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A FIXED-SIZE BATCH SERVICE QUEUE WITH VACATIONS[1]

HO WOO LEE
Sung Kyun Kwan University
Department of Industrial Engineering
Su Won, Korea 440-746

SOON SEOK LEE
ETRI, Switching Method Section
Tae Jon, Korea 305-606

K.C. CHAE
KAIST
Department of Management Science
Tae Jeon, Korea 305-701

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ABSTRACT

The paper deals with batch service queues with vacations in which customers arrive according to a Poisson process. Decomposition method is used to derive the queue length distributions both for single and multiple vacation cases. The authors look at other decomposition techniques and discuss some related open problems.

Key words: Fixed Size Batch Service Queue, Vacations, Decomposition.

AMS (MOS) subject classifications: 60K10, 60K15, 60K25, 90B22, 90B25.

1. Introduction

Batch service queues have numerous applications to traffic, transportation, production, and manufacturing systems. The first study on batch service queues was due to Bailey [1]. He obtained the transform solution to the fixed-size batch service queue with Poisson arrivals. Miller [23] studied the batch arrival batch service queues and Jaiswal [14] considered batch service queues in which service size is random. Neuts [24] proposed the "general bulk service rule" in which service initiates only when a certain number of customers in the queue is available. His general bulk service rule was extended by Borthakur and Medhi [2]. Studies on waiting time in a batch service queue were also rendered by Downton [8], Cohen [5], Medhi [22] and Powell [25]. Fakinos [11] derived the relation between limiting queue size distributions at arrival and departure epochs. Briere and Chaudhry [3], Grassmann and Chaudhry [13], and Kambo and Chaudhry [15] used numerical approaches to obtain the performance measures. Numerical methods were found to be effective especially for batch service queues, because the transform solution of the queue length (number of customers in the system including those in service) in batch service systems contains

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some unknown values. For more extensive study on batch arrival/service queues, refer to Chaudhry and Templeton [4].

Vacation queues have been extensively studied by many researchers. Comprehensive surveys can be found in Doshi [7] and Takagi [26]. Most of the studies on vacation queues have been concerned with single-unit service systems such as M/G/1 or M^X/G/1 queues. A well-known result concerning vacation queues is the "decomposition property" (Fuhrmann and Cooper [12]) which states that the probability generating function (PGF) of the queue length of a vacation system can be factorized into the queue length of ordinary queue without vacation and "something else", the "something else" depends on the system characteristics. Lee et al. [18] and Lee et al. [19] analyzed the operating characteristics of batch arrival queues with N-policy and vacations, and obtained the queue length and waiting time distributions.

For batch service queues with vacations, there have been a few related works. Dhas [6] considered Markovian batch service systems and obtained the queue length distributions by matrix-geometric methods. Lee et al. [16] obtained various performance measures for M/G^B/1 queue with single vacation. Dshalalow and Yellen [10] considered a non-exhaustive batch service system with multiple vacations in which the server starts a multiple vacation whenever the queue drops below a level r and resumes service at the end of a vacation segment when the queue accumulates to at least r. They called such a system (r, R)-*quorum system*, R ($\geq r$) being the service capacity of the server. They applied the theory of the first excess level (Dshalalow [9]).

Lee et al. [17] showed that for some batch service queues, mean queue length may even decrease in systems with server vacations. This has an implication that for some batch service queues, customers do not have to complain about unavailability of the server. Instead, they would rather force the server to take a vacation.

In this paper, we are going to concentrate on a very specific batch service queues called the fixed-size batch service queues with vacations. We first analyze the fixed-size batch service queue without vacations.

2. The M/G^K/1/FS Queue

In this section we consider the fixed-size batch service queue without vacations. Consider a batch service queueing system in which the server can take in a maximum of K customers into his service. If less than K customers are in the queue just after a service completion, the server waits in the system until the queue size reaches K (Figure 1). We will denote this queueing system by M/G^K/1/FS queue in which 'FS' stands for 'fixed-size'. We are going to derive the decomposition of the queue length distribution at an arbitrary time point. By queue length we mean the number of customers in the system including those in service.

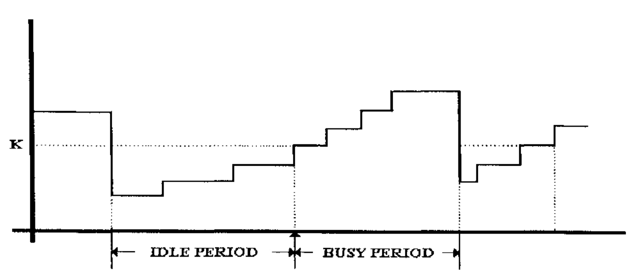

Figure 1. The M/G^K/1/FS queue

2.1 Queue length distribution at an arbitrary time epoch

The PGF of the steady-state queue length of $M/G^K/1/FS$ queueing system can be found in Chaudhry and Templeton [4] and is given by

$$P(z) = \frac{(z^K - 1)S^*(\lambda - \lambda z)\sum_{n=0}^{K-1} P_{n,0} z^n}{z^K - S^*(\lambda - \lambda z)}, \tag{2.1}$$

in which $S^*(\cdot)$ is the Laplace-Stieltjes transform (LST) of the service time distribution, and $P_{n,0}$ is the joint probability that the server is idle and there are n customers in the system, $n = 1, 2, \ldots, K-1$. $P_{n,0}$ can be seen as the probability of state n before a busy period begins. Note that $P(z)$ contains K unknown values, $P_{n,0}$, $n = 0, \ldots, K-1$. These unknowns occur in any type of batch service queues (see Chaudhry and Templeton [4]) and can be found by applying the well-known Rouche's theorem.

Theorem 2.1: *The PGF of the queue length distribution given by equation* (2.1) *decomposes into*

$$P(z) = P_U(z) \cdot P_I(z), \tag{2.2}$$

where

$$P_U(z) = \frac{(1-\rho)(z^K - 1)S^*(\lambda - \lambda z)}{z^K - S^*(\lambda - \lambda z)}, \tag{2.3}$$

and

$$P_I(z) = \frac{\sum_{n=0}^{K-1} \pi_n z^n}{\sum_{j=0}^{K-1} \pi_j}. \tag{2.4}$$

Here π_j is the probability that the idle process ever enters state j, $j = 0, 1, \ldots, K-1$.

Proof: From the definition of $P_{n,0}$, we see that

$$\sum_{n=0}^{K-1} P_{n,0} = 1 - \rho, \quad \left(\rho = \frac{\lambda}{K\mu}\right), \tag{2.5}$$

where ρ is the probability that the server is busy. Then $P_{n,0}/\sum_{j=0}^{K-1} P_{j,0}$ is the probability that the server is idle with n customers in the system under the condition that the server is idle. Define I_j as

$$I_j = \begin{cases} 1 & \text{if the idle process ever enters state } j, j = 0, 1, 2, \ldots, K-1, \\ 0 & o/w. \end{cases}$$

Then defining π_j as the probability that the idle process ever enters state j, we see that $\pi_j = Pr(I_j = 1)$, and $E\left\{\sum_{j=0}^{K-1} I_j\right\} = \sum_{j=0}^{K-1} \pi_j$ is the mean number of states the idle process enters until the server begins to be busy. Since the arrival process is Poisson with rate λ, the mean time for the idle process to stay in a state is $1/\lambda$, and thus $\sum_{j=0}^{K-1} \pi_j/\lambda$ becomes the mean length of the idle period. Thus we have $\frac{(\pi_n/\lambda)}{\sum_{j=0}^{K-1} \pi_j/\lambda} = P_{n,0}/\sum_{j=0}^{K-1} P_{j,0}$. The statement follows from equations (2.1) and (2.5). □

Remark 2.1: Equation (2.2) shows that at an arbitrary point of time, the queue length of the $M/G^K/1/FS$ queue is the sum of two random variables. The behavior of the system represented by the PGF, $P_U(z)$, is not clear at this point of our analysis and it will be left as an open problem for future study. $P_U(z)$ plays a very important role in the subsequent analysis and will be called "the basic stochastic system (BSS)."

Remark 2.2: (Decomposition of the queue length of $M/G^K/1/FS$ queue) Since $\sum_{j=0}^{K-1} P_{j,0}$ is the probability that the server is idle, $p_{n,0}/\sum_{j=0}^{K-1} P_{j,0}$ is the probability that there are n customers in the system under the condition that the server is idle. Thus $P_I(z)$, given by equation (2.4), can be interpreted as the PGF of the queue length given that the server is idle. Then we see that the PGF of the queue length distribution of the $M/G^K/1/FS$ queue decomposes into:

1. the BSS represented by $P_U(z)$, and
2. the queue length during the idle period represented by $P_I(z)$.

It is well known that the queue length of an M/G/1 vacation queue decomposes into two random variables one of which is the queue length of the ordinary M/G/1 queue (Fuhrmann and Cooper [12]). We will see in the upcoming sections that the role of the ordinary M/G/1 queue is played by the BSS in the decomposition of the fixed-size batch service queue.

Remark 2.3: In the $M/G^K/1/FS$ queue, let $P^+(z)$ be the PGF of the queue length embedded at departure epochs. Then from Chaudhry and Templeton [4],

$$P^+(z) = \frac{S^*(\lambda - \lambda z)\sum_{n=0}^{K-1}(z^K - z^n)P_n^+}{z^K - S^*(\lambda - \lambda z)}.$$

In order for the system state to enter n during the idle period, it suffices to have n or less customers at a departure point. Thus we have $\pi_n = \sum_{i=0}^{n} P_i^+$. Then, after some manipulations, we get

$$P^+(z) = \frac{S^*(\lambda - \lambda z)\sum_{n=0}^{K-1}(z^K - z^n)P_n^+}{z^K - S^*(\lambda - \lambda z)} = \frac{(z-1)S^*(\lambda - \lambda z)\sum_{n=0}^{K-1}\pi_n z^n}{z^K - S^*(\lambda - \lambda z)} = P(z)\cdot\frac{K(1-z)}{1-z^K}.$$

Therefore, we have the following relationship between the queue length at an arbitrary time point and a departure point:

$$P(z) = \frac{1-z^K}{K(1-z)}\cdot P^+(z). \tag{2.6}$$

This agrees with the result of Fakinos [11]. Observe that $(1-z^K)/[K(1-z)]$ is the PGF of the backward recurrence time of a renewal interval H in a discrete renewal process with $Pr(H = K) = 1$. Equation (2.6) states that a departing customer is more likely to find the system empty than an arriving customer.

3. A Fixed-Size Batch Service Queue with Vacations

In this section, we analyze the fixed-size batch service queues with server vacations. We consider two types of server vacations: single and multiple. The systems are described as follows:

1. Single vacation queue (Figure 2): everytime a service is finished, if less than K customers are in the queue, the server leaves for a vacation of random length V. When he returns from the vacation, and finds K or more customers waiting, he begins to process K of them. Otherwise, he

remains dormant in the system until the queue length reaches K. This system will be denoted as $\mathrm{M/G}^K/1/\mathrm{FS/SV}$ in which 'SV' stands for 'single vacation'.

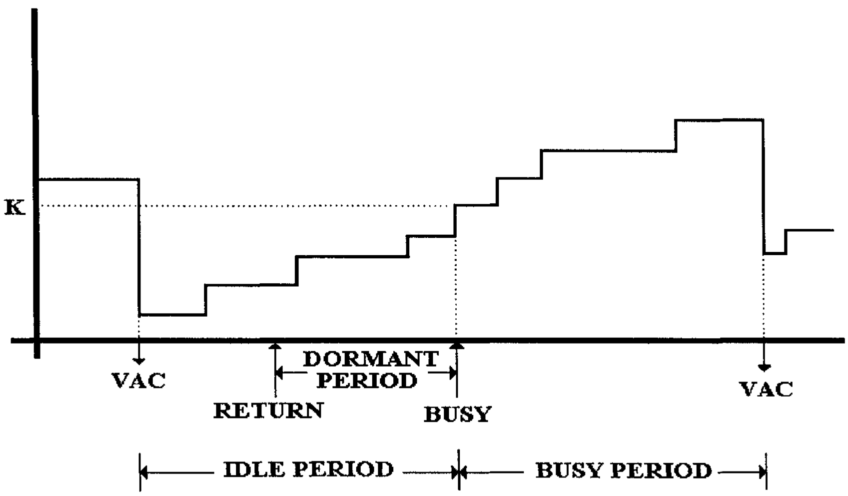

Figure 2. The $\mathrm{M/G}^K/1/\mathrm{FS/SV}$ queue.

2. Multiple vacation queue (Figure 3): every time a service is finished, and there are less than K customers in the queue, the server leaves for a vacation of random length V_1. If there are less than K customers in the queue upon his return from the vacation, he immediately leaves for another vacation of random length V_2, and so on until he finally finds K or more customers in the queue. We assume that $\{V_j, j \geq 1\}$ constitutes iid sequence with generic representation V. This system will be denoted by $\mathrm{M/G}^K/1/\mathrm{FS/MV}$ where 'MV' stands for 'multiple vacation'.

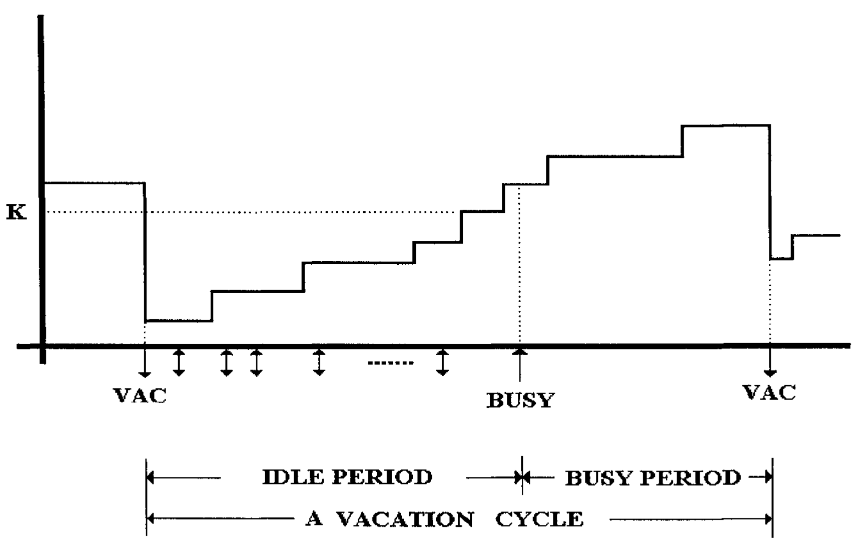

Figure 3. The $\mathrm{M/G}^K/1/\mathrm{FS/MV}$ queue.

We will use the following notations and probabilities:

λ	arrival rate
μ	service rate
K	service size
ρ	traffic intensity $(=\frac{\lambda}{K\mu})$
S	service time random variable
V	vacation time random variable
$s(x), S(x), S^*(\theta)$	pdf, DF, LST of S
$v(x), V(x), V^*(\theta)$	pdf, DF, LST of V
$S^0(t)$	remaining service time for the customer in service at time t
$V^0(t)$	remaining vacation time for the server on vacation at time t
$Y = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$	if the server is in dormancy if the server is busy in the system if the server is on vacation
$N(t)$	system size at time t
v_n	probability that n customers arrive during a vacation
$P_n(x,t)\Delta t =$	$Pr[N(t)=n, x \le S^0(t) \le x+\Delta t, Y=1]$ $(n \ge K)$
$P_n =$	$\lim_{t\to\infty} Pr[N(t)=n, Y=1]$
$Q_n(x,t)\Delta t =$	$Pr[N(t)=n, x \le V^0(t) \le x+\Delta t, Y=2]$ $(n \ge 0)$
$Q_n =$	$\lim_{t\to\infty} Pr[N(t)=n, Y=2]$
$R_n(t) =$	$Pr[N(t)=n, Y=0]$ $(0 \le n \le K-1)$
$R_n =$	$\lim_{t\to\infty} R_n(t)$

3.1 The M/G^K/1/FS/SV queue

In this section, we analyze the M/G^K/1/FS/SV queue. First we model the system by using the residual service and vacation times as supplementary variables. Using the above notations, we easily derive the following steady-state system of equations:

$$0 = -\lambda R_0 + Q_0(0), \tag{3.1.1}$$

$$0 = -\lambda R_n + \lambda R_{n-1} + Q_n(0), \quad (n = 1, 2, \ldots, K-1), \tag{3.1.2}$$

$$-\frac{d}{dx}P_K(x) = -\lambda P_K(x) + P_{2K}(0)s(x) + \lambda s(x)R_{K-1} + Q_K(0)s(x), \tag{3.1.3}$$

$$\begin{aligned} -\frac{d}{dx}P_{K+n}(x) = &-\lambda P_{K+n}(x) + \lambda P_{K+n-1}(x) \\ &+ P_{2K+n}(0)s(x) + Q_{K+n}(0)s(x), \; (n \ge 1) \end{aligned} \tag{3.1.4}$$

$$-\frac{d}{dx}Q_0(x) = -\lambda Q_0(x) + P_K(0)v(x), \tag{3.1.5}$$

$$-\frac{d}{dx}Q_m(x) = -\lambda Q_m(x) + \lambda Q_{m-1}(x) + P_{K+m}(0)v(x), \quad (m = 1, 2, \ldots, K-1), \tag{3.1.6}$$

$$-\frac{d}{dx}Q_n(x) = -\lambda Q_n(x) + \lambda Q_{n-1}(x), \quad (n \ge K). \tag{3.1.7}$$

Let us define the following Laplace transforms and generating functions:

$$P_n^*(\theta) = \int_0^\infty e^{-\theta x} P_n(x)dx, \quad (n \ge K),$$

$$Q_n^*(\theta) = \int_0^\infty e^{-\theta x} Q_n(x)dx, \quad (n \ge 0),$$

$$P^*(z,\theta)=\sum_{n=K}^{\infty}P_n^*(\theta)z^n,\quad P(z,0)=\sum_{n=K}^{\infty}P_n(0)z^n,$$

$$Q^*(z,\theta)=\sum_{n=0}^{\infty}Q_n^*(\theta)z^n,\quad Q(z,0)=\sum_{n=0}^{\infty}Q_n(0)z^n,$$

$$R(z)=\sum_{n=0}^{K-1}R_nz^n.$$

From equations (3.1.1) and (3.1.2), we get

$$R(z)=\frac{\sum_{n=0}^{K-1}(z^K-z^n)Q_n(0)}{\lambda(z-1)}. \tag{3.1.8}$$

After applying Laplace transforms and generating functions to equations (3.1.5), (3.1.6) and (3.1.7), we have

$$Q^*(z,\theta)=\frac{[V^*(\lambda-\lambda z)-V^*(\theta)]\sum_{n=0}^{K-1}P_{K+n}(0)z^n}{\theta-\lambda+\lambda z}. \tag{3.1.9}$$

Using Laplace transforms and generating functions to equations (3.1.3) and (3.1.4), we get

$$P^*(z,\theta)=\frac{z^K[S^*(\lambda-\lambda z)-S^*(\theta)]}{[z^K-S^*(\lambda-\lambda z)][\theta-\lambda+\lambda z]}\cdot Y(z), \tag{3.1.10}$$

where

$$Y(z)=[V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1}z^nP_{K+n}(0)+\sum_{n=0}^{K-1}(z^K-z^n)Q_n(0).$$

Now we use $P(z)=R(z)+P^*(z,0)+Q^*(z,0)$ to obtain the PGF of the queue length distribution at an arbitrary point of time. From equations (3.1.8), (3.1.9), (3.1.10) and $P(1)=1$,

$$P(z)=\frac{(1-\rho)(z^K-1)S^*(\lambda-\lambda z)}{z^K-S^*(\lambda-\lambda z)}\cdot\frac{Y(z)}{(z-1)E(Y)} \tag{3.1.11}$$

where $E(Y)=\frac{dY(z)}{dz}$ evaluated at $z=1$.

We need to take a closer look at $Y(z)$. First, $Q_n(0)$ can be expressed in terms of the number of customers at a vacation initiation point and the number of customers that arrive during the vacation as

$$Q_n(0)=\sum_{i=0}^{n}v_i\cdot P_{K+n-i}(0),\ (n=0,1,2,\ldots,K-1). \tag{3.1.12}$$

Then, $Y(z)$ becomes

$$Y(z)=\sum_{n=0}^{K-1}z^nP_{K+n}(0)\left\{[V^*(\lambda-\lambda z)-1]+(z-1)\sum_{m=0}^{K-n-1}z^mA_m\right\}, \tag{3.1.13}$$

where $A_m=\sum_{i=0}^{m}v_i=Pr$ (m or less customers arrive during a vacation). See Appendix 1-(a) for the derivation of equation (3.1.13).

Theorem 3.1.1: *The PGF of the queue length distribution given by equation* (3.1.11) *can be rewritten as*

$$P(z)=\frac{(1-\rho)(z^K-1)S^*(\lambda-\lambda z)}{z^K-S^*(\lambda-\lambda z)} \tag{3.1.14}$$

$$\times\left\{\frac{E(V)\sum_{n=0}^{K-1}\xi_n z^n}{E(V)+\frac{1}{\lambda}\sum_{j=0}^{K-1}\xi_j\sum_{m=0}^{K-j-1}A_m}\cdot\frac{1-V^*(\lambda-\lambda z)}{E(V)[\lambda-\lambda z]}+\frac{\frac{1}{\lambda}\sum_{n=0}^{K-1}\xi_n z^n\sum_{m=0}^{K-n-1}A_m z^{m+n}}{E(V)+\frac{1}{\lambda}\sum_{j=0}^{K-1}\xi_j\sum_{m=0}^{K-j-1}A_m}\right\}$$

where $\xi_n = \dfrac{P_{K+n}(0)}{\sum_{j=0}^{K-1} P_{K+j}(0)}$.

Proof: From $Y(z)$ given by (3.1.13), we get

$$E(Y)=\sum_{n=0}^{K-1}P_{K+n}(0)\left\{\lambda E(V)+\sum_{m=0}^{K-n-1}A_m\right\}.$$

The theorem follows from equations (3.1.11) and (3.1.13). □

Remark 3.1.1: Since there are n customers in the queue right after leaving for a vacation with $K+n$ customers in the system just before the service completion, it is easily seen that ξ_n equals the probability that there are n customers in the queue just after leaving for a vacation.

To interpret the terms in the bracket in equation (3.1.14), we need the following theorems.

Theorem 3.1.2: *Let* Γ_n *be the event that there are* n $(n=0,1,2,\ldots,K-1)$ *customers in the system just after the server leaves for a vacation, i.e.,* $Pr(\Gamma_n)=\xi_n$ *from remark* 3.1.1. *Then under* Γ_n, $A_m(m=0,1,2,\ldots,K-n-1)$ *is the probability that the system ever enters state* $n+m$ *during the dormant period.*

Proof: Define

$$I_m=\begin{cases}1 & \text{if the server is dormant with } n+m \text{ customers in the queue,}\\ 0 & o/w.\end{cases}$$

In order for the system to enter state $n+m$ during the dormant period, it is necessary that m or less customers arrive during the vacation. Thus we have $Pr(I_m=1)=\sum_{i=0}^{m}v_i=A_m$. □

Theorem 3.1.3: $\sum_{n=0}^{K-1}\xi_n\left[E(V)+\frac{1}{\lambda}\sum_{m=0}^{K-n-1}A_m\right]=E(V)+\frac{1}{\lambda}\sum_{n=0}^{K-1}\xi_n\sum_{m=0}^{K-n-1}A_m$ *is the mean length of the idle period* (= *vacation* + *dormant period*).

Proof: From theorem 3.1.2, it is easily seen that $\sum_{m=0}^{K-n-1}I_m$ is the number of states that are entered under Γ_n before the server gets busy. Thus $E\left\{\sum_{m=0}^{K-n-1}I_m\right\}=\sum_{m=0}^{K-n-1}Pr(I_m=1)$ $=\sum_{m=0}^{K-n-1}A_m$ is the mean number of states that are entered under Γ_n. Since the arrival process is Poisson, $\sum_{m=0}^{K-n-1}A_m/\lambda$ is the mean length of the dormant period under Γ_n. Thus $E(V)+\sum_{m=0}^{K-n-1}A_m/\lambda$ is the mean length of the idle period under Γ_n. Relaxing the condition on Γ_n completes the proof. □

Remark 3.1.2: Now we are ready to interpret the terms in the bracket of equation (3.1.14).

$$\Omega_1=\frac{E(V)}{E(V)+\frac{1}{\lambda}\sum_{n=0}^{K-1}\xi_n\sum_{m=0}^{K-n-1}A_m}$$

is the conditional probability that the server is idle due to vacations. On the other hand,

$$\Omega_2 = \frac{\frac{1}{\lambda}\sum_{n=0}^{K-1}\xi_n\left\{\sum_{m=0}^{K-n-1}A_m\right\}}{E(V)+\frac{1}{\lambda}\sum_{n=0}^{K-1}\xi_n\sum_{m=0}^{K-n-1}A_m}$$

is the conditional probability that the server is idle due to dormancy. Therefore, the first term in the bracket is the sum of the number of customers left behind in the queue at the vacation initiation point and the number of customers that arrive during the residual vacation. This event occurs with probability Ω_1. The second term is nothing but the system process during the dormant period which occurs with probability Ω_2.

Now we are ready to interpret the decomposition of the queue length distribution.

Remark 3.1.3: (Decomposition of the queue length) The queue length of M/G^K/1/FS/SV queue can be decomposed into

1. the BSS,
2. the queue length at the vacation initiation point, and
3. the queue length during the idle period.

Remark 3.1.4: The decomposition of the queue length distribution is different from that for the single-unit service cases like M/G/1 queues with vacations. In those cases, the decomposition contains the queue length of "the ordinary queueing system without vacations". But in our case, the decomposition contains the BSS whose PGF is given by $P_U(z) = \frac{(1-\rho)(z^K-1)S^*(\lambda-\lambda z)}{z^K - S^*(\lambda-\lambda z)}$. The authors failed to identify the stochastic behavior of the BSS. This will be left as an open problem for further study.

3.2 The M/G^K/1/FS/MV queue

In this section we derive the decomposition of the M/G^K/1/FS/MV queue. In addition to the earlier notations, we define

$$Q_{n,j}(x,t)\Delta t = Pr[N(t)=n, x \le V^0(t) \le x+\Delta t, N_V(t)=j, Y=2],\ (j \ge 1),$$

where $N_V(t)$ is the current index of the vacation the server is on during a vacation cycle. For example, if the server is on the 2nd vacation during a vacation cycle, we have $N_V(t)=2$. Note that $N_V(t)$ is reset every time a new vacation cycle begins. Then we have the following steady-state system of equations:

$$-\frac{d}{dx}P_K(x) = -\lambda P_K(x) + P_{2K}(0)s(x) + \sum_{j=1}^{\infty} Q_{K,j}(0)s(x), \tag{3.2.1}$$

$$-\frac{d}{dx}P_{K+n}(x) = -\lambda P_{K+n}(x) + \lambda P_{K+n-1}(x)$$

$$+ P_{2K+n}(0)s(x) + \sum_{j=1}^{\infty} Q_{K+n,j}(0)s(x), (n \ge 1), \tag{3.2.2}$$

$$-\frac{d}{dx}Q_{0,1}(x) = -\lambda Q_{0,1}(x) + P_K(0)v(x), \tag{3.2.3}$$

$$-\frac{d}{dx}Q_{m,1}(x) = -\lambda Q_{m,1}(x) + \lambda Q_{m-1,1}(x) + P_{K+m}(0)v(x), (m=1,2,\ldots,K-1), \tag{3.2.4}$$

$$-\frac{d}{dx}Q_{n,1}(x) = -\lambda Q_{n,1}(x) + \lambda Q_{n-1,1}(x), (n \ge K), \tag{3.2.5}$$

$$-\frac{d}{dx}Q_{0,j}(x) = -\lambda Q_{0,j}(x) + Q_{0,j-1}(0)v(x), \ (j \geq 2), \tag{3.2.6}$$

$$-\frac{d}{dx}Q_{m,j}(x) = -\lambda Q_{m,j}(x) + \lambda Q_{m-1,j}(x) + Q_{m,j-1}(0)v(x),$$
$$(m = 0, 1, \ldots, K-1), \ (j \geq 2), \tag{3.2.7}$$

$$-\frac{d}{dx}Q_{n,j}(x) = -\lambda Q_{n,j}(x) + \lambda Q_{n-1,j}(x), \quad (n \geq K), \quad (j \geq 2). \tag{3.2.8}$$

Let us define the following Laplace transforms and generating functions:

$$Q^*_{n,j}(\theta) = \int_0^\infty e^{-\theta x} Q_{n,j}(x)dx,$$

$$Q^*_j(z,\theta) = \sum_{n=0}^\infty Q^*_{n,j}(\theta)z^n,$$

$$Q_j(z,0) = \sum_{n=0}^\infty Q_{n,j}(0)z^n,$$

$$P^*(z,\theta) = \sum_{n=K}^\infty P^*_n(\theta)z^n,$$

$$P(z,0) = \sum_{n=K}^\infty P_n(0)z^n.$$

From equations (3.2.3), (3.2.4), (3.2.5), we get

$$Q_1(z,0) = V^*(\lambda - \lambda z)\sum_{n=0}^{K-1} z^n P_{K+n}(0), \tag{3.2.9}$$

and

$$Q^*_1(z,\theta) = \frac{[V^*(\lambda - \lambda z) - V^*(\theta)]\sum_{n=0}^{K-1} P_{K+n}(0)}{\theta - \lambda + \lambda z}. \tag{3.2.10}$$

From equations (3.2.6), (3.2.7) and (3.2.8),

$$Q_j(z,0) = V^*(\lambda - \lambda z)\sum_{n=0}^{K-1} z^n Q_{n,j-1}(0), \quad (j \geq 2), \tag{3.2.11}$$

$$Q^*_j(z,\theta) = \frac{[V^*(\lambda - \lambda z) - V^*(\theta)]Q_{j-1}(z,0)}{\theta - \lambda + \lambda z}, \ (j \geq 2). \tag{3.2.12}$$

From equations (3.2.1), (3.2.2), (3.2.9), we obtain

$$P^*(z,\theta) = \frac{z^K[S^*(\lambda - \lambda z) - S^*(\theta)][V^*(\lambda - \lambda z) - 1]}{[z^K - S^*(\lambda - \lambda z)][\theta - \lambda + \lambda z]} \cdot X(z), \tag{3.2.13}$$

where

$$X(z) = \sum_{n=0}^{K-1} z^n P_{K+n}(0) + \sum_{j=1}^\infty \ \sum_{n=0}^{K-1} z^n Q_{n,j}(0).$$

From equations (3.2.9) and (3.2.11), we have

$$Q_{n,j}(0) = \sum_{i=0}^n v_i \cdot Q_{n-i,j-1}(0), \ (j \geq 1) \tag{3.2.14}$$

where $Q_{n,0}(0) = P_{K+n}(0)$ for $n < K$. Now let $\phi_n = P_{K+n}(0) + \sum_{j=0}^{\infty} Q_{n,j}(0)$. Then, by mathematical induction, we see that

$$\phi_n = \frac{1}{1-v_0}\left\{P_{K+n}(0) + \sum_{i=1}^{n-1}\phi_i v_{n-i}\right\}, \quad n = 0,1,2,\ldots,K-1. \tag{3.2.15}$$

Let us define β_n as

$$\beta_0 = 1, \ \ \beta_n = \sum_{i=1}^{n}\frac{v_i}{1-v_0}\beta_{n-i}, \quad n = 0,1,2,\ldots,K-1. \tag{3.2.16}$$

Then, we get

$$\phi_n = \frac{1}{1-v_0}\sum_{j=0}^{n} P_{K+j}(0)\cdot\beta_{n-j}, \ \ n = 0,1,2,\ldots,K-1. \tag{3.2.17}$$

Thus,

$$X(z) = \frac{1}{1-v_0}\sum_{n=0}^{K-1} z^n P_{K+n}(0)\cdot\left\{\sum_{j=0}^{K-n-1}\beta_j z^j\right\}. \tag{3.2.18}$$

See Appendix 1-(b) for the derivation of equation (3.2.18).

From $P(z) = P^*(z,0) + \sum_{j=1}^{\infty} Q_j^*(z,0)$ and $P(1) = 1$, we have the following theorem without proof.

Theorem 3.2.1: *The PGF of the queue length distribution in the $M/G^K/1/FS/MV$ system satisfies the formula*

$$P(z) = \frac{(1-\rho)(z^K-1)S^*(\lambda-\lambda z)}{z^K - S^*(\lambda-\lambda z)}\cdot\frac{1-V^*(\lambda-\lambda z)}{E(V)[\lambda-\lambda z]} \cdot \frac{\sum_{n=0}^{K-1}\xi_n^0 z^n\cdot\left\{\sum_{j=0}^{K-n-1}\beta_j z^j\right\}}{\sum_{n=0}^{K-1}\xi_n^0 \sum_{j=0}^{K-n-1}\beta_j}, \tag{3.2.19}$$

where $\xi_n^0 = \dfrac{P_{K+n}(0)}{\sum_{j=0}^{K-1} P_{K+j}(0)}$.

Remark 3.2.1: ξ_n^0 is the probability that there are n customers in the queue right after leaving for the first vacation. This is easy to see from the fact that there are n customers left behind in the queue with $K+n$ customers being before the server takes his vacation.

Remark 3.2.2: Equation (3.2.19) shows that the queue length of M/G^K/1/FS/MV system equals the sum of three random variables: the first is the BSS and the second is the number of customers that arrive during the residual vacation time. The interpretation of the third term is now in order. To identify the third term of equation (3.2.19), we need to define a "super vacation" and the "super vacation process (SVP)" (Figure 4). A "super vacation" is defined as the period of time from the time point when the server leaves for a vacation until a change in the system state is observed upon his return from the vacation. Thus a super vacation consists of one or more vacations. In the M/G^K/1/FS/MV queue, if there are $n(<K)$ customers at the beginning of the idle period, there are $(K-n)$ or less super vacations during that idle period. The SVP is a sequence of period of times starting at the super vacation initiation points. Thus, a vacation cycle consists of a SVP and a busy period. In Figure 4, the SVP is depicted in bold solid lines and it consists of three super vacations G_1, G_2, and G_3. Levy and Yechiali [21] derived the distribution of the length of the idle period of M/G/1 queue with multiple vacations which is

equivalent to the length of a super vacation in our system. But their derivation was wrong. See Takagi [26], or Lee and Lee [20] for the corrected distribution.

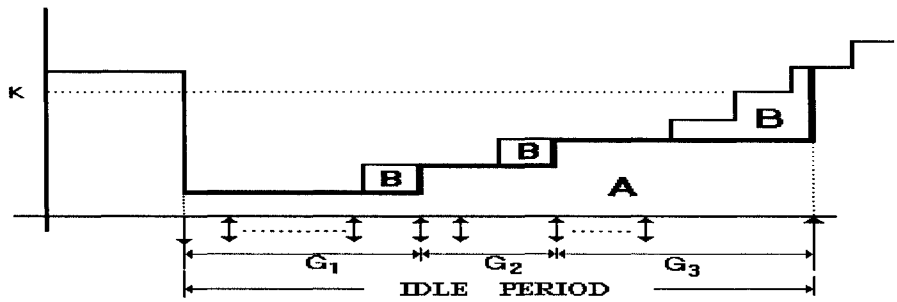

Figure 4. The super vacation process of M/G^K/1/FS/MV queue.

Theorem 3.2.2: *Let Π_n be the event that there are n customers in the queue at the idle period initiation point (or just after leaving for the first vacation). Under Π_n, β_j, $j=0,1,2,\ldots,K-n-1$, given by equation (3.2.16), is the probability for the SVP to ever enter state $(n+j)$.*

Proof: Under Π_n, the SVP enters state n with probability 1. Thus we have $\beta = 1$. Now define $I_{i,j}$ as

$$I_{i,j}=\begin{cases}1 & \text{if there are } n+j \text{ customers just after leaving for } i\text{th super vacation,} \\ & \qquad (i\le j+1), \quad (j\ge 0), \\ 0 & o/w.\end{cases}$$

Conditioning on the queue length at the previous super vacation initiation point, we have

$$Pr(I_{i,j}=1)=\sum_{k=1}^{j}\frac{v_k}{1-v_0}Pr(I_{i-1,j-k}=1), \quad (1\le j-k+2).$$

Then,

$$\begin{aligned}\sum_{i=1}^{j+1}Pr(I_{i,j}=1) &= \sum_{i=1}^{j+1}\sum_{k=1}^{i}Pr(I_{i-1,j-k}=1)\cdot\frac{v_k}{1-v_0} \\ &= \sum_{k=1}^{j}\sum_{i=2}^{j-k+2}Pr(I_{i-1,j-k}=1)\cdot\frac{v_k}{1-v_0} \\ &= \sum_{k=1}^{j}\sum_{i=1}^{j-k+1}Pr(I_{i,j}=1)\cdot\frac{v_k}{1-v_0}.\end{aligned}$$

Letting $\sum_{i=1}^{j+1}Pr(I_{i,j}=1)=\beta_j$, we have $\beta_j=\sum_{k=1}^{j}\beta_{j-k}\frac{v_k}{1-v_0}$ which satisfies equation (3.2.16). Then we see that $\beta_j=\sum_{i=1}^{j+1}Pr(I_{i,j}=1)$ is the probability that the SVP ever enters state $n+j$. □

Theorem 3.2.3: *Under Π_n, $\sum_{j=0}^{K-n-1}\beta_j$ is the mean number of super vacations during an idle period.*

Proof: Since only the one super vacation sees state j during a SVP, we have

$$\sum_{i=1}^{j+1} I_{i,j} = \begin{cases} 0 & \text{if } I_{j-k+1,j} = 0 \text{ for all } k,\ k = 0,1,2,\ldots,j \\ 1 & o/w. \end{cases}$$

Thus,

$$E\left(\sum_{i=1}^{j+1} I_{i,j}\right) = Pr\left(\sum_{i=1}^{j+1} I_{i,j} = 1\right)$$

$$= Pr(I_{j-k+1,j} = 1 \text{ for } \textit{some } k,\ k = 0,1,\ldots,j)$$

$$= \sum_{i=1}^{j} Pr(I_{i,j} = 1) = \beta_j.$$

But $\sum_{j=0}^{K-n-1} \sum_{i=1}^{j+1} I_{i,j}$ is the number of states the SVP enters. Taking expectation completes the proof. □

Remark 3.2.3: From theorem 3.2.3, $\sum_{n=0}^{K-1} \xi_n^0 \sum_{j=0}^{K-n-1} \beta_j$ is the mean number of super vacations during an idle period. Since the mean length of a super vacation is $E(V)/(1-v_0)$, $\frac{E(V)}{1-v_0} \cdot \sum_{n=0}^{K-1} \xi_n^0 \sum_{j=0}^{K-n-1} \beta_j$ is the mean length of the idle period. $\frac{E(V)\beta_j}{1-v_0}$ is the mean length of the super vacation that sees $(n+j)$ customers. Thus, $\beta_j / \left\{ \sum_{n=0}^{K-1} \xi_n^0 \sum_{j=0}^{K-n-1} \beta_j \right\}$ is the portion of state $n = j$ contributes to the idle period. Thus we see that the third term of equation (3.2.19) is the PGF of the queue length observed by the returning server from a vacation.

Remark 3.2.4: (Interpretation of the decomposition) In summary, the queue length of $M/G^K/1/FS/MV$ queue decomposes into

1. the BSS,
2. the queue size observed by the returning server ("A" in Figure 4), and
3. the queue size unobserved by the server on vacation, i.e., the number of customers that arrive during the residual vacation ("B" in Figure 4).

4. Summary and Suggestions for Further Study

In this paper, we considered fixed-size batch service queue with single and multiple vacations. We derived the decompositions of the queue length distributions and provided relevant interpretations. The decompositions contain "the basic stochastic system" whose stochastic behavior is not clarified yet. Identifying the operational characteristics of the basic stochastic system could lead to finding more of the decompositions of the fixed-size batch service queue.

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Appendix

(a) Derivation of equation (3.1.13)

$$\begin{aligned}
Y(z) &= [V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1} z^n P_{K+n}(0) + \sum_{n=0}^{K-1}(z^K - z^n)Q_n(0) \\
&= [V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1} z^n P_{K+n}(0) + (z-1)\sum_{n=0}^{K-1} z^n \sum_{j=0}^{n} Q_j(0) \\
&= [V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1} z^n P_{K+n}(0) + (z-1)\sum_{n=0}^{K-1} z^n \sum_{j=0}^{n}\sum_{i=0}^{j} v_i \cdot P_{K+j-i}(0) \\
&= [V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1} z^n P_{K+n}(0) + (z-1)\sum_{n=0}^{K-1} z^n P_{K+n}(0)\sum_{j=0}^{K-n-1} z^j \sum_{i=0}^{j} v_i \\
&= [V^*(\lambda-\lambda z)-1]\sum_{n=0}^{K-1} z^n P_{K+n}(0) + (z-1)\sum_{n=0}^{K-1} z^n P_{K+n}(0)\sum_{j=0}^{K-n-1} z^j A_j \\
&= \sum_{n=0}^{K-1} z^n P_{K+n}(0)\left\{[V^*(\lambda-\lambda z)-1] + (z-1)\sum_{j=0}^{K-n-1} z^j A_j\right\}.
\end{aligned}$$

(b) Derivation of equation (3.2.18)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{K-1} z^n P_{K+n}(0) + \sum_{j=1}^{\infty}\sum_{n=0}^{K-1} z^n Q_{n,j}(0) \\
&= \sum_{n=0}^{K-1} z^n \left\{P_{K+n}(0) + \sum_{j=1}^{\infty} Q_{n,j}(0)\right\} \\
&= \sum_{n=0}^{K-1} z^n \phi_n \\
&= \frac{1}{1-v_0}\sum_{n=0}^{K-1} z^n \sum_{j=0}^{n} P_{K+j}(0)\beta_j \\
&= \frac{1}{1-v_0}\sum_{n=0}^{K-1} z^n P_{K+n}(0)\left\{\sum_{j=0}^{K-n-1}\beta_j z^j\right\}.
\end{aligned}$$

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