A FIXED-POINT THEOREM FOR INWARD AND OUTWARD MAPS

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The Schauder-Tychonoff theorem states that a continuous function from a compact convex subset of a locally convex topological vector space into itself must have a fixed point ([1, Chapter V, 10.5], or [2]). Using this theorem, we obtain here a stronger result, stating that a map from such a set into the surrounding vector space has a fixed point if the *directions* in which the points are moved satisfy a certain "inwardness" condition.

It follows immediately that a symmetrical "outwardness" condition also implies the existence of a fixed point. We find also that under the latter condition the image of the map necessarily includes the original set!

1. **Definitions.** Let X be a topological vector space, and K a compact convex subset of X.

We shall call a map $F: K \to X$ "inner" if $F(K) \subset K$.

Given $x \in K$, let us define the "inward set" of x with respect to K as the set of points of the form $(1-\alpha)x+\alpha y$, for $y \in K$, $\alpha \geqq 0$. It can be thought of as the union of all rays originating at x and drawn so as to pass through some other point y of K. For $z \in K$, $\neq x$, a necessary and sufficient condition for z to lie in the inward set of x is that the line *segment* connecting x and z meet K in some point other than x.

A map $F: K \to X$ will be called "inward" if for all $x \in K$, $F(x)$ belongs to the inward set of x. The class of inward maps clearly includes the class of inner maps.

Similarly, the "outward set" of x with respect to K will mean the set of points $(1-\alpha)x+\alpha y$ for $y \in K$ and (N.B.) $\alpha \leqq 0$, and F will be called outward if $F(x)$ always belongs to the outward set of x.

The "weakly inward" and "weakly outward" sets of a point x will be defined as the closures of the inward and outward sets of x, respectively. "Weakly inward" and "weakly outward" maps will mean maps taking every x to a member of the appropriate set.

We note that if F is a (weakly) inward map, then the map $x \mapsto 2x-F(x)$ is (weakly) outward, and conversely. Also, x is a fixed point of one map if and only if it is a fixed point of the other. Hence fixed-point results for (weakly) inward maps

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are equivalent to such results for (weakly) outward maps. We shall derive our results by considering maps of the former type.

2. **The strictly-convex-normed case.** Suppose that the X considered above is a strictly convex normed linear space. Then to every point $y \in X$ there corresponds a unique point $N_K(y) \in K$ whose distance to y is minimal. The function N_K so defined is a continuous retraction of X onto K.

Given $x \in K$, let us define the "normal-outward set of x" to be the set of points $y \neq x$ such that $N_K(y) = x$.

Given any $x \in K$, the weakly-inward set of x and the normal-outward set of x are disjoint. To show this, we must find, given y in the normal-outward set of x, a neighborhood of y containing no members of the *inward* set of x. We claim that the open ball of radius $\|x-y\|$ about y has this property. For, given z in this ball, all points of the segment joining z and x, other than x itself, belong to this ball, and hence are nearer to y than x is. Since x is the point of K nearest to y, no point of this segment can belong to K, hence z is not in the inward set of x.

Let us call a map $F: K \to X$ "nowhere normal-outward" if $F(x)$ belongs to the normal-outward set of x for no x. It is clear from the above that the class of maps so defined includes the weakly inward maps.

THEOREM 2.1. *Let X be a strictly convex normed linear space, K a compact convex subset of X, and F a nowhere normal-outward map from K into X. Then F has a fixed point.*

Proof. $N_K F$ is a continuous map of K into itself. Hence by the Schauder-Tychonoff theorem, there exists x in K such that $N_K F(x) = x$. Looking at the definition of the "normal-outward set of x", we see that $F(x)$ must belong to that set unless $F(x) = x$. Hence $F(x) = x$. ∎

3. **The case $X = \boldsymbol{R}^\infty$, and a Fibering Lemma.** Let $\boldsymbol{R}^\infty$ designate the space of all sequences $x = (x_1, x_2, \dots)$ of real numbers, with the product topology. Let $p_i: \boldsymbol{R}^\infty \to \boldsymbol{R}$ be the ith-coordinate map.

LEMMA 3.1. *Let K be a compact convex subset of $\boldsymbol{R}^\infty$, and F a weakly inward map from K into $\boldsymbol{R}^\infty$. Then F has a fixed point.*

Proof. Suppose F has no fixed point. Then the sets $U_i = \{x \in K \mid p_i F(x) \neq p_i(x)\}$ cover K, hence some finite number of them—say $U_1, \dots, U_n$—cover K. Thus the function $\sup_{i=1,\dots,n} |p_i F(x) - p_i(x)|$ is nowhere zero, hence is bounded away from zero. We can clearly assume it is everywhere $\geqq 1$.

By the assumption that F is weakly inward, we can, for every $x \in K$, find $y \in K$ and $\alpha \geqq 0$, such that the first n coordinates of $u(x) = (1-\alpha)x + \alpha y$ differ from those of $F(x)$ by less than $1/2^n$. Now it is clear that the y and α chosen for a given x will work for all x' in a neighborhood of x in K. Hence by compactness of K, we can handle all points of K by choosing y and α from some finite set; u can thus be chosen as a function which, though not necessarily continuous, will take on all its values in a compact set. Hence for each $i>0$, we can find a real number B_i such that $\forall x \in K$, $|p_i(x)|<B_i$, $|p_iF(x)|<B_i$, and $|p_i(F(x)-u(x))|<B_i$. Multiplication of each coordinate by an independent constant is a linear homeomorphism on $\boldsymbol{R}^\infty$, hence preserves all the structure we are considering. Consequently, we may assume $B_i \leqq 2^{-i}$ for all $i>n$. (We have already put conditions on the first n coordinates.)

Now let H designate the space of all L_2 (square-summable) sequences of real numbers under the L_2 norm—a vector subspace of $\boldsymbol{R}^\infty$, but with a stronger topology. Let B be the set of all $x=(x_1, x_2, \ldots)$ such that $\forall i\,|x_i| \leqq B_i$. B lies in H, and it is easily shown that the $\boldsymbol{R}^\infty$ and H-topologies agree on B. Hence K is a compact convex subset of H and F is continuous in the topology of H.

We note that for all $x \in K$, $u(x)$ is at a distance less than 1 from $F(x)$, since $|p_i(u(x)-F(x))|<1/2^i$ (for i both $\leqq n$ and $>n$). On the other hand, $F(x)$ is at a distance at least 1 from x (see first paragraph). Hence $u(x)$, a point on a ray drawn from x, through some other point of K, is nearer to $F(x)$ than x is. Some point on the line segment between x and $u(x)$ will both be closer to $F(x)$ than x is and be in K. Hence x is not the point of K nearest $F(x)$. So F is a nowhere normal-outward map without a fixed point, contradicting Theorem 2.1. ∎

We shall obtain our most general form of the fixed-point theorem from the above by the Fibering Lemma and the corollary below. (This is a strengthened form of the argument used in the Dunford-Schwartz lemma [1, Chapter V, 10.4]—the analogous step in the proof of the Schauder-Tychonoff theorem.) Note that our lemma merely requires K to be Lindelöf (every open covering has a countable subcovering), though in the case we are interested in, it is compact.

LEMMA 3.2 (FIBERING LEMMA). *Let X be a topological vector space whose topology is induced by linear functionals, let K be a Lindelöf subset of X, and let F: $K \to X$ be a continuous map. Then, given any countable family G_0 of continuous linear functionals on X, there is a continuous linear map p: $X \to \boldsymbol{R}^\infty$, and a continuous map F': $p(K) \to \boldsymbol{R}^\infty$ such that:*

(1) *$F'p=pF$,*

(2) *For each $f \in G_0$ there exists an f': $\boldsymbol{R}^\infty \to \boldsymbol{R}$ such that $f=f'p$.*

Proof. We shall first show that any continuous real-valued function g on K is (in a sense to be made clear) "continuously determined" by a countable family of continuous linear functionals of X.

Given g we can, by the Lindelöf assumption, find for each $\varepsilon>0$ a countable covering of K by open sets $(U_\alpha)_{\alpha \in A_\varepsilon}$, such that for each α in the index-set A_ε, and $x, x' \in U_\alpha$, $|g(x)-g(x')|<\varepsilon$. By the assumption on the topology of X, each U_α can be assumed of the form

$$\{x \in K \mid f_{\alpha 1}(x) \in (a_{\alpha 1}, b_{\alpha 1}), \ldots, f_{\alpha n_\alpha}(x) \in (a_{\alpha n_\alpha}, b_{\alpha n_\alpha})\}$$

where the $f_{\alpha i}$ are continuous linear functionals on X, and the $a_{\alpha i}<b_{\alpha i}$ are real numbers.

Let A be the union of index-sets $A_1 \cup A_{1/2} \cup A_{1/4} \cup \cdots$. Then it follows from our construction that g is a continuous function of the family of maps $(f_{\alpha i})_{\alpha \in A, i=1,\ldots,n_\alpha}$. I.e., g can be written $g_0 f$, where f is the product map of the $f_{\alpha i}$, sending X into $\boldsymbol{R}^{\{(\alpha,i) \mid \alpha \in A, i \leq n_\alpha\}}$, and g_0 is a continuous function on $f(K)$, as we desired.

We are given a set of functionals G_0; for $j>0$ let us, inductively, make G_j a countable family of linear functionals which, for every $f \in G_{j-1}$, "continuously determines" fF. Let $G = \bigcup_{j=0}^{\infty} G_j$, which we can reindex $(f_i)_{i=1,2,\ldots}$, since it is countable. (If G is finite, we let $f_i = 0$ for large i.) Let $p\colon X \to \boldsymbol{R}^\infty$ be the product of this family of maps. It is clear from our construction that for all i, $f_i F$ is continuous determined by p, thus pF is continuously determined by p, i.e., we can write $pF = F'p$, where F' is continuous. On the other hand, for every $f \in G_0$, f will equal some f_i; letting f' be the ith projection map we have $f = f'p$. ∎

Corollary 3.3. *Lemma* 3.2 *still holds if the hypothesis "the topology of X is determined by linear functionals," is replaced by "linear functionals distinguish points of X, and K has compact closure."*

Proof. Let X' designate X with the topology induced by the continuous linear functionals. Since the closure of K is compact in X, the closure of K, and hence K itself, has the same topology in X' as in X. Hence continuous maps on K remain continuous in the X' topology; and we get our results by applying Lemma 3.3 in X'. ∎

4. The general fixed-point theorems.

Theorem 4.1. *Let X be a topological vector space such that continuous linear functionals distinguish points. Let K be a compact convex subset of X, and F: K → X a weakly inward map. Then F has a fixed point.*

Proof. Given any continuous linear functional f on X let S_f be the set of $x \in K$ such that $fF(x) = f(x)$. We claim that any finite intersection of the sets S_f is nonempty.

Indeed, given any finite set, G_0, of such functionals, we apply Corollary 3.3, getting maps $p\colon X \to \boldsymbol{R}^\infty$ and $F'\colon p(K) \to \boldsymbol{R}^\infty$. The set $p(K)$ is compact and convex. Further, F' is weakly inward, for it is easy to see that the property of lying in the weakly inward set of a point is preserved under continuous linear maps of the vector space.

So by Lemma 3.1, F' has a fixed point $p(z)$, $z \in X$. We claim that $z \in \bigcap_{G_0} S_f$. For given any $f \in G_0$, we compute

$$fF(z) = f'pF(z) = f'F'p(z) = f'p(z) = f(z).$$

It follows from the compactness of K that the intersection of the sets S_f over *all* linear functionals f is nonempty. Clearly, a point in this intersection is a fixed point of F. ∎

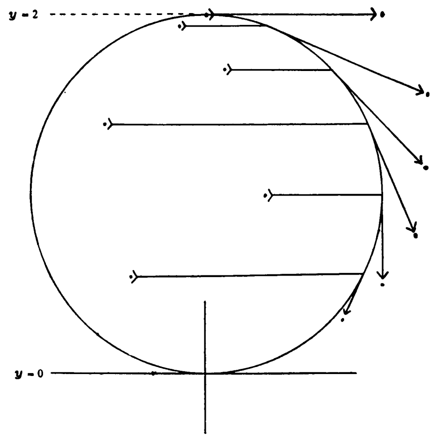

FIGURE 1

Given a map F on a compact convex set K in a topological space X, let us call a coset C of a closed subspace of X a nonempty section of K if $C \cap K \neq \varnothing$; and let us call it an invariant section if $F(C \cap K) \subset C$. The proof in Dunford-Schwartz [1] of the general case of the Schauder-Tychonoff theorem makes use of the fact that the inverse image of a fixed point under a fibering is a nonempty invariant section, and that if C is a nonempty invariant section and F is an inner map, then $(F \mid K \cap C)\colon (K \cap C) \to C$ is again an inner map. (Zorn's lemma is then used.) The latter result also holds for *inward* maps, but *not* for weakly inward maps! In fact, for F weakly inward, a nonempty invariant section need not contain a fixed point. For example, let K be the disc $x^2+(y-1)^2 \leqq 1$ in the xy-plane, and let F send the point (x, y) to the end of the clockwise segment of length $y/2$ tangent to K at the point $((1-(y-1)^2)^{1/2}, y)$. Then $y=0$ and $y=2$ are both invariant sections, but only $y=0$ has a fixed point. (See Figure 1.)

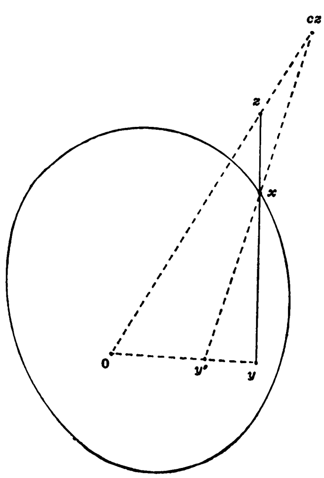

FIGURE 2

It was this observation that forced us to look, not at sets based on an arbitrary choice, but sets such as the S_f's which cannot exclude any potential fixed points.

LEMMA 4.2. *Suppose a compact convex subset K of a topological vector space X contains the point* 0. *Then the outward set of any point x is closed under multiplication by constants $c>1$. Hence so is the weakly outward set.*

(**Proof.** See Figure 2, where $z=(1-\alpha)x+\alpha y$ ($\alpha<0$) is an arbitrary point of the outward set of x. The reader can easily supply the numerical argument, getting cz in the form $(1-\alpha')x+\alpha' y'$. ∎)

THEOREM 4.3. *Let X be a topological vector space such that continuous linear functionals distinguish points. Let K be a compact convex subset of X, and $F: K \to X$ a weakly outward map. Then*:

(1) *F has a fixed point,*

(2) *$F(K)\supset K$.*

Proof. (1) is clear from Theorem 4.1 and our original discussion of the relationship between "inwardness" and "outwardness" conditions.

To show (2), let us suppose the contrary. Clearly, we can assume that 0 is a point of $K-F(K)$. The complement U of $F(K)$ is a neighborhood of 0, so we can choose $c>1$ such that $cU\supset K$. Then $cF(K)$ is disjoint from K, and so the map cF can have no fixed points. But, by Lemma 4.2 it is clear that cF is weakly outward. Contradiction.

REFERENCES

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