x_{n+1} = A_n x_n$ *with any given initial condition* x_0 *where* $\{A_n\}$ *is matrix function sequence of square matrices of n-th order* $A_n = A_n(x_j : 0 \le j \le n)$ *by simply taking matrix norms with replacements of the type* $|a| \to \|A\|$, $|1-a| \to \|I-A\|$ *where I is the identity matrix of n-th order.*

Remark 2.3. *Finally, note that Theorem 2.2 can be also reformulated with no difficulty to the previous formulations discussed in Section 1, in particular under Theorems 1.1 to 1.5, by giving the necessary changes in the corresponding contractive conditions.*

The problem of common fixed points and the related one of coupled fixed points are of wide interest nowadays. See, for instance, [4], [15], [23], [25]-[26] and some references therein. We now propose a further question and a new conjecture as a theorem as follows:

Question 2.1. *Can we extend Theorem 2.1 for a pair of maps? Indeed, we conjecture the following:*

Theorem 2.3. *Let X be a complete metric space and* $S, T : X \to X$ *such that for every* $x, y \in X$,

$$\frac{1}{2}\min\{d(x,Sx), d(y,Ty)\} \le d(x,y)$$

implies

$$\psi(d(Sx,Ty)) \le \psi(m(x,y)) - \varphi(m(x,y)),$$

where ψ *and* φ *are defined as in Theorem 1.5, and*

$$m(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}.$$

Then S and T have a unique common fixed point.

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