A FIXED POINT THEOREM IN EUCLIDEAN BUILDINGS

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Abstract. We establish a fixed point theorem for a certain type of non-expanding maps in Euclidean buildings, which is inspired by a theorem of Laffaille in p-adic Hodge theory [10, Theorem 3.2].

1. Introduction

1.1. We establish a fixed point theorem for certain non-expanding self-mappings of a Euclidean building X. The maps we consider are of the form $\alpha = \mathscr{F} \circ \Phi$, where Φ moves every point a distance t towards a point ξ at infinity, for some fixed t and ξ, and $\mathscr{F}$ is an isometry satisfying a decency condition. The results and methods are similar to those of [7] and [1], who dealt with maps of the form $\Phi_1 \circ \cdots \circ \Phi_n$, with all Φ_i's as above. However, our initial motivation was the generalization of a theorem of Laffaille [10, Theorem 3.2] in p-adic Hodge theory: we framed it as a fixed point theorem, axiomatized the situation, and arrived at the building theoretical proof given here. The actual applications of our fixed point theorem to p-adic Hodge theory are thoroughly explained in [4], but we briefly describe the relevant framework in Remark 3 below: X is a Bruhat-Tits building, $\mathscr{F}$ is a Frobenius and Φ is a filtration. The notion of a decent isometry introduced here gets its name from the related notion of a decent isocrystal [14].

1.2. Let thus X be a Euclidean building, as defined in [9, §4.1.2]. Recall that an isometry $\mathscr{F}$ of X is called semi-simple precisely when $\mathrm{Min}(\mathscr{F}) \neq \emptyset$, where

$$\begin{aligned} \mathrm{Min}(\mathscr{F}) &= \{x \in X : \mathrm{dist}(x, \mathscr{F}x) = \min(\mathscr{F})\} \text{ with} \\ \min(\mathscr{F}) &= \inf \{\mathrm{dist}(x, \mathscr{F}x) : x \in X\}. \end{aligned}$$

We say that $\mathscr{F}$ is decent if it is semi-simple and its minimal set $\mathrm{Min}(\mathscr{F})$ is a locally compact subbuilding of X such that the following property holds:

$$(1.1) \quad \forall c > 0, \exists c' > 0 \text{ s.t. } \forall x \in X : \quad \mathrm{dist}(x, \mathscr{F}x) \leq c \Longrightarrow \mathrm{dist}(x, \mathrm{Min}(\mathscr{F})) \leq c'.$$

See section 2.3.3 below for some comments on (1.1).

1.3. Let ∂X be the visual boundary of X and let $\mathscr{C}(\partial X)$ be the cone on ∂X, so that $\mathscr{C}(\partial X) = \partial X \times \mathbb{R}_{\geq 0} / \sim$ where the equivalence relation contracts $\partial X \times \{0\}$ to a single point, the origin 0 of the cone $\mathscr{C}(\partial X)$. There is an "action"

$$\mathscr{C}(\partial X) \times X \longrightarrow X$$

which sends $\Phi = (\zeta, t) \in \partial X \times \mathbb{R}_{\geq 0}$ to the non-expanding function

$$\Phi \text{ or } e(\Phi) : X \longrightarrow X$$

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which maps a point x in X to the unique point

$$\Phi x = e(\Phi)(x) = x + t\zeta$$

at distance t from x on the geodesic ray from x to ζ. For $\xi \in \partial X$, let

$$B_\xi : X \times X \longrightarrow \mathbb{R}$$

be the Busemann function defined by

$$B_\xi(x, y) = \varinjlim (\mathrm{dist}(y, z + t\xi) - \mathrm{dist}(x, z + t\xi))$$

for an arbitrary z in X, see [2, II.8.20].

1.4. We fix a decent isometry $\mathscr{F}$ of X and some Φ in $\mathscr{C}(\partial X)$. We define

$$\begin{aligned} \mu(\mathscr{F}, \Phi; \xi) &= \inf \{ B_\xi(\mathscr{F} \circ \Phi(x), x) : x \in X \} \\ \mu(\mathscr{F}, \Phi) &= \sup \{ \mu(\mathscr{F}, \Phi; \xi) : \xi \in \partial \mathrm{Min}(\mathscr{F}) \} \\ \mathrm{Max}(\mathscr{F}, \Phi) &= \{ \xi \in \partial \mathrm{Min}(\mathscr{F}) : \mu(\mathscr{F}, \Phi; \xi) = \mu(\mathscr{F}, \Phi) \} \\ X_{\mathscr{F}}(\Phi) &= \{ x \in X : \mathscr{F} \circ \Phi(x) = x \} \end{aligned}$$

Theorem 1. *With notations as above, $\mu(\mathscr{F}, \Phi) \in \{-\infty\} \cup \mathbb{R}$ and*

$$\mu(\mathscr{F}, \Phi) \leq 0 \iff X_{\mathscr{F}}(\Phi) \neq \emptyset.$$

Moreover: $X_{\mathscr{F}}(\Phi)$ is a closed convex subset of X and

(1) *$\mu(\mathscr{F}, \Phi) = -\infty$ if and only if $\mathscr{F}$ has a unique fixed point $\star$ on X.*
(2) *$\mu(\mathscr{F}, \Phi) < 0$ if and only if $X_{\mathscr{F}}(\Phi)$ is non-empty and bounded.*
(3) *$\mu(\mathscr{F}, \Phi) = 0$ if and only if $X_{\mathscr{F}}(\Phi)$ is non-empty and unbounded. Then $\partial X_{\mathscr{F}}(\Phi)$ equals $\mathrm{Max}(\mathscr{F}, \Phi)$, and this is a convex subset of $\partial \mathrm{Min}(\mathscr{F})$.*
(4) *If $\mu(\mathscr{F}, \Phi) > 0$, then $\mathrm{Max}(\mathscr{F}, \Phi)$ is a singleton.*

The proof will be given in section 3 after some preliminaries on CAT(0)-spaces, Busemann functions and angles in Euclidean buildings.

Remark 2. The implication $X_{\mathscr{F}}(\Phi) \neq \emptyset \Longrightarrow \mu(\mathscr{F}, \Phi) \leq 0$ is obvious, since

$$\forall x \in X_{\mathscr{F}}(\Phi), \forall \xi \in \partial X : \qquad \mu(\mathscr{F}, \Phi; \xi) \leq B_\xi(\mathscr{F} \circ \Phi(x), x) = 0.$$

Put $\alpha = \mathscr{F} \circ \Phi$, so that $\alpha : X \to X$ is a non-expanding map and

$$X_{\mathscr{F}}(\Phi) = \mathrm{Fix}(\alpha) := \{ x \in X : \alpha(x) = x \}.$$

In particular, $X_{\mathscr{F}}(\Phi)$ is a closed and convex subset of X by [3, Theorem 1.3]. It is a well-known property of non-expanding maps in complete CAT(0) spaces that

$$\begin{aligned} \mathrm{Fix}(\alpha) \neq \emptyset &\iff \exists x \in X : n \mapsto \alpha^n(x) \text{ is bounded,} \\ &\iff \forall x \in X : n \mapsto \alpha^n(x) \text{ is bounded.} \end{aligned}$$

See [8, Remark p. 1452][1]. The main content of the theorem is the implication

$$\mu(\mathscr{F}, \Phi) \leq 0 \Longrightarrow \alpha \text{ has bounded orbits.}$$

The overall strategy of the proof is similar to that of the main theorem of [1].

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[1]Here is Karlsson's proof that a non-expanding map α with a bounded orbit $x_n = \alpha^n(x)$ has a fixed point. There is a non-empty invariant bounded closed convex subset in X, for instance the closure of $\cup_{k \geq 1} \cap_{j \geq k} B(x_j, d)$ where $d = \sup\{\mathrm{dist}(x_n, x_m) : n, m \geq 1\}$ and $B(x, d)$ is the closed ball of radius d centered at x. Using [12, 2.12] and Zorn's lemma, we find that there is a minimal such set, call it C. Then C is equal to the convex closure of $\alpha(C)$. Let z and r be the circumcenter and minimal radius of C, so that $C \subset B(z, r)$. Then $\alpha(C) \subset \alpha(B(z, r)) \subset B(\alpha(z), r)$, thus also $C \subset B(\alpha(z), r)$ and $\alpha(z) = z$ by uniquess of the circumcenter.

Remark 3. In the sequel [4] to this paper, we apply our theorem to the study of certain objects naturally arising in p-adic Hodge theory, namely filtered G-isocrystals, see [6]. There G is a reductive group over the p-adic completion $\mathbb{Q}_p$ of $\mathbb{Q}$, X is the Bruhat-Tits building of G over the fraction field $K = W(k)[\frac{1}{p}]$ of the Witt ring of a perfect field k of characteristic p, the isometry $\mathscr{F}$ is given by the action of an element (b, σ) in $G(K) \rtimes \langle\sigma\rangle$ on X where σ is the Frobenius automorphism of K, and Φ corresponds to an $\mathbb{R}$-filtration on the fiber functor $\mathsf{Rep}_{\mathbb{Q}_p}(G) \to \mathsf{Vect}_K$ from the category of algebraic representations of G on finite dimensional $\mathbb{Q}_p$ -vector spaces to the category of finite dimensional K-vector spaces. When k is algebraically closed, the main result of [5] implies that any such $\mathscr{F}$ is a decent isometry of X. More precisely, it shows that $\mathrm{Min}(\mathscr{F})$ is a non-empty locally compact subbuilding of X and (1.1) follows from [12, Théorème 4.1], as explained in section 2.3.3 below or [5, Proposition 8]. For every k, the assumption $\mu(\mathscr{F}, \Phi) \leq 0$ is equivalent to the (weak)-admissibility of $(\mathscr{F}, \Phi)$ and our theorem 1 thus implies that $(\mathscr{F}, \Phi)$ is (weakly)-admissible if and only if $X_{\mathscr{F}}(\Phi) \neq \emptyset$. For $G = GL_n$ and Φ given by a $\mathbb{Z}$-filtration, this statement is equivalent to Laffaille's theorem [10, 3.2].

2. Preliminaries on CAT(0)-spaces and Euclidean buildings

Reference: [2, II, §1-3,6,8,9], [9, §2-4], [12, §1,2], [15].

2.1. **Angles and Busemann functions.** Let X be a complete CAT(0)-space.

2.1.1. Let x, y, z be three points of X. A comparison triangle for (x, y, z) is a triangle $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in the Euclidean plane $\mathbf{E}^2$ such that

$$\mathrm{dist}(x, y) = \mathrm{dist}(\mathbf{x}, \mathbf{y}), \quad \mathrm{dist}(y, z) = \mathrm{dist}(\mathbf{y}, \mathbf{z}) \quad \text{and} \quad \mathrm{dist}(x, z) = \mathrm{dist}(\mathbf{x}, \mathbf{z}).$$

If $x \neq y, z$, we denote by $\angle_x^c(y, z)$ the angle at $\mathbf{x}$ in any comparison triangle.

2.1.2. For x in X and ξ_1, ξ_2 in ∂X, the function

$$(t_1, t_2) \in \mathbb{R}_{>0}^2 \mapsto \angle_x^c(x + t_1\xi_1, x + t_2\xi_2) \in [0, \pi]$$

is non-decreasing in both variables. We define

$$\begin{aligned} \angle_x(\xi_1, \xi_2) &= \inf\{\angle_x^c(x + t_1\xi_1, x + t_2\xi_2) : t_1, t_2 > 0\}, \\ \angle^x(\xi_1, \xi_2) &= \sup\{\angle_x^c(x + t_1\xi_1, x + t_2\xi_2) : t_1, t_2 > 0\}. \end{aligned}$$

Then $\angle^x(\xi_1, \xi_2) = \angle(\xi_1, \xi_2) = \sup\{\angle_x(\xi_1, \xi_2) : x \in X\}$ is independant of x and

$$\angle(\xi_1, \xi_2) = \lim_{t\to\infty} \angle_{x+t\xi_1}(\xi_1, \xi_2) = \lim_{t\to\infty} \angle_{x+t\xi_2}(\xi_1, \xi_2).$$

Remark. It is not true that $\angle(\xi_1, \xi_2)$ is also the limit of $\angle_x(\xi_1, \xi_2)$ as x converges to ξ_1 (or ξ_2) in the cone topology of $X \cup \partial X$, as one checks easily when X is a tree.

2.1.3. The function $\angle : \partial X \times \partial X \to [0, \pi]$ is a CAT(1)-distance on ∂X. It induces a "scalar product" and a CAT(0)-distance on $\mathscr{C}(\partial X)$, respectively given by

$$\begin{aligned} \langle(\zeta_1, \ell_1), (\zeta_2, \ell_2)\rangle &= \ell_1\ell_2 \cos(\angle(\zeta_1, \zeta_2)) \\ \mathrm{dist}(\Phi_1, \Phi_2) &= \sqrt{|\Phi_1|^2 + |\Phi_2|^2 - 2\langle\Phi_1, \Phi_2\rangle} \end{aligned}$$

where the length function $|-| : \mathscr{C}(\partial X) \to \mathbb{R}_{\geq 0}$ maps (ζ, ℓ) to ℓ. We occasionaly identify ∂X with the unit sphere $\{\Phi \in \mathscr{C}(\partial X) : |\Phi| = 1\}$.

Proposition 4. *The function $\langle\Phi, -\rangle : \mathscr{C}(\partial X) \to \mathbb{R}$ is homogeneous and concave.*

Proof. Homegeneity means that for any $\lambda \geq 0$ and $\Psi = (\zeta, \ell)$ in $\mathscr{C}(\partial X)$,

$$\langle \Phi, \lambda \cdot \Psi \rangle = \lambda \cdot \langle \Phi, \Psi \rangle$$

where $\lambda \cdot \Psi = (\zeta, \lambda\ell)$. This is obvious from the definitions. Concavity means that for any Ψ_0, Ψ_1 in $\mathscr{C}(\partial X)$ and $t \in [0,1]$, if $\Psi_t = t \cdot \Psi_1 + (1-t) \cdot \Psi_0$ is the unique point at distance t from Ψ_0 on the geodesic segment $[\Psi_0, \Psi_1]$ of $\mathscr{C}(\partial X)$, then

$$\langle \Phi, \Psi_t \rangle \geq t \cdot \langle \Phi, \Psi_1 \rangle + (1-t) \cdot \langle \Phi, \Psi_0 \rangle . \tag{2.1}$$

The following proof was indicated to me by G. Rousseau. Let $(0, \phi, \psi_0, \psi_1)$ be a comparison triedron in $\mathbf{E}^3$ for $(0, \Phi, \Psi_0, \Psi_1)$, by which we mean that the length of the edges containing 0 and the angle between them are the same for both tetrahedron. Then the length of the other three edges are also the same, since every triangle $(0, x, y)$ in $\mathscr{C}(\partial X)$ is flat. Put $\psi_t = t \cdot \psi_1 + (1-t) \cdot \psi_0$, so that

$$\langle \phi, \psi_t \rangle = t \cdot \langle \phi, \psi_1 \rangle + (1-t) \cdot \langle \phi, \psi_0 \rangle$$

is equal to the right hand side of (2.1), and we have to show that

$$\langle \Phi, \Psi_t \rangle \geq \langle \phi, \psi_t \rangle .$$

We already know that $|\Phi| = |\phi|$, and also $|\Psi_t| = |\psi_t|$ by flatness of $(0, \Psi_0, \Psi_1)$. On the other hand, (ϕ, ψ_0, ψ_1) is a comparison triangle for (Φ, Ψ_0, Ψ_1), therefore $\mathrm{dist}(\Phi, \Psi_t) \leq \mathrm{dist}(\phi, \psi_t)$ since $\mathscr{C}(\partial X)$ is a CAT(0)-space. The required inequality then easily follows from the flatness of $(0, \Phi, \Psi_t)$. □

Remark 5. Scalar products on Euclidean cones were introduced in [11], from whose Proposition 2.6 another proof of the above proposition can also be derived.

2.1.4. Recall that for $x, y \in X$, $\xi \in \partial X$ and any $z \in X$, the limit

$$B_\xi(x,y) = \lim_{t\to\infty} \mathrm{dist}\,(y, z + t\xi) - \mathrm{dist}(x, z+t\xi)$$

is well-defined and independent of z. The triangle inequality implies that

$$-\mathrm{dist}(x,y) \leq B_\xi(x,y) \leq \mathrm{dist}(x,y).$$

If $x \neq y$, we may thus define an angle $\angle_y^B(x,\xi) \in [0,\pi]$ by the formula

$$B_\xi(x,y) = \mathrm{dist}(x,y) \cdot \cos\left(\angle_y^B(x,\xi)\right) .$$

It may also be computed as follows. For $y \neq x, z$ in X, one checks easily that

$$\mathrm{dist}(y,z) - \mathrm{dist}(x,z) = \mathrm{dist}(x,y) \cdot \left(\cos \angle_y^c(x,z) - \sin \angle_y^c(x,z) \cdot \tan\left(\tfrac{1}{2}\angle_z^c(x,y)\right)\right) .$$

Replacing z by $z + t\xi$ and letting $t \to \infty$, we find that since $\angle_{z+t\xi}^c(x,y) \to 0$,

$$\angle_y^B(x,\xi) = \lim_{t\to\infty} \angle_y^c(x, z+t\xi).$$

2.1.5. For $\Phi = (\zeta, \ell)$ in $\mathscr{C}(\partial X)$ with $\ell \neq 0$, we define $\angle_x^B(\Phi, \xi)$ by

$$B_\xi(\Phi(x), x) = |\Phi| \cos\left(\angle_x^B(\Phi,\xi)\right) .$$

Therefore $\angle_x^B(\Phi,\xi) = \angle_x^B(\Phi(x),\xi) = \lim_{t\to\infty} \angle_x^c(x + \ell\zeta, x + t\xi)$ and using 2.1.2,

$$\angle_x(\zeta,\xi) \leq \angle_x^B(\Phi,\xi) \leq \angle(\zeta,\xi).$$

This yields yet another set of formulae for $\angle(\zeta,\xi)$, namely

$$\angle(\zeta,\xi) = \sup\left\{\angle_x^B(\Phi,\xi) : x \in X\right\} = \lim_{t\to\infty} \angle_{x+t\xi}^B(\Phi,\xi)$$

together with the corresponding formulae for the scalar product:

$$\langle \Phi, \xi \rangle = \inf \{B_\xi(\Phi(x), x) : x \in X\} = \lim_{t\to\infty} B_\xi(\Phi(x + t\xi), x + t\xi).$$

2.2. **Semi-simple isometries.** Let $\mathscr{F}$ be an isometry of X. We assume that $\mathscr{F}$ is semi-simple, i.e. $\mathrm{Min}(\mathscr{F}) \neq \emptyset$. Then $\mathrm{Min}(\mathscr{F})$ is a closed convex subset of X.

2.2.1. The translation vector of $\mathscr{F}$ is an element $\nu = \nu(\mathscr{F}) \in \mathscr{C}(\partial X)$ which is defined as follows. If $\mathscr{F}$ is elliptic, i.e. $\min(\mathscr{F}) = 0$, then $\nu = 0$. Otherwise $\mathscr{F}$ is hyperbolic, i.e. $\min(\mathscr{F}) > 0$, in which case $\mathrm{Min}(\mathscr{F})$ is the union of the $\mathscr{F}$-stable geodesic lines in X, any two of these lines are asymptotic, and $\mathscr{F}$ acts on them by a translation of lenght $\min(\mathscr{F})$. Then $\nu = (\zeta, \ell)$ where $\ell = \min(\mathscr{F})$ and ζ is the asymptotic class of any $\mathscr{F}$-stable geodesic ray in any of these $\mathscr{F}$-stable geodesic lines. In both cases, one checks that ν is the unique element of $\mathscr{C}(\partial X)$ whose restriction to $\mathrm{Min}(\mathscr{F})$ equals $\mathscr{F}|\mathrm{Min}(\mathscr{F})$.

2.2.2. A Clifford translation of X is an isometry Φ of X such that $\mathrm{Min}(\Phi) = X$, i.e. such that $x \mapsto \mathrm{dist}(x, \Phi x)$ is constant. The map $\Phi \mapsto \nu(\Phi)$ identifies the group of all Clifford translations of X with a closed, convex subspace $\mathscr{C}_0(\partial X)$ of $\mathscr{C}(\partial X)$, namely the Euclidean factor of the pointed CAT(0)-space $\mathscr{C}(\partial X)$. For a Clifford translation $\Phi \in \mathscr{C}_0(\partial X)$, the function $\langle \Phi, - \rangle$ of Proposition 4 is also convex, since more precisely $\langle \Phi, - \rangle + \langle \Phi', - \rangle \equiv 0$ on $\mathscr{C}(\partial X)$ where Φ' is the inverse (or opposite) Clifford translation. The latter formula easily follows from 2.1.5 or [2, II.6.15].

Returning to our semi-simple isometry $\mathscr{F}$ of X, we see that its restriction to $\mathrm{Min}(\mathscr{F})$ is a Clifford translation of this CAT(0)-subspace of X, corresponding to the translation vector $\nu \in \mathscr{C}_0(\partial \mathrm{Min}(\mathscr{F})) \subset \mathscr{C}(\partial \mathrm{Min}(\mathscr{F}) \subset \mathscr{C}(\partial X)$.

2.2.3. We still denote by $\mathscr{F}$ the induced isometries of ∂X and $\mathscr{C}(\partial X)$. We have

$$\mathscr{F}(x + t\xi) = \mathscr{F}(x) + t\mathscr{F}(\xi) \quad \text{and} \quad B_{\mathscr{F}\xi}(\mathscr{F}x, \mathscr{F}y) = B_\xi(x, y).$$

Suppose $\xi \in \partial\mathrm{Min}(\mathscr{F})$. Then $\mathscr{F}(\xi) = \xi$ and $(x, y) \mapsto B_\xi(x, y)$ is $\mathscr{F}$-invariant. Also

$$B_\xi(\mathscr{F}x, x) = B_\xi(\mathscr{F}x, \mathscr{F}y) + B_\xi(\mathscr{F}y, y) + B_\xi(y, x) = B_\xi(\mathscr{F}y, y)$$

for any $x, y \in X$, i.e. the function $x \mapsto B_\xi(\mathscr{F}x, x)$ is actually constant. Its value is easily computed on $\mathrm{Min}(\mathscr{F}) \neq \emptyset$, where $\mathscr{F} = \nu$, using e.g. 2.1.5. We obtain

$$\forall x \in X : \qquad B_\xi(\mathscr{F}x, x) = \langle \nu, \xi \rangle .$$

2.2.4. Using 2.1.5 and 2.2.3, we find that for any $\xi \in \partial\mathrm{Min}(\mathscr{F})$ and $\Phi \in \mathscr{C}(\partial X)$,

$$\mu(\mathscr{F}, \Phi; \xi) = \inf \{B_\xi(\mathscr{F} \circ \Phi(x), x) : x \in X\} = \langle \Phi, \xi \rangle + \langle \nu, \xi \rangle . \tag{2.2}$$

This allows us to extend $\mu(\mathscr{F}, \Phi; -)$ from $\partial\mathrm{Min}(\mathscr{F})$ to $\mathscr{C}(\partial\mathrm{Min}(\mathscr{F}))$ as follows:

$$\mu(\mathscr{F}, \Phi; -) : \mathscr{C}(\partial\mathrm{Min}(\mathscr{F})) \to \mathbb{R}, \qquad \mu(\mathscr{F}, \Phi; \Psi) = \langle \Phi, \Psi \rangle + \langle \nu, \Psi \rangle .$$

Proposition 6. *This function is homogeneous and concave on $\mathscr{C}(\partial\mathrm{Min}(\mathscr{F}))$.*

Proof. This follows from Proposition 4. □

2.3. **Euclidean buildings.**

2.3.1. *Some definitions.* Let $\mathbf{A} = \mathbf{E}^n$ be a finite dimensional Euclidean space. A subgroup W of $\mathrm{Isom}(\mathbf{A})$ is an affine Weyl group if and only if it is generated by affine reflections and the image $\Omega = \partial W$ of W in $\mathrm{Isom}(\partial\mathbf{A}) = O(\mathbf{A})$ is finite. A Euclidean building modelled on $(\mathbf{A}, W)$ (in the sense of [9, 4.1]) is a CAT(0)-space X equipped with an atlas of isometrical embeddings $\mathcal{A} = \{\iota : \mathbf{A} \hookrightarrow X\}$. These embeddings are called charts, and their images appartments. Among various other conditions, it is required that any geodesic segment, ray or line is contained in some appartement, and that for any pair of charts ι_1, ι_2 in $\mathcal{A}$, the change of coordinate

$$\iota_2^{-1}(\iota_1(\mathbf{A})) \xrightarrow{\iota_2} \iota_1(\mathbf{A}) \cap \iota_2(\mathbf{A}) \xrightarrow{\iota_1^{-1}} \iota_1^{-1}(\iota_2(\mathbf{A}))$$

is equal to the restriction of some element w in W. This yields a well-defined map

$$d : X \times X \to \mathscr{C}(X) \quad \text{where} \quad \mathscr{C}(X) = \Omega\backslash\mathbf{A}$$

sending (x_1, x_2) to the class of $y_2 - y_1 \in \mathbf{A}$ if $x_i = \iota(y_i)$ for some $\iota \in \mathcal{A}$. Note that $\mathscr{C}(X) = \Omega\backslash\mathbf{A}$ may be identified with any closed Weyl chamber $\overline{\mathfrak{C}} \subset \mathbf{A}$. It is a polysimplicial cone and a partially ordered set (for the usual dominance order $\leq$), and A. Parreau shows in [13, Proposition 3.5] that for every x, y, z in X,

$$d(x, z) \leq d(x, y) + d(y, z).$$

The initial CAT(0)-distance on X is retrieved by $\mathrm{dist}(x, y) = |d(x, y)|$, where

$$|-| : \mathscr{C}(X) \to \mathbb{R}_{\geq 0}$$

is the obvious length function on $\mathscr{C}(X)$. There is also a type map

$$t : \mathscr{C}(\partial X) \to \mathscr{C}(X)$$

such that for every $\Phi \in \mathscr{C}(\partial X)$ and $x \in X$,

$$d(x, \Phi(x)) = t(\Phi).$$

We refer to [12, §1 and 2] for a detailed comparison of various definitions of Euclidean buildings, including some non-complete generalisations. The rank of X is the dimension of $\mathbf{A}$. The building is essential (resp. trivial) if and only if $\mathbf{A}^\Omega = \{0\}$ (resp. $\Omega = \{1\}$, in which case $X = \mathbf{A}$). Every Euclidean building has a unique decomposition $X = X_e \times X_0$ with X_e essential and X_0 trivial [12, Corollaire 2.4]. The semi-simple rank $s(X)$ of X is the dimension of its essential part X_e.

2.3.2. *Rigidity of angles.* Given nonzero vectors a, b in $\mathbf{A}$, let $D(a, b)$ be the finite set of angles between elements of $\Omega \cdot a$ and elements of $\Omega \cdot b$. A key property of Euclidean buildings is the following axiom: for every $x \neq y, z$ in X,

$$\angle_x(y, z) \text{ belongs to } D(a, b) \text{ where } a = d(x, y) \text{ and } b = d(x, z).$$

2.3.3. *Parreau's theorem.* A subset of X is a Weyl chamber if and only if it equals $\iota(a + \mathfrak{C})$ for some $\iota \in \mathcal{A}$, some $a \in \mathbf{A}$ and some (vectorial) Weyl chamber $\mathfrak{C} \subset \mathbf{A}$.

Theorem 7. *(*Parreau, [12, Th. 4.1]*) Suppose $\mathbf{A}^\Omega = \{0\}$. There exists $\alpha \in]0, \pi]$ such that for every Euclidean building X modelled on $(\mathbf{A}, W)$, every isometry $\mathscr{F}$ of X which preserves the Weyl chambers is semi-simple and moreover satisfies*

$$\forall x \in X : \qquad \mathrm{dist}(x, \mathscr{F}x) \geq \sin(\tfrac{\alpha}{2}) \cdot \mathrm{dist}(x, \mathrm{Min}(\mathscr{F})).$$

Corollary 8. *For any Euclidean building X, every isometry $\mathscr{F}$ of X which preserves the Weyl chambers is semi-simple and satisfies (1.1):*

$$\forall c > 0, \exists c' > 0 \text{ s.t. } \forall x \in X : \quad \mathrm{dist}(x, \mathscr{F}x) \leq c \Longrightarrow \mathrm{dist}(x, \mathrm{Min}(\mathscr{F})) \leq c'.$$

Proof. Use [12, Prop 2.8] to reduce to either the trivial case – which is easy, see section 3.1 below – or to the essential case, which follows from the theorem. □

Remark 9. If the spherical building ∂X is thick, then every isometry of X preserves the Weyl chambers, and therefore satisfies (1.1). See [12, Proposition 2.27].

3. Proof of theorem 1

3.1. **The affine case.** Let X be a Euclidean affine space with underlying Euclidean vector space V. Then $V \simeq \mathscr{C}(\partial X)$ and $\Phi \in V$ acts on X by $x \mapsto x + \Phi$. Moreover, every isometry $\mathscr{F}$ of X is decent [2, II.6.5]: if $\mathscr{G} = \partial\mathscr{F} \in O(V)$ is the rotational part of $\mathscr{F}$, then $\mathrm{Min}(\mathscr{F})$ is an affine subspace of X with underlying vector space $\ker(\mathscr{G} - \mathrm{Id})$. The translation vector $\nu = \nu(\mathscr{F})$ belongs to $\ker(\mathscr{G} - \mathrm{Id})$ and

$$\forall (x, v) \in \mathrm{Min}(\mathscr{F}) \times V : \qquad \mathscr{F}(x + v) = x + \nu + \mathscr{G}(v).$$

It follows that $X_{\mathscr{F}}(\Phi) \neq \emptyset$ if and only if Φ belongs to $\mathrm{Im}(\mathscr{G} - \mathrm{Id}) - \nu$. If this is indeed the case, i.e. $\Phi = \mathscr{G}(w) - w - \nu$ for some $w \in V$, then

$$X_{\mathscr{F}}(\Phi) = \mathrm{Min}(\mathscr{F}) - \mathscr{G}(w) \quad \text{and} \quad \partial X_{\mathscr{F}}(\Phi) = \partial \mathrm{Min}(\mathscr{F}).$$

On the other hand, $\mathscr{C}(\partial \mathrm{Min}(\mathscr{F})) = \ker(\mathscr{G} - \mathrm{Id})$ is the orthogonal complement of $\mathrm{Im}(\mathscr{G} - \mathrm{Id})$ inside V, thus Φ belongs to $\mathrm{Im}(\mathscr{G} - \mathrm{Id}) - \nu$ if and only if $\langle \Phi, \xi \rangle = -\langle \nu, \xi \rangle$ for every $\xi \in \mathscr{C}(\partial \mathrm{Min}(\mathscr{F}))$. The Busemann functions are given by

$$\forall (x, y, \xi) \in X^2 \times \partial X : \qquad B_\xi(x, y) = \langle x - y, \xi \rangle .$$

One checks easily that for every $\xi \in \partial \mathrm{Min}(\mathscr{F})$ and $x \in X$,

$$B_\xi(\mathscr{F} \circ \Phi(x), x) = \langle \Phi + \nu, \xi \rangle = \mu(\mathscr{F}, \Phi; \xi).$$

We now have three cases. If $\ker(\mathscr{G} - \mathrm{Id}) = 0$, then $\mu(\mathscr{F}, \Phi) = -\infty$ for every Φ and $X_{\mathscr{F}}(\Phi)$ is a singleton. If $\ker(\mathscr{G} - \mathrm{Id}) \neq 0$ and Φ belongs to $\mathrm{Im}(\mathscr{G} - \mathrm{Id}) - \nu$, then $\mu(\mathscr{F}, \Phi) = 0$, $X_{\mathscr{F}}(\Phi)$ is an affine subspace of X with underlying vector space $\ker(\mathscr{G} - \mathrm{Id})$ and $\mathrm{Max}(\mathscr{F}, \Phi) = \partial X_{\mathscr{F}}(\Phi) = \partial \mathrm{Min}(\mathscr{F}) = \ker(\mathscr{G} - \mathrm{Id})$. Finally if Φ does not belong to $\mathrm{Im}(\partial\mathscr{F} - \mathrm{Id}) - \nu$, then $X_{\mathscr{F}}(\Phi)$ is empty, $\mu(\mathscr{F}, \Phi) > 0$ is the length of the orthogonal projection $\mu(\mathscr{F}, \Phi)\xi_0 \neq 0$ of $\Phi + \nu$ to $\ker(\mathscr{G} - \mathrm{Id})$ and $\mathrm{Max}(\mathscr{F}, \Phi) = \{\xi_0\}$. This completes the proof of the theorem in the affine case.

3.2. **Products.** Suppose that $X = X_1 \times X_2$ is a product of Euclidean buildings with $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$. Then $\mathscr{C}(\partial X) = \mathscr{C}(\partial X_1) \times \mathscr{C}(\partial X_2)$ and thus also $\Phi = (\Phi_1, \Phi_2)$ and $X_{\mathscr{F}}(\Phi) = X_{\mathscr{F}_1}(\Phi_1) \times X_{\mathscr{F}_2}(\Phi_2)$. Moreover $\mathrm{Min}(\mathscr{F}) = \mathrm{Min}(\mathscr{F}_1) \times \mathrm{Min}(\mathscr{F}_2)$ and $\mathscr{C}(\partial \mathrm{Min}\mathscr{F}) = \mathscr{C}(\partial \mathrm{Min}\mathscr{F}_1) \times \mathscr{C}(\partial \mathrm{Min}(\mathscr{F}_2))$ with $\nu(\mathscr{F}) = (\nu(\mathscr{F}_1), \nu(\mathscr{F}_2))$ and

$$\forall (x_i, y_i) \in \mathscr{C}(\partial \mathrm{Min}\mathscr{F}_i)^2 : \qquad \langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle .$$

For $\xi = (x_1, x_2)$ in $\partial \mathrm{Min}(\mathscr{F})$ with $x_i = |x_i|\, \xi_i$ in $\mathscr{C}(\partial \mathrm{Min}(\mathscr{F}_i))$, we find

$$\mu(\mathscr{F}, \Phi; \xi) = |x_1|\, \mu(\mathscr{F}_1, \Phi_1; \xi_1) + |x_2|\, \mu(\mathscr{F}_2, \Phi_2; \xi_2).$$

It follows that the theorem holds for $(X, \mathscr{F}, \Phi)$ if it holds for $(X_i, \mathscr{F}_i, \Phi_i)$, $i \in \{1, 2\}$.

3.3. **The case $\mu(\mathscr{F},\Phi)=-\infty$.** Since $\mathrm{Min}(\mathscr{F})$ is an $\mathscr{F}$-stable subbuilding of X,

$$\mu(\mathscr{F},\Phi)=-\infty \iff \partial\mathrm{Min}(\mathscr{F})=\emptyset \iff \mathrm{Min}(\mathscr{F})=\mathrm{Fix}(\mathscr{F})=\{\star\}.$$

Assuming this, fix some x in X and put $x_n=\alpha^n(x)$ where $\alpha=\mathscr{F}\circ\Phi$. Since

$$\begin{aligned}\mathrm{dist}\,(\mathscr{F}x_n,x_n) &= \mathrm{dist}\,(x_n,\Phi x_{n-1})\\ &\leq \mathrm{dist}\,(x_n,x_{n-1})+\mathrm{dist}\,(x_{n-1},\Phi x_{n-1})\\ &\leq \mathrm{dist}\,(x_1,x_0)+|\Phi|\end{aligned}$$

and $\mathscr{F}$ satisfies (1.1), x_n remains at bounded distance from $\mathrm{Min}(\mathscr{F})=\{\star\}$. Therefore α has bounded orbits and $X_{\mathscr{F}}(\Phi)=\mathrm{Fix}(\alpha)\neq\emptyset$ by Remark 2. Since

$$\forall x\in X_{\mathscr{F}}(\Phi):\qquad \mathrm{dist}\,(\mathscr{F}x,x)=\mathrm{dist}(x,\Phi x)=|\Phi|\,,$$

the same argument shows that $X_{\mathscr{F}}(\Phi)$ is bounded.

3.4. **Unbounded orbits.** Let again $\alpha=\mathscr{F}\circ\Phi$. For $(\star,x)\in X^2$, define

$$\begin{aligned}\mathbb{N}(\star,x) &= \left\{n\in\mathbb{N}:\mathrm{dist}\,(\star,\alpha^n(x))>\mathrm{dist}\left(\star,\alpha^{n-1}(x)\right)\right\}\\ L(\star,x) &= \{\xi\in\partial X:\exists(n_i)\in\mathbb{N}\,(\star,x)\ \text{ with }\alpha^{n_i}(x)\to\xi\}\,.\end{aligned}$$

Lemma 10. *Suppose that the orbits of α are unbounded. Then for every $(\star,x)\in X^2$, the subset $L(\star,x)$ of ∂X is non-empty, contained in $\partial\mathrm{Min}(\mathscr{F})$ and*

$$\forall\xi\in L(\star,x):\quad \mu\,(\mathscr{F},\Phi;\xi)\geq 0\quad\textit{and}\quad\forall z\in X:\,B_\xi\,(\star,\alpha^n(z))\to-\infty.$$

Proof. Let y_n be the orthogonal projection of $x_n=\alpha^n(x)$ to $\mathrm{Min}(\mathscr{F})$. Since again

$$\mathrm{dist}(\mathscr{F}x_n,x_n)\leq\mathrm{dist}(x_1,x_0)+|\Phi|$$

also $\mathrm{dist}(x_n,y_n)$ is bounded by (1.1). Since $\mathrm{dist}(\star,x_n)$ is unbounded, there exists a sequence (n_i) in $\mathbb{N}(\star,x)$ with $\mathrm{dist}(\star,x_{n_i})\to\infty$, and then also $\mathrm{dist}(\star,y_{n_i})\to\infty$. Upon passing to a subsequence, we may assume that $y_{n_i}\to\xi$ for some $\xi\in\partial\mathrm{Min}(\mathscr{F})$ since $\mathrm{Min}(\mathscr{F})\cup\partial\mathrm{Min}(\mathscr{F})$ is compact in the cone topology. Then also $x_{n_i}\to\xi$, i.e. $L(\star,x)\neq\emptyset$. For any $\xi\in L(\star,x)$, with $x_{n_i}\to\xi$ also $x_{n_i-1}\to\xi$ and $y_{n_i}\to\xi$, therefore ξ belongs to $\partial\mathrm{Min}(\mathscr{F})$. Moreover for any $z\in X$,

$$\begin{aligned}B_\xi\,(\star,\alpha(z)) &= \lim_i\,(\mathrm{dist}\,(\alpha(z),x_{n_i})-\mathrm{dist}\,(\star,x_{n_i}))\\ &\leq \lim_i\,(\mathrm{dist}\,(z,x_{n_i-1})-\mathrm{dist}\,(\star,x_{n_i-1}))\quad=\quad B_\xi\,(\star,z)\,.\end{aligned}$$

Therefore $B_\xi\,(\alpha(z),z)\geq 0$ and $\mu(\mathscr{F},\Phi;\xi)\geq 0$. The above inequality also shows that for $z_n=\alpha^n(z)$, the sequence $n\mapsto B_\xi\,(\star,z_n)$ is non-increasing. If it were bounded, then $B_\xi\,(\star,x_n)$ would also be bounded since

$$|B_\xi\,(\star,z_n)-B_\xi\,(\star,x_n)|=|B_\xi\,(x_n,z_n)|\leq\mathrm{dist}\,(x_n,z_n)\leq\mathrm{dist}\,(x,z)\,.$$

However $x_{n_i}\to\xi$, therefore $B_\xi(\star,x_{n_i})\to-\infty$ and thus also $B_\xi(\star,z_n)\to-\infty$. □

Corollary 11. *If $\mu(\mathscr{F},\Phi)<0$, then $X_{\mathscr{F}}(\Phi)\neq\emptyset$.*

3.5. **Unbounded $X_{\mathscr{F}}(\Phi)$.**

Lemma 12. *Every unbounded sequence x_n in $X_{\mathscr{F}}(\Phi)$ has a subsequence which converges to a point ξ in $\partial\mathrm{Min}(\mathscr{F})$ with $\mu(\mathscr{F},\Phi;\xi)=0$.*

Proof. Let y_n be the orthogonal projection of x_n to $\mathrm{Min}(\mathscr{F})$. Since

$$\mathrm{dist}(\mathscr{F}x_n, x_n) = \mathrm{dist}(x_n, \Phi x_n) = |\Phi|$$

also $\mathrm{dist}(x_n, y_n)$ is bounded by (1.1), therefore y_n is also unbounded. Passing to a subsequence, we may assume that $y_n \to \xi$ for some $\xi \in \partial\mathrm{Min}(\mathscr{F})$, in which case also $x_n \to \xi$. Since $X_{\mathscr{F}}(\Phi)$ is closed and convex, it thus contains $z_t = x_0 + t\xi$ for every $t \geq 0$. If t is sufficently large, we find using 2.1.5 and 2.2.3 that

$$\langle \Phi, \xi\rangle = B_\xi(\Phi(z_t), z_t) = B_\xi(z_t, \mathscr{F}(z_t)) = -\langle \nu, \xi\rangle\,.$$

Therefore $\mu(\mathscr{F},\Phi;\xi) = \langle\Phi,\xi\rangle + \langle\nu,\xi\rangle = 0$. □

Corollary 13. *If $X_{\mathscr{F}}(\Phi)$ is unbounded, then*

$$\mu(\mathscr{F},\Phi) = 0 \quad \textit{and} \quad \partial X_{\mathscr{F}}(\Phi) \subset \mathrm{Max}(\mathscr{F},\Phi).$$

3.6. **Subbuildings.** Let (ξ_+, ξ_-) be a pair of antipodal points in $\partial\mathrm{Min}(\mathscr{F})$. Let X' be the union of all appartments A in X whose boundary ∂A contains ξ_+ and ξ_-. This is an $\mathscr{F}$-stable, full-rank subbuilding of X. Put $\mathscr{F}' = \mathscr{F}|X'$.

Lemma 14. *$\mathscr{F}'$ is a decent isometry of X'.*

Proof. Let d be a geodesic line from ξ_+ to ξ_- in $\mathrm{Min}(\mathscr{F})$. Then X' is the parallel set of d in X. Let Y be the parallel set of d in $\mathrm{Min}(\mathscr{F})$, so that Y is a non-empty, locally compact, full-rank subbuilding of $\mathrm{Min}(\mathscr{F})$ [9, 4.8.1]. Clearly

$$Y \subset \mathrm{Min}(\mathscr{F}) \cap X' = \mathrm{Min}(\mathscr{F}').$$

Conversely, let x be a point in $\mathrm{Min}(\mathscr{F}')$. Since x belongs to X', there is a unique line d' in X' passing trough x and parallel to d. Since $\mathscr{F}$ is an isometry, $\mathscr{F}d'$ is parallel to $\mathscr{F}d$ which is parallel to d since d is contained in $\mathrm{Min}(\mathscr{F})$ on which $\mathscr{F}$ acts by translation. Therefore $\mathscr{F}d'$ is parallel to d', which implies that the function $y \mapsto \mathrm{dist}(y, \mathscr{F}y)$ is constant on d'. Since it equals $\min(\mathscr{F})$ for $y = x$, the whole of d' is contained in $\mathrm{Min}(\mathscr{F})$. Thus x belongs to the parallel set Y of d in $\mathrm{Min}(\mathscr{F})$. □

Lemma 15. *There is a retraction $r : X \twoheadrightarrow X'$ given by the following formula:*

$$r(x) = \lim_{t\to\infty} e(t\xi_-) \circ e(t\xi_+)(x).$$

It is $\mathscr{F}$-equivariant, non-expanding, and induces an isometry on any appartment A of X such that $\xi_+ \in \partial A$. It induces non-expanding, $\mathscr{F}$-equivariant retractions

$$r : \partial X \twoheadrightarrow \partial X' \quad \textit{and} \quad r : \mathscr{C}(\partial X) \twoheadrightarrow \mathscr{C}(\partial X').$$

For any $\Phi \in \mathscr{C}(\partial X)$, the action of $r(\Phi) = \Phi' \in \mathscr{C}(\partial X')$ on X' is given by

$$\Phi'(x) = \lim_{t\to\infty} e(t\xi_-) \circ \Phi \circ e(t\xi_+)(x).$$

Moreover, for all $x \in X$, $\Phi'(x + t\xi_+) = \Phi(x + t\xi_+)$ for all $t \gg 0$.

Proof. For $x \in X$, the function $t \mapsto f(t) = \angle_{x+t\xi_+}(\xi_-, \xi_+)$ is non-decreasing, takes finitely many values, and converges to $\angle(\xi_-, \xi_+) = \pi$. Let $[t_x, \infty[= f^{-1}(\pi)$. The geodesic rays from $y = x + t_x\xi_+$ to ξ_- and ξ_+ form a geodesic line d in X. The latter is contained in an appartement A of X whose boundary ∂A contains ξ_+ and ξ_-, thus $d \subset A \subset X'$. For every $t \geq t_x$, we have $e(t\xi_-) \circ e(t\xi_+)(x) = y + t_x\xi_-$. Therefore also $r(x) = y + t_x\xi_-$, and this belongs to $d \subset X'$. If x already belongs to X', then $t_x = 0$ and $x = y = r(x)$, so that r is a retraction of $X' \hookrightarrow X$. Since $\mathscr{F} \circ e(t\xi_\pm) = e(t\xi_\pm) \circ \mathscr{F}$ for every t, also $r \circ \mathscr{F} = \mathscr{F}' \circ r$. For x, y in X and $t \geq t_x, t_y$,

$$\operatorname{dist}(r(x), r(y)) = \operatorname{dist}(e(t\xi_-) \circ e(t\xi_+)(x), e(t\xi_-) \circ e(t\xi_+)(y)) \leq \operatorname{dist}(x, y).$$

If x and y both belong to an appartment A of X with $\xi_+ \in \partial A$, the above inequality becomes an equality since $e(t\xi_+)$ is an isometry on A and $e(t\xi_-)$ is an isometry on X'. Our retraction thus maps germs of geodesic rays to germs of geodesic rays and being non-expanding, it induces a map $r : \partial X \to \partial X'$ which is an $\mathscr{F}$-equivariant retraction of $\partial X' \hookrightarrow \partial X$. For x in X and ξ_1, ξ_2 in ∂X, one checks that for $t \gg 0$,

$$\begin{array}{rcl} \operatorname{dist}(r(x + t\xi_1), r(x + t\xi_2))^2 & \sim & 2t^2 (1 - \cos\angle(r(\xi_1), r(\xi_2))) \\ \leq \quad \operatorname{dist}(x + t\xi_1, x + t\xi_2)^2 & \sim & 2t^2 (1 - \cos\angle(\xi_1, \xi_2)) \end{array}$$

therefore $\angle(r(\xi_1), r(\xi_2)) \leq \angle(\xi_1, \xi_2)$. The induced retraction $r : \mathscr{C}(\partial X) \twoheadrightarrow \mathscr{C}(\partial X')$ is also $\mathscr{F}$-equivariant and non-expanding. We have $r(x + t\zeta) = r(x) + tr(\zeta)$ for all $t \geq 0$ and all x in any appartment A of X such that ζ and ξ_+ belong to ∂A. In other words: for $\Phi \in \mathscr{C}(\partial X)$ and $r(\Phi) = \Phi' \in \mathscr{C}(\partial X')$, we have $r \circ \Phi = \Phi' \circ r$ on any appartement A of X such that $\zeta = \Phi(\infty)$ and ξ_+ belong to ∂A. It follows that for any $x \in X$, there exists $t'_x \geq 0$ such that for all $t \geq t'_x$, $\Phi(x + t\xi_+) = \Phi'(x + t\xi_+)$, which equals $\Phi'(x) + t\xi_+$ whenever $x \in X'$. □

Lemma 16. *With notations as above, there exists a constant $\kappa > 0$ such that*

$$\forall \xi \in \partial X' : \qquad \angle(\xi, \xi_+) \leq \kappa \implies \langle \Phi, \xi \rangle = \langle \Phi', \xi \rangle .$$

Proof. We may assume that $\Phi \neq 0$. For $x \in X'$, $x_t = x + t\xi_+$ and $t \gg 0$, we have

$$B_\xi(\Phi'(x), x) = B_\xi(e(t\xi_-) \circ \Phi \circ e(t\xi_+)(x), x) = B_\xi(\Phi(x_t), x_t) . \tag{3.1}$$

Therefore $\langle \Phi, \xi \rangle \leq \langle \Phi', \xi \rangle$ for every ξ. Choose $\kappa > 0$ such that for any appartement A of X, $\partial A \cap \{\xi : \angle(\xi, \xi_+) \leq \kappa\} \neq \emptyset$ implies $\xi_+ \in \partial A$. For $\xi \in \partial X'$ with $\angle(\xi, \xi_+) \leq \kappa$, choose an appartment A of X such that ξ and $\Phi(\infty)$ belong to ∂A. Then also $\xi_+ \in \partial A$ and $X' \cap A \neq \emptyset$ since for any element x of A, $x + t\xi_+$ belongs to $X' \cap A$ for $t \gg 0$. Thus for $x \in X' \cap A$, $B_\xi(\Phi(x_t), x_t) = \langle \Phi, \xi \rangle$ for all $t \geq 0$. Then (3.1) shows that also $\langle \Phi', \xi \rangle \leq \langle \Phi, \xi \rangle$. □

Lemma 17. *Suppose that $\xi_+ \in \operatorname{Max}(\mathscr{F}, \Phi)$ and $\mu(\mathscr{F}, \Phi) \geq 0$. Then*

$$\xi_+ \in \operatorname{Max}(\mathscr{F}', \Phi') \quad \text{and} \quad \mu(\mathscr{F}', \Phi') = \mu(\mathscr{F}, \Phi).$$

Proof. The previous lemma implies that $\mu(\mathscr{F}, \Phi; -) = \mu(\mathscr{F}', \Phi'; -)$ in a neighbourghood of ξ_+ in $\partial\operatorname{Min}(\mathscr{F}')$. Since ξ_+ is a non-negative global maximum of $\mu(\mathscr{F}, \Phi; -)$, it is also a non-negative local maximum of $\mu(\mathscr{F}', \Phi'; -)$ on $\partial\operatorname{Min}(\mathscr{F}')$. But $\mu(\mathscr{F}', \Phi'; -)$ is concave and homegeneous on $\mathscr{C}(\partial\operatorname{Min}(\mathscr{F}'))$. Therefore ξ_+ is a global maximum of $\mu(\mathscr{F}', \Phi'; -)$ on $\partial\operatorname{Min}(\mathscr{F}')$, which proves the lemma. □

3.7. **End of the proof.** The formula 2.2 shows that $\mu(\mathscr{F},\Phi) \leq |\nu| + |\Phi| < \infty$. If $\mu(\mathscr{F},\Phi) > 0$, then $X_{\mathscr{F}}(\Phi) = \emptyset$ by remark 2 and $\mathrm{Max}(\mathscr{F},\Phi)$ is a singleton by Proposition 6. Given the corollaries 11 and 13, it remains to show that

$$\mu(\mathscr{F},\Phi) = 0 \quad \Longrightarrow \quad X_{\mathscr{F}}(\Phi) \neq \emptyset \quad \text{and} \quad \mathrm{Max}(\mathscr{F},\Phi) \subset \partial X_{\mathscr{F}}(\Phi).$$

We prove this by induction on the semi-simple rank $s(X)$ of X. Using 3.1 and 3.2, we may assume that X is semi-simple with $s(X) > 0$. Since $\mu(\mathscr{F},\Phi) = 0$, $\mathrm{Max}(\mathscr{F},\Phi) \neq \emptyset$. Pick $\xi_+ \in \mathrm{Max}(\mathscr{F},\Phi)$. Since $\mathrm{Min}(\mathscr{F})$ is a subbuilding of X, there exists an antipodal point ξ_- in $\partial\mathrm{Min}(\mathscr{F})$. Define $(X', \mathscr{F}', \Phi')$ as above with respect to (ξ_-, ξ_+). Then $s(X') < s(X)$ by [9, 4.8] and $\mu(\mathscr{F}', \Phi') = \mu(\mathscr{F},\Phi) = 0$ by the previous lemma and $X_{\mathscr{F}'}(\Phi') \neq \emptyset$ by induction. For $x \in X_{\mathscr{F}'}(\Phi')$ and $t \gg 0$,

$$\Phi(x + t\xi_+) = \Phi'(x + t\xi_+) = \Phi'(x) + t\xi_+ = \mathscr{F}'^{-1}(x) + t\xi_+ = \mathscr{F}^{-1}(x + t\xi_+)$$

therefore $x + t\xi_+ \in X_{\mathscr{F}}(\Phi)$. Thus $X_{\mathscr{F}}(\Phi) \neq \emptyset$ and $\xi_+ \in \partial X_{\mathscr{F}}(\Phi)$.

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