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A FIXED POINT THEOREM FOR PLANE HOMEOMORPHISMS

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The purpose of this note is to outline a proof of "every homeomorphism of the plane into itself that leaves a continuum M invariant has a fixed point in $T(M)$". That is, the orientation preserving condition in the Cartwright Littlewood fixed point theorem [3] is unnecessary.

All sets will be assumed to be subsets of the plane unless otherwise indicated.

DEFINITION. If A is a bounded set then $T(A)$ is the smallest compact set that contains A and has a connected complement.

THEOREM 1. *Let $f: D \longrightarrow R^2$ be a map defined on a simple closed curve D. If there is a partition of D, $\{x_0, x_1, x_2, \ldots, x_n = x_0\}$ and arcs $A_1, A_2, A_3, \ldots, A_n$ in $T(D)$ such that A_i joins $f(x_{i-1})$ to $f(x_i)$ and $x_{i-1}x_i \cap T(f[x_{i-1}x_i] \cup A_i) = \emptyset$, then every extension of f to a map defined on $T(D)$ has a fixed point.*

PROOF. Suppose there is a fixed point free extension of f to a map g defined on $T(D)$. Then find mutually disjoint (except for endpoints) arcs $K_1, K_2, \ldots, K_n$ in $T(D)$ such that K_i joins x_{i-1} to x_i and $T(K_i \cup x_{i-1}x_i) \cap T(f(K_i) \cup A_i) = \emptyset$. Then using the Tietze extension theorem, piece together a map $g': T(D) \longrightarrow R^2$ for which $g'(z) = g(z)$ if $z \notin \bigcup \{T(x_{i-1}x_i \cup K_i): i = 1, 2, \ldots, n\}$, $g'(x_{i-1}x_i) \subset A_i$, and $g'(T(x_{i-1}x_i \cup K_i)) \cap T(x_{i-1}x_i \cup K_i) = \emptyset$. If r is a retract of R^2 onto

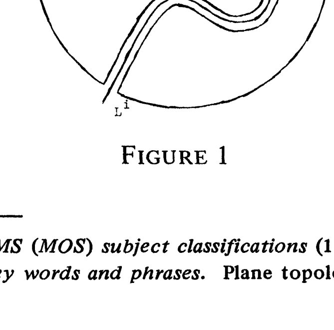

FIGURE 1

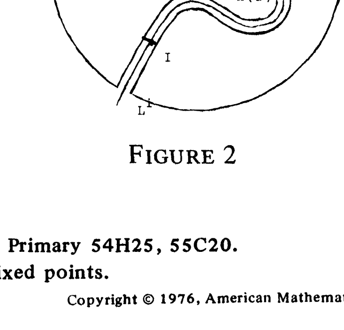

FIGURE 2

AMS (MOS) subject classifications (1970). Primary 54H25, 55C20.

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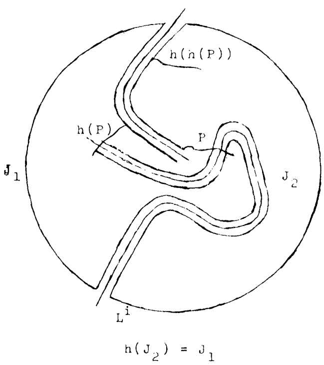

FIGURE 3

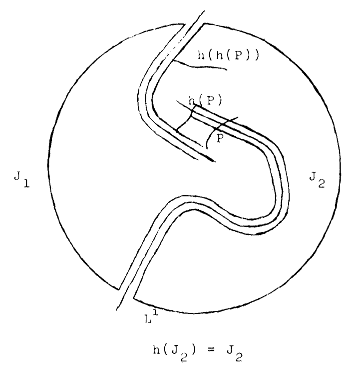

FIGURE 4

$T(D)$ such that $r(z) \in D$ if $z \notin T(D)$, then $r \circ g'$ is a fixed point free map of $T(D)$ into $T(D)$, contrary to the Brouwer fixed point theorem.

Figure 1 is meant to portray a continuum M obtained by removing two infinitely long open channels, an "in channel" and an "out channel", with center lines L^o and L^i respectively. Also the boundary of M is $\overline{L^o} - L^o = \overline{L^i} - L^i$ and the set of accessible points of M has two arc components J_1 and J_2.

THEOREM 2. *There does not exist a homeomorphism of the plane into itself that leaves M invariant and has the property that small arcs that cut across the out channel have images that cut across the out channel further out and small arcs that cut across the in channel have images that cut across the in channel further in.*

PROOF. Let D be a simple closed curve that is constructed using a small arc that cuts across the out channel A^o, a small arc that cuts across the in channel A^i, and parts of J_1 and J_2. Let P be a very small arc in $T(D)$ that has one endpoint on $J_2 \cap D$, separates M, and is such that neither P, $h(P)$, or $h(h(P))$ intersects the out channel center line up until A^o or the in channel center line up until A^i. Let C_1 be the first component of $h(P) - M$ that cuts across the in channel. Then $h^{-1}(C_1)$ is a component of $P - M$ that cuts across the in channel before C_1 does and no component of $h(h(P)) - M$ cuts across the in channel before C_1 does. Notice that $h(h(P))$ is attached to J_2 so that, by the Jordan Curve Theorem, the out channel must pass through P on its way to $h(h(P))$. But before the out channel can pass through P it must pass through $h(P)$, that is, after it has passed through $h(h(P))$, clearly an impossible situation.

For the remainder of this note we suppose that h is a homeomorphism of the plane into itself that leaves a continuum M invariant. We may assume, without loss of generality, that M is a minimal nonseparating invariant nondegenerate continuum.

Let $d(x, M)$ denote the distance from a point x to M, $S(x, M)$ denotes $\{z: |z - x| = d(x, M)\}$, and $E(M)$ denotes $\{e: M \cap S(e, M)$ has at least two points$\}$. For $e \in E(M)$ let $\mathcal{I}_e(M)$ be the set of open intervals (a, b) for which there is a component of $S(e, M) - M$ with endpoints a and b. Let $\mathcal{I}(M) = \bigcup \{I\colon I \in \mathcal{I}_e(M)$ for some $e \in E(M)\}$. Choose $r_0 > 0$ so that if $\mathrm{dia}(A) \leqslant r_0$ then $A \cap h(A) = \varnothing$. Let $Y' = T(\bigcup \{I \in \mathcal{I}(M)\colon \mathrm{dia}(I) \leqslant r_0$ and $I \cap T[h(I) \cup M] = \varnothing = h(I) \cap T(I \cup M)\})$.

Theorem 1 yields

THEOREM 3. *Y' is not a topological two-cell.*

By leaning heavily on the results in [1] and [2] we have

THEOREM 4. *If Y' is not a topological two-cell then the continuum M and the map h resemble the continuum M and map h of Theorem 2 well enough to apply the same technique of proof.*

That is, there are subsets of $E(M)$, L^{o} and L^{i}, such that (i) L^{o} and L^{i} are each homeomorphic to the set of real numbers, (ii) for each $e \in L^{o}$ there is an $I \in \mathcal{I}_e(M)$ such that $I \subset T[h(I) \cup M]$, and (iii) for each $e \in L^{i}$ there is an $I \in \mathcal{I}_e(M)$ such that $h(I) \subset T(I \cup M)$. It follows from (ii) and (iii) that $\overline{L^{o}} - L^{o}$ and $\overline{L^{i}} - L^{i}$ are invariant subcontinua of bdry(M), from which it follows that $\overline{L^{o}} - L^{o} = \mathrm{bdry}(M) = \overline{L^{i}} - L^{i}$. L^{o} serves as the center line of the out channel and L^{i} serves as the center line of the in channel.

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